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Natural Dualities in Partnership

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Abstract Traditionally in natural duality theory the algebras carry no topology and the objects on the dual side are structured Boolean spaces. Given a duality, one may ask when the topology can be swapped to the other side to yield a partner duality (or, better, a dual equivalence) between a category of topological algebras and a category of structures. A prototype for this procedure is provided by the passage from Priestley duality for bounded distributive lattices to Banaschewski duality for ordered sets. Moreover, the partnership between these two dualities yields as a spin-off a factorisation of the functor sending a bounded distributive lattice to its natural extension, *alias*, in this case, the canonical extension or profinite completion. The main theorem of this paper validates topology swapping as a uniform way to create new dual adjunctions and dual equivalences: we prove that, for every finite algebra of finite type, each dualising alter ego gives rise to a partner duality. We illustrate the theorem via a variety of natural dualities, some classic and some less familiar.

Dedicated to the 75th birthday of Professor Tibor Katriňák.

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For lattice-based algebras this leads immediately, as in the Priestley–Banaschewski example, to a concrete description of canonical extensions.

Keywords Natural duality · Natural extension · Canonical extension

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1 Introduction

Priestley duality for the variety $\mathcal{D}$ of bounded distributive lattices sets up a dual equivalence between $\mathcal{D}$ and the category of Priestley spaces, which we shall denote by $\mathcal{P}_{\mathcal{T}}$ [36]. Banaschewski's duality sets up a dual equivalence between the category $\mathcal{D}_{\mathcal{T}}$ of Boolean topological distributive lattices and the category $\mathcal{P}$ of ordered sets [1]. These partner dualities are set up by hom-functors into a pair of structures: a structure (with no topology) and a topological structure acting as its alter ego. Thus the two dualities are related to one another by 'topology swapping': each can be obtained from its partner by removing the topology from the alter ego and applying it to the untopologised structure. This connection was revealed by Davey et al. [13], in the context of an investigation of canonical extensions. Earlier, and providing the initial impetus for [13], Haviar and Priestley [27] had used a similar technique to derive new and very natural descriptions of canonical extensions for Stone algebras and double Stone algebras. In this paper, building on work of Hoffmann [30], Davey [10] and Davey et al. [16] we demonstrate that the above examples illustrate a very general procedure that takes a duality and swaps the topology from the structure side to the algebra side to obtain a new partner duality paired with the original one. In many, but by no means all cases, one or both of the paired dualities will be full, so providing a tight relationship between the four categories involved in the partnership.

Our focus will be on paired dualities as a phenomenon within natural duality theory. We do not pursue here the implications of the existence of such pairings in the context of canonical extensions. We propose instead to discuss in a companion paper the significance of our work for the algebraic and relational semantics for logics modelled algebraically by finitely generated lattice-based varieties.

The stepping-off point for our presentation is the concept of the *natural extension*, $n_{\mathcal{A}}(\mathbf{A})$, of an algebra $\mathbf{A}$ in a prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ generated by a family $\mathcal{M}$ of finite algebras. This was introduced by Davey et al. [11]. They observed that a multisorted natural duality between the class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ and a class $\mathcal{X}_{\mathcal{T}}$ of Boolean topological structures can be used to simplify the description of the natural extension [11, Theorem 4.3]. The natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A} \in \mathcal{A}$ carries a Boolean topology and this enables us to view the natural extension as a functor $n_{\mathbf{A}}$ from $\mathcal{A}$ to a class $\mathcal{A}_{\mathcal{T}}$ of Boolean topological $\mathcal{A}$-algebras.

We obtain a new, and widely applicable, topology-swapping theorem (TopSwap Theorem 2.4). By applying this theorem along with basic results from the theory of natural dualities and from the associated theory of standard topological quasi-varieties, we obtain a hierarchy of increasingly rich results on natural extensions, valid under progressively more stringent assumptions.

(1) At the base level of the hierarchy, we obtain the natural extension functor $n_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\mathcal{T}}$ as a composite of three functors—two hom-functors and a forgetful functor—along with a duality between the category $\mathcal{A}_{\mathcal{T}}$ and a category $\mathcal{X}$ of discrete structures.
(2) At the next level, we will seek to set up a dual equivalence between the category $\mathcal{A}_{\mathcal{T}}$ and the category $\mathcal{X}$.
(3) Finally, at the top level we have (first-order) descriptions of the topological algebras in $\mathcal{A}_{\mathcal{T}}$ and of the structures in dually equivalent category $\mathcal{X}$.

These levels are considered in turn in Sections 3, 4 and 5. When applied to lattice-based algebras, the top level provides a rich environment within which the canonical extension lives and has the potential to provide worthwhile algebraic and relational semantics for an associated logic.

A natural duality for a prevariety of the form $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ generated by a finite family $\mathcal{M}$ of finite algebras will involve a dual category in which the objects are multisorted topological structures with a sort for each algebra in the family $\mathcal{M}$. To keep our presentation as simple as possible, we concentrate on the case in which $\mathcal{M}$ consists of a single algebra, but note along the way that the whole discussion extends to the case where $\mathcal{M}$ is a finite set of finite algebras. The final example in Section 4 illustrates the multisorted case.

2 Natural Extensions via Natural Dualities

Davey et al. [11] defined the natural extension functor $n_{\mathcal{A}}$ on the prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ generated by a (possibly infinite) family $\mathcal{M}$ of finite algebras. They proved that the natural extension $n_{\mathcal{A}}(\mathbf{A})$ is isomorphic to the $\mathcal{A}$-profinite completion of $\mathbf{A}$. Combined with results of Harding [26] and Gouveia [25] on profinite completions, it follows that if $\mathcal{A}$ is a finitely generated variety of lattice-based algebras, then the natural extension agrees with the canonical extension in $\mathcal{A}$. (A direct proof of this fact, avoiding profinite completions, is given by Davey and Priestley [18, 19].) As indicated above, we will restrict to the case where $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ is the quasivariety generated by a single finite algebra $\mathbf{M}$. Nevertheless, we note that, modulo some slightly cumbersome notation, our results extend to the case where $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ is the prevariety generated by a *finite* family $\mathcal{M}$ of finite algebras.

Let $\mathbf{M}$ be a finite algebra. We shall associate with $\mathbf{M}$ two naturally defined classes—a class $\mathcal{A}$ of algebras and a class $\mathcal{A}_{\mathcal{T}}$ of Boolean topological algebras. As usual, the class $\mathcal{A}$ is the quasivariety $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ generated by $\mathbf{M}$. To define the class $\mathcal{A}_{\mathcal{T}}$ we first let $\mathbf{M}_{\mathcal{T}}$ denote $\mathbf{M}$ equipped with the discrete topology, and then define $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_{c}\mathbb{P}(\mathcal{M}_{\mathcal{T}})$, the class of isomorphic copies of topologically closed subalgebras of powers of $\mathbf{M}_{\mathcal{T}}$. We make $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$ into categories in the expected way: the morphisms are the homomorphisms and the continuous homomorphisms, respectively.

Let $\mathbf{A} \in \mathcal{A}$. We define a map $e_{\mathbf{A}} : \mathbf{A} \to \mathbf{M}_{\mathcal{T}}^{\mathcal{A}(\mathbf{A},\mathbf{M})}$ by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and $x \in \mathcal{A}(\mathbf{A}, \mathbf{M})$. As $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, the homomorphism $e_{\mathbf{A}}$ is an embedding (ignoring the topology on the codomain). The *natural extension* $n_{\mathcal{A}}(\mathbf{A})$ of $\mathbf{A}$ in $\mathcal{A}$ is then defined to be the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\mathbf{M}_{\mathcal{T}}^{\mathcal{A}(\mathbf{A},\mathbf{M})}$. Clearly $n_{\mathcal{A}}(\mathbf{A}) \in \mathcal{A}_{\mathcal{T}}$, and it is proved in [11] that $n_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\mathcal{T}}$ is the object half of a functor that is independent of the choice of the generator $\mathbf{M}$ of the quasivariety $\mathcal{A}$. The following theorem comes from [11] where it is established in greater generality. We refer to [11] for the missing definitions and proofs.

Theorem 2.1 ([11, Theorems 3.6 and 3.8]) *Let* $\mathbf{M}$ *be a finite algebra, let* $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ *and let* $\mathbf{A} \in \mathcal{A}$. *The natural extension* $n_{\mathcal{A}}(\mathbf{A})$ *of* $\mathbf{A}$ *is (isomorphic to) the* $\mathcal{A}$*-profinite completion of* $\mathbf{A}$. *If* $\mathbf{M}$ *is a lattice-based algebra, then* $n_{\mathcal{A}}(\mathbf{A})$ *is a canonical extension of* $\mathbf{A}$ *in* $\mathcal{A}$.

We now wish to show how a natural duality for the quasivariety $\mathcal{A}$ leads to a simple description of the maps $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}) \to M$ that belong to $n_{\mathcal{A}}(\mathbf{A})$. For this it would suffice to work with the usual natural-duality setting (*á la* Clark and Davey [3]) in which we have algebras on the discrete side and Boolean topological structures on the other. Since we also want to show that $n_{\mathcal{A}}$ factors in a natural way as a composite of three functors, we need the more general setting (*á la* Hoffmann [30], Davey [10] and Davey et al. [16]) in which structures are allowed on both the discrete and the topological side. We briefly review the requisite theory and refer to [3, 10, 16, 30] for the missing details.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle$ and $\mathbf{M}_2 = \langle M; G_2, H_2, R_2\rangle$ be finite structures on the same underlying set. Here the G_i, H_i and R_i are respectively sets of finitary operations, partial operations and relations on M. Assume that $\mathbf{M}_2$ is *compatible with* $\mathbf{M}_1$, that is, each (n-ary) operation $g \in G_2$ is a homomorphism from $\mathbf{M}_1^n$ to $\mathbf{M}_1$, for each (n-ary) partial operation $h \in H_2$, the domain of h forms a substructure $\mathrm{dom}(h)$ of $\mathbf{M}_1^n$ and h is a homomorphism from $\mathrm{dom}(h)$ to $\mathbf{M}_1$, and each (n-ary) relation $r \in R_2$ forms a substructure of $\mathbf{M}_1^n$. The topological structure $(\mathbf{M}_2)_{\mathcal{T}}$ obtained by adding the discrete topology to $\mathbf{M}_2$ is denoted by $\underset{\sim}{\mathbf{M}}_2$ and is referred to as an *alter ego* of $\mathbf{M}_1$. We define the category $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ of structures and the category $\mathcal{X}_{\mathcal{T}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ of Boolean topological structures. (Note that the class operator $\mathbb{P}$ allows empty indexed products and so yields the total one-element structure while $\mathbb{P}^+$ does not, and that the operator $\mathbb{S}$ excludes the empty structure while $\mathbb{S}^0$ includes the empty structure when the type does not include nullary operations.)

There are naturally defined hom-functors $\mathrm{D}\colon \mathcal{A} \to \mathcal{X}_{\mathcal{T}}$ and $\mathrm{E}\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{A}$, given on objects by

$$\mathrm{D}(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \le \underset{\sim}{\mathbf{M}}_2^{A} \text{ and } \mathrm{E}(\mathbf{X}) := \mathcal{X}_{\mathcal{T}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_2) \le \mathbf{M}_1^{X},$$

for all $\mathbf{A} \in \mathcal{A}$ and all $\mathbf{X} \in \mathcal{X}_{\mathcal{T}}$. The evaluation maps

$$e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{ED}(\mathbf{A}) \text{ and } \varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$$

are always embeddings and $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$. We say that $\underset{\sim}{\mathbf{M}}_2$ *yields a duality* on $\mathcal{A}$ if, for all $\mathbf{A} \in \mathcal{A}$, the map $e_{\mathbf{A}}$ is an isomorphism, that is, the only continuous homomorphisms $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to \underset{\sim}{\mathbf{M}}_2$ are the evaluations maps $e_{\mathbf{A}}(a)$, for $a \in A$. We also say that $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$. If, in addition, $\varepsilon_{\mathbf{X}}$ is an isomorphism, for all $\mathbf{X} \in \mathcal{X}_{\mathcal{T}}$, we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality* on $\mathcal{A}$. If $e_{\mathbf{A}}$ is an isomorphism for all finite structures $\mathbf{A}$ in $\mathcal{A}$, we say that $\underset{\sim}{\mathbf{M}}_2$ yields a *finite-level duality* between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$. A finite-level duality such that $\varepsilon_{\mathbf{X}}$ is an isomorphism, for all finite $\mathbf{X} \in \mathcal{X}_{\mathcal{T}}$, is referred to as a *finite-level full duality* between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$.

Since compatibility of structures is symmetric (see [10, Lemma 2.1] and [16, Lemma 1.3]), we can swap the topology to the other side and repeat the construction using the alter ego $\underset{\sim}{\mathbf{M}}_1$ of the structure $\mathbf{M}_2$. In order to have well-defined forgetful functors relative to the original dual adjunction between $\mathcal{A} = \mathbb{ISP}(\mathbf{M}_1)$ and $\mathcal{X}_{\mathcal{T}} = \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$, we now define new categories $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$ of Boolean topological structures and $\mathcal{X} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_2)$ of structures. In this situation, our emphasis will be on the category $\mathcal{A}_{\mathcal{T}}$ rather than on the category $\mathcal{X}$, and our notation will reflect it. We have hom-functors $\mathrm{F}\colon \mathcal{A}_{\mathcal{T}} \to \mathcal{X}$ and $\mathrm{G}\colon \mathcal{X} \to \mathcal{A}_{\mathcal{T}}$, given on objects by

$$\mathrm{F}(\mathbf{A}) := \mathcal{A}_{\mathcal{T}}(\mathbf{A}, \underset{\sim}{\mathbf{M}}_1) \leq \mathbf{M}_2^A \text{ and } \mathrm{G}(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \mathbf{M}_2) \leq \underset{\sim}{\mathbf{M}}_1^X,$$

and evaluation maps $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{GF}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{FG}(\mathbf{X})$, for all $\mathbf{A} \in \mathcal{A}_{\mathcal{T}}$ and all $\mathbf{X} \in \mathcal{X}$, giving rise to a new dual adjunction $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$. We refer to $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ and $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ as *paired adjunctions*. (Here we use generic notation for the evaluation maps arising in the two adjunctions; the precise definitions in each case are clear from the context.) If $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{GF}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{A}_{\mathcal{T}}$, then we say that $\mathbf{M}_2$ *yields a duality on* $\mathcal{A}_{\mathcal{T}}$. (The terminology $\underset{\sim}{\mathbf{M}}_1$ *yields a co-duality on* $\mathcal{X}$ is also used, but we will avoid this as we wish to place the emphasis on the topological category $\mathcal{A}_{\mathcal{T}}$.)

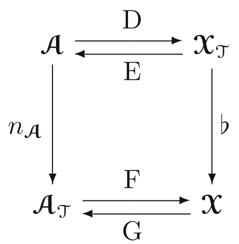
Remark 2.2 The decision whether to include or exclude empty structures and total one-element structures is one of convenience and personal preference. All results remain valid modulo small but sometimes annoying changes, like having to include an empty nullary operation in order to get a strong duality—see the appendix to [16] for a detailed discussion.

Now assume that $\mathbf{M}_1$ is an algebra (that is, $H_1 = R_1 = \varnothing$) and consider the diagram of functors in Fig. 1 arising from the paired adjunction constructed above. Here $^\flat\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{X}$ is the natural forgetful functor. We can now show how the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A} \in \mathcal{A}$ can be described via the paired adjunctions. In all the examples presented below we shall see that, at a minimum, the conditions for Theorem 2.3 are met. The theorem gives information about the natural extension $n_{\mathcal{A}}(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{A}$ in the quasivariety $\mathcal{A}$ of algebras under consideration. We stress that to obtain this information, a duality, rather than a full duality, between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$ is involved, and that even a duality valid at the finite level will suffice.

Theorem 2.3 *Let* $\mathbf{M}_1$ *be a finite algebra and let* $\mathbf{M}_2$ *be a structure compatible with* $\mathbf{M}_1$ *and define* $\mathcal{A}$ *and* $\mathcal{X}_{\mathcal{T}}$ *as above. The following are equivalent:*

(i) *the outer square of Fig. 1 commutes, that is,* $n_{\mathcal{A}}(\mathbf{A}) = \mathrm{G}(\mathrm{D}(\mathbf{A})^\flat)$*, for all* $\mathbf{A} \in \mathcal{A}$*;*
(ii) $n_{\mathcal{A}}(\mathbf{A})$ *consists of all maps* $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ *that preserve the structure on* $\mathbf{M}_2$*, for all* $\mathbf{A} \in \mathcal{A}$*;*
(iii) $\underset{\sim}{\mathbf{M}}_2$ *yields a finite-level duality between* $\mathcal{A}$ *and* $\mathcal{X}_{\mathcal{T}}$*.*

Fig. 1 The paired adjunctions



Moreover, if the type of $\mathbf{M}_2$ *is finite, then* (i)–(iii) *are equivalent to*

(iv) $\underset{\sim}{\mathbf{M}}_2$ *yields a duality between* $\mathcal{A}$ *and* $\mathcal{X}_{\mathcal{T}}$*.*

Proof Let $\mathbf{A} \in \mathcal{A}$. Since $\mathrm{G}(\mathrm{D}(\mathbf{A})^\flat)$ consists of all maps $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ that preserve the structure on $\mathbf{M}_2$, the equivalence of (i) and (ii) is immediate.

By definition, the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of $\mathbf{A}$ is the topological closure of $e_{\mathbf{A}}(A)$ in the power $\underset{\sim}{\mathbf{M}}_2^{\mathcal{A}(\mathbf{A},\mathbf{M}_1)}$. Thus $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ belongs to $n_{\mathcal{A}}(\mathbf{A})$ if and only if α is locally an evaluation, that is, for every finite subset Y of $\mathcal{A}(\mathbf{A}, \mathbf{M}_1)$, there exists $a \in A$ such that $\alpha{\restriction}_Y = e_{\mathbf{A}}(a){\restriction}_Y$. By Clark and Davey [3, Theorem 10.5.1] (see also Pitkethly and Davey [35, Lemma 1.4.4]), α is locally an evaluation if and only it preserves every finitary relation compatible with $\mathbf{M}_1$. Hence (ii) says precisely that, for all $\mathbf{A} \in \mathcal{A}$, each map $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ that preserves the structure on $\mathbf{M}_2$ in fact preserves every finitary compatible relation on $\mathbf{M}_1$. If $\mathbf{A}$ is finite, then the only maps $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ that preserve every finitary compatible relation on $\mathbf{M}_1$ are the evaluations (see [3, Theorem 2.3.1]). Hence (ii) implies (iii).

Now assume that $\underset{\sim}{\mathbf{M}}_2$ yields a finite-level duality on $\mathcal{A}$. Let $\mathbf{A} \in \mathcal{A}$ and let $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \to M$ be a map that preserves the structure on $\mathbf{M}_2$. By the discussion above, to prove (ii) it remains to show that α preserves every finitary compatible relation on $\mathbf{M}_1$. Let r be an n-ary compatible relation on $\mathbf{M}_1$ and let $\mathbf{r}$ be the corresponding subalgebra of $\mathbf{M}_1^n$. Assume that $x_1, \dots, x_n \in \mathcal{A}(\mathbf{A}, \mathbf{M}_1)$ with $(x_1, \dots, x_n) \in r^{\mathcal{A}(\mathbf{A},\mathbf{M}_1)}$. It follows that there is a well-defined homomorphism $z\colon \mathbf{A} \to \mathbf{r}$ given by $z(a) := (x_1(a), \dots, x_n(a))$, for all $a \in A$. As $\mathrm{D}(z)\colon \mathrm{D}(\mathbf{r}) \to \mathrm{D}(\mathbf{A})$ is an $\mathcal{X}_{\mathcal{T}}$-morphism and $\alpha\colon \mathrm{D}(\mathbf{A})^\flat \to \mathbf{M}_2$ is an $\mathcal{X}$-morphism, and since $\mathrm{D}(\mathbf{r})$ is finite, the composite $\alpha \circ \mathrm{D}(z)\colon \mathrm{D}(\mathbf{r}) \to \underset{\sim}{\mathbf{M}}_2$ is an $\mathcal{X}_{\mathcal{T}}$-morphism. As $\underset{\sim}{\mathbf{M}}_2$ yields a finite-level duality, there exists $c \in r$ with $\alpha \circ \mathrm{D}(z) = e_{\mathbf{r}}(c)$, that is, $\alpha(u \circ z) = u(c)$, for all $u \in \mathcal{A}(\mathbf{r}, \mathbf{M}_1)$. Hence, since $x_i = \rho_i \circ z$, for all i, where $\rho_i\colon \mathbf{r} \to \mathbf{M}_1$ is the i-th projection, we have

$$
\begin{aligned}
(\alpha(x_1), \dots, \alpha(x_n)) &= (\alpha(\rho_1 \circ z), \dots, \alpha(\rho_n \circ z)) \\
&= (\rho_1(c), \dots, \rho_n(c)) \\
&= c \in r.
\end{aligned}
$$

Hence α preserves r, whence (iii) implies (ii).

Finally, (iv) always implies (iii), and (iii) implies (iv) if the type of $\underset{\sim}{\mathbf{M}}_2$ is finite, by the Duality Compactness Theorem [3, 2.2.11]. □

We now show that if we add to Theorem 2.3 the assumption that the algebra $\mathbf{M}_1$ is of finite type, then whenever $\mathbf{M}_2$ yields a description of natural extensions in $\mathcal{A}$, it also yields a duality on the category $\mathcal{A}_{\mathcal{T}}$ within which the natural extensions live. Note that the theorem holds not only when $\mathbf{M}_1$ is an algebra of finite type, but also when it is a *total structure* of finite type, that is, $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ where G_1 is a finite set of total operations and R_1 is a finite set of relations. The authors acknowledge with thanks several conversations with Jane Pitkethly which led them to the proof of this result.

TopSwap Theorem 2.4 *Let* $\mathbf{M}_1$ *be a finite total structure of finite type, let* $\mathbf{M}_2$ *be a structure compatible with* $\mathbf{M}_1$ *and define the categories* $\mathcal{A}$*,* $\mathcal{A}_{\mathcal{T}}$*,* $\mathcal{X}$ *and* $\mathcal{X}_{\mathcal{T}}$ *as above.*

(1) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a finite-level duality between* $\mathcal{A}$ *and* $\mathfrak{X}_{\mathcal{T}}$*, then* $\mathbf{M}_2$ *yields a duality between* $\mathcal{A}_{\mathcal{T}}$ *and* $\mathfrak{X}$*.*

(2) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a finite-level full duality between* $\mathcal{A}$ *and* $\mathfrak{X}_{\mathcal{T}}$*, then the adjunction* $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ *is a dual equivalence between the categories* $\mathcal{A}_{\mathcal{T}}$ *and* $\mathfrak{X}$*.*

Proof Let $\mathbf{M}_2'$ be any structure that is compatible with $\mathbf{M}_1$, has $\mathbf{M}_2$ as a reduct and fully dualises $\mathbf{M}_1$ at the finite level, and define the corresponding categories $\mathfrak{X}' := \mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_2')$ and $\mathfrak{X}_{\mathcal{T}}' := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2')$, functors $\mathrm{F}'\colon \mathcal{A}_{\mathcal{T}} \to \mathfrak{X}'$ and $\mathrm{G}'\colon \mathfrak{X}' \to \mathcal{A}_{\mathcal{T}}$ and evaluation maps $e_{\mathbf{A}}'\colon \mathbf{A} \to \mathrm{G}'\mathrm{F}'(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}'\colon \mathbf{X} \to \mathrm{F}'\mathrm{G}'(\mathbf{X})$, for all $\mathbf{A} \in \mathcal{A}_{\mathcal{T}}$ and $\mathbf{X} \in \mathfrak{X}$. (For example, we could take $\mathbf{M}_2' = \langle M; G_\omega, H_\omega, R_\omega\rangle$, where G_ω, H_ω and R_ω are respectively the sets of *all* finitary total operations, partial operations and relations compatible with $\mathbf{M}_1$, in which case $\underset{\sim}{\mathbf{M}}_2'$ is the strong brute force alter ego of $\mathbf{M}_1$ and so yields a finite-level full duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}'$ [16, Lemma 4.6].) By the Sesqui Full Duality Theorem [10, 6.4], $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $\mathfrak{X}'$ and $\mathcal{A}_{\mathcal{T}}$ and hence the adjunction $\langle \mathrm{F}', \mathrm{G}', e', \varepsilon'\rangle$ is a dual equivalence between the categories $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}'$. If $\underset{\sim}{\mathbf{M}}_2$ yields a finite-level full duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$, we may choose $\mathbf{M}_2' = \mathbf{M}_2$ and conclude that the adjunction $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ is a dual equivalence between the categories $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}$, which establishes part (2) of the theorem.

We now turn our attention back to the structures $\mathbf{M}_1$ and $\mathbf{M}_2$ and the proof that $\mathbf{M}_2$ yields a duality between $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}$. Let $\mathbf{A} \in \mathcal{A}_{\mathcal{T}}$. We must prove that every $\mathfrak{X}$-morphism from $\mathcal{A}_{\mathcal{T}}(\mathbf{A}, \underset{\sim}{\mathbf{M}}_1)$ to $\mathbf{M}_2$ is an evaluation. Let $\alpha\colon \mathcal{A}_{\mathcal{T}}(\mathbf{A}, \underset{\sim}{\mathbf{M}}_1) \to \mathbf{M}_2$ be an $\mathfrak{X}$-morphism. As $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between the categories $\mathfrak{X}'$ and $\mathcal{A}_{\mathcal{T}}$, every $\mathfrak{X}'$-morphism from $\mathcal{A}_{\mathcal{T}}(\mathbf{A}, \underset{\sim}{\mathbf{M}}_1)$ to $\mathbf{M}_2'$ is an evaluation, so it suffices to show that α is an $\mathfrak{X}'$-morphism. The map α preserves a (partial) operation h in the type of $\mathbf{M}_2'$ if and only if α preserves the relation graph(h), so it certainly suffices to show that α preserves every compatible relation on $\mathbf{M}_1$. The following standard entailment argument completes the proof.

Let s be an n-ary compatible relation on $\mathbf{M}_1$ and let $\mathbf{s}$ be the corresponding substructure of $\mathbf{M}_1^n$. As $\underset{\sim}{\mathbf{M}}_2$ yields a finite level duality on $\mathcal{A}$, the structure $\underset{\sim}{\mathbf{M}}_2$ entails every compatible relation on $\mathbf{M}_1$ and so entails s. By the Dual Entailment Theorem [16, 3.6] (see also [3, Theorem 9.1.2]), there is a primitive positive formula $(\exists u_1, \ldots, u_m)\, \Phi(v_1, \ldots, v_n, u_1, \ldots, u_m)$ in the language of $\mathbf{M}_2$ such that

$$s = \{\, (a_1, \ldots, a_n) \in M^n \mid (\exists c_1, \ldots, c_m)\, \mathbf{M}_2 \models \Phi(a_1, \ldots, a_n, c_1, \ldots, c_m)\,\} \qquad (1)$$

and there exist homomorphisms $w_1, \ldots, w_m\colon \mathbf{s} \to \mathbf{M}_1$ such that $\mathcal{A}(\mathbf{s}, \mathbf{M}_1)$ satisfies $\Phi(\rho_1, \ldots, \rho_n, w_1, \ldots, w_m)$, where $\rho_i\colon \mathbf{s} \to \mathbf{M}_1$ is the ith projection. Now let $(x_1, \ldots, x_n) \in s^{\mathcal{A}_{\mathcal{T}}(\mathbf{A}, \underset{\sim}{\mathbf{M}}_1)}$ and define continuous homomorphisms $y_i\colon \mathbf{A} \to \underset{\sim}{\mathbf{M}}_1$ by

$$y_i(a) := w_i(x_1(a), \ldots, x_n(a)).$$

Since $\mathcal{A}(\mathbf{s}, \mathbf{M}_1)$ satisfies $\Phi(\rho_1, \ldots, \rho_n, w_1, \ldots, w_m)$, it follows that $\mathcal{A}(\mathbf{A}, \mathbf{M}_1)$ satisfies $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$. As the formula $\Phi(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is a conjunct of atomic formulæ in the language of $\mathbf{M}_2$ and α preserves the structure on $\mathbf{M}_2$, we conclude that $\mathbf{M}_2$ satisfies $\Phi(\alpha(x_1), \ldots, \alpha(x_n), \alpha(y_1), \ldots, \alpha(y_m))$. It follows by (1) that $(\alpha(x_1), \ldots, \alpha(x_n)) \in s$. Hence α preserves s, as required. □

If $\mathbf{M}_1$ is an algebra of finite type and $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$, then part (1) of this theorem tells us that, in addition, $\mathbf{M}_2$ yields a duality on $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}^+(\mathbf{M}_1)$; we refer to these as *paired dualities* and say that the structure $\mathbf{M}_2$ *yields* *paired dualities* on $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$. If $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\mathcal{A}$, then part (2) of the theorem tells us that both of the dual adjunctions of Fig. 1 are dual equivalences, and we refer to them as *paired full dualities*.

Remark 2.5 Several remarks should be made about this theorem.

(i) It is natural to seek generalisations of this theorem and ask: *if* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are compatible structures and* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$*, does it follow that* $\mathbf{M}_2$ *yields a duality on* $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$*?* By Example 5.4 of Davey [10], in the presence of partial operations in the type of $\mathbf{M}_1$, the answer in general is no.
(ii) In applications where $\mathbf{M}_1$ is a lattice-based algebra, the assumption that $\mathbf{M}_1$ be of finite type is not a real restriction. Every finite lattice-based algebra is term equivalent to an algebra of finite type since every clone on a finite set that contains a near-unanimity function is finitely generated (see, for example, Szendrei [40, Corollary 1.26]).
(iii) The astute reader will have noticed that there was no mention of *named constants* in the TopSwap Theorem, while the Sesqui Full Duality Theorem [10, 6.4], which is used in the proof of the TopSwap Theorem, requires that the algebra $\mathbf{M}_1$ has named constants. We can avoid this technical requirement as we have intentionally excluded the empty structure from the class $\mathcal{A}_{\mathcal{T}}$.
(iv) There is an obvious multisorted variant of the TopSwap Theorem that applies when we replace the total structure $\mathbf{M}_1$ by a set $\mathcal{M}_1 = \{\mathbf{M}_1, \dots, \mathbf{M}_k\}$ of finite algebras and replace the structure $\mathbf{M}_2$ by a k-sorted structure $\mathcal{M}_2$ compatible with $\mathcal{M}_1$. See the preamble to Example 4.6 below for a brief discussion of multisorted structures and dualities and [3, Section 7.1] and [11] for more details.

The remainder of the paper consists of a catalogue of examples of the application of Theorem 2.3 and the TopSwap Theorem. We will progressively work our way up the hierarchy described in the introduction.

3 The Base Level: Paired Adjunctions

In this section we concentrate on examples of algebras $\mathbf{M}_1$ where a dualising alter ego $\underset{\sim}{\mathbf{M}}_2$ is known but

(a) there is no fully dualising alter ego, or
(b) there is a known fully dualising alter ego but it is more complex than $\underset{\sim}{\mathbf{M}}_2$, or
(c) it is not known if there is a fully dualising alter ego.

In each case, an application of Theorem 2.3 yields a description, for each algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}_1)$, of the natural extension $n_{\mathcal{A}}(\mathbf{A})$ and hence, via Theorem 2.1, of the $\mathcal{A}$-profinite completion of $\mathbf{A}$, and, in the lattice-based case, of the canonical extension of $\mathbf{A}$. Then an application of part (1) of the TopSwap Theorem shows that we have paired dualities on $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$.

Example 3.1 Our first example, originating with Hyndman and Willard [31], falls under (a) above. Consider the unary algebra $\mathbf{3}_1 = \langle\{0, 1, 2\}; u, d\rangle$ where $u(0) =$

1, $u(1) = u(2) = 2$ and $d(2) = 1$, $d(1) = d(0) = 0$. Since u and d are endomorphisms of the three-element lattice $\mathbf{3} = \langle\{0, 1, 2\}; \vee, \wedge\rangle$ with $0 < 1 < 2$, the Lattice Endomorphism Theorem [7] (see also [35, 2.1.2]) implies that the alter ego $\underset{\sim}{\mathbf{3}}_2' = \langle\{0, 1, 2\}; \vee, \wedge, R_6, \mathcal{T}\rangle$ dualises $\mathbf{3}_1$, where R_6 is the set of all 6-ary compatible relations on $\mathbf{3}_1$. Hyndman and Willard proved that the simpler alter ego $\underset{\sim}{\mathbf{3}}_2 = \langle\{0, 1, 2\}; \vee, \wedge, r, s, \mathcal{T}\rangle$ also dualises $\mathbf{3}_1$, where r and s are given by

$$r = \{\, (x, y) \mid x \leqslant y \ \ \& \ \ (x, y) \neq (0, 2)\,\}, \text{ and}$$

$$s = \{\, (x, y, z, w) \mid x \leqslant y \leqslant z \leqslant w \ \ \& \ \ (x = y \text{ or } z = w)\,\}.$$

The relation r can be interpreted as the set of order-preserving maps from **2** to **3** excluding the map onto $\{0, 2\}$ or as the directed, looped path of length 2. The relation s can be interpreted as the set of order-preserving maps from **4** to **3** excluding the map corresponding to $(0, 1, 1, 2)$. By Theorem 2.3 and part (1) of the TopSwap Theorem, the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A}$ in $\mathbb{ISP}(\mathbf{3}_1)$ consists of all lattice homomorphisms $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{3}_1) \to \mathbf{3}$ that preserve r and s, and $\mathbf{3}_2$ yields paired dualities on $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$.

Example 3.2 This next, classic, example falls under category (b). It is typical of situations in which there is a duality with no partial operations in the alter ego, but where the alter ego needs to be augmented with partial endomorphisms to achieve a full duality.

Let $\mathbf{n}_1$ be the n-element chain regarded as a Heyting algebra. Then the class $\mathcal{L}^{(n)} := \mathbb{ISP}(\mathbf{n}_1)$ is a variety of relative Stone algebras and every proper subvariety of the variety of relative Stone algebras is of this form (Hecht and Katriňák [29]). Davey [8, 9] proved that $\underset{\sim}{\mathbf{n}}_2 := \langle n; \mathrm{End}(\mathbf{n}_1), \mathcal{T}\rangle$ dualises $\mathbf{n}_1$. This duality has played a seminal role in the general theory and is re-proved several times in the Clark–Davey text [3]. For $n \geqslant 4$, the duality is not full, but can be upgraded to a full duality at the expense of adding the partial endomorphisms to the type of $\underset{\sim}{\mathbf{n}}_2$ (Clark and Davey [3, Theorem 4.2.3]). By Theorems 2.1 and 2.3, the Heyting algebra consisting of all maps $\alpha\colon \mathcal{L}^{(n)}(\mathbf{A}, \mathbf{n}_1) \to n$ that preserve the action of $\mathrm{End}(\mathbf{n}_1)$ is the natural extension of $\mathbf{A}$ and hence is both the profinite completion and a canonical extension of $\mathbf{A}$, for each $\mathbf{A} \in \mathcal{L}^{(n)}$. By part (1) of the TopSwap Theorem, $\mathrm{End}(\mathbf{n}_1)$ yields paired dualities on $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(n)}_{\mathcal{T}}$.

Example 3.3 We now give an example from category (c). The semilattice-based algebra $\mathbf{S}_1 = \langle\{0, 1, 2\}; \wedge, u, d\rangle$ is obtained by adding the operation of the three-element semilattice $\mathbf{3}_\wedge = \langle\{0, 1, 2\}; \wedge\rangle$, with $0 < 1 < 2$, to the algebra $\mathbf{3}_1$ of Example 3.1. Since u and d are endomorphisms of the $\mathbf{3}_\wedge$, the Semilattice-Based Duality Theorem of Davey et al. [15, 3.3] implies that the alter ego $\underset{\sim}{\mathbf{S}}_2 = \langle\{0, 1, 2\}; \wedge, R_4, \mathcal{T}\rangle$ dualises $\mathbf{S}_1$, where R_4 is the set of all 4-ary compatible relations on $\mathbf{S}_1$. Theorem 2.3 now supplies a description of $n_{\mathcal{A}}(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{A} := \mathbb{ISP}(\mathbf{S}_1)$ and the TopSwap Theorem provides paired dualities. (Whether the algebra $\mathbf{S}_1$ is fully dualisable has not been studied.)

4 The Next Level: Paired Full Dualities

We now move up to examples where there is a known full duality, at least at the finite level.

Example 4.1 The TopSwap Theorem subsumes the results for particular categories that were its forerunners. The Priestley and Banaschewski dualities are paired full dualities, and were, as in Theorem 2.3, used by Davey et al. [13] to recapture the description of the canonical extension of a bounded distributive lattice originally given by Gehrke and Jónsson [23]. Similarly, Haviar and Priestley [27] used dualities arising as in the TopSwap Theorem to describe the canonical extensions of Stone algebras and of double Stone algebras.

It would be erroneous to give the impression that, in the context of distributive-lattice-based algebras, the TopSwap Theorem does no more than provide a unified treatment of examples previously investigated in a more ad hoc way. There are many dualities in the literature known to be full, and so in particular full at the finite level, and to which the theorem applies. We mention for example varieties of MV and BL algebras generated by chains (Niederkorn [33], Di Nola and Niederkorn [22]), as well as the varieties of Kleene algebras and De Morgan algebras whose partner dualities were already available as applications of the theorems in Section 6 of Davey [10, page 25].

There is one general situation in which fullness can immediately be guaranteed and which encompasses both the MV and Kleene examples mentioned above. Assume that $\mathbf{M}_1$ is a finite lattice-based algebra such that each subalgebra of $\mathbf{M}_1$ is subdirectly irreducible and the only homomorphisms between subalgebras of $\mathbf{M}_1$ are identity maps. Then the standard NU Duality Theorem [3, 2.3.4] already supplies a strong, and hence full, duality on $\mathcal{A} = \mathbb{ISP}(\mathbf{M}_1)$ without a need to upgrade. (In fact, this observation is a special case of the characterisation of finite algebras with a purely relational, strongly dualising alter ego—see Pitkethly and Davey [35, Theorem A.7.8].) Therefore any alter ego $\underset{\sim}{\mathbf{M}}_2$ of $\mathbf{M}_1$ for which $G_2 = H_2 = \varnothing$ and $R_2 = \mathbb{S}(\mathbf{M}_1^2)$, or a subset thereof which entails every element of $\mathbb{S}(\mathbf{M}_1^2)$, brings $\mathcal{A}$ within the scope of part (2) of the TopSwap Theorem.

We now provide one novel example to which part (2) of the TopSwap Theorem applies in the manner described above.

Example 4.2 We shall enrich Example 3.1 further by forming the distributive-lattice-based algebra $\mathbf{L}_1 = \langle\{0, 1, 2\}; \vee, \wedge, u, d\rangle$, where $\vee$ and $\wedge$ are the operations of the three-element lattice $\mathbf{3}$. The NU Duality Theorem [3, 2.3.4] implies that the alter ego $\underset{\sim}{\mathbf{L}}'_2 = \langle\{0, 1, 2\}; R_2, \mathcal{T}\rangle$ dualises $\mathbf{L}_1$, where R_2 is the set of all binary compatible relations on $\mathbf{L}_1$. A simple analysis of the subalgebras of $\mathbf{L}_1^2$ shows that the simpler alter ego $\underset{\sim}{\mathbf{L}}_2 = \langle\{0, 1, 2\}; \leqslant, \sim, \mathcal{T}\rangle$ dualises $\mathbf{L}_1$, where $\leqslant$ is the order relation on $\mathbf{3}$ and $\sim = \{0, 1, 2\}^2 \setminus \{(0, 2), (2, 0)\}$ is the binary relation that also arises in the natural duality for Kleene algebras—see for example [3, 4.3.9]. (In fact, this duality is optimal: neither relation can be removed without destroying the duality.) Since $\mathbf{L}_1$ is simple, has no proper subalgebras and no non-identity endomorphisms, we deduce that this duality is strong and therefore full, and that part (2) of the TopSwap

Theorem applies. In particular the functors $\mathrm{F}\colon \boldsymbol{\mathcal{A}}_{\mathcal{T}} \to \boldsymbol{\mathfrak{X}}$ and $\mathrm{G}\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}_{\mathcal{T}}$ yield a dual equivalence between the categories $\boldsymbol{\mathcal{A}}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{L}}_1)$ and $\boldsymbol{\mathfrak{X}} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{L}_2)$.

Example 4.3 We end this collection of distributive-lattice-based examples with one in which the starting finite-level duality does not lift to a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}_{\mathcal{T}}$. Let $\mathbf{D}_1 = \langle \{0, d, 1\}; \vee, \wedge, 0, 1\rangle$ be the three-element bounded lattice. Thus, $\boldsymbol{\mathfrak{D}} := \mathbb{ISP}(\mathbf{D}_1)$ is the class of all bounded distributive lattices. Davey et al. [14] proved that the alter ego $\underset{\sim}{\mathbf{D}}_2 := \langle \{0, d, 1\}; f, g, h, \mathcal{T}\rangle$, where f, g and h are as given in Fig. 2, yields a duality between $\boldsymbol{\mathfrak{D}}$ and the category $\boldsymbol{\mathfrak{X}}_{\mathcal{T}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{D}}_2)$ that is full at the finite level but not full.

By part (2) of the TopSwap Theorem, we may swap the topology from $\underset{\sim}{\mathbf{D}}_2$ to $\mathbf{D}_1$ and conclude that the hom-functors induced by $\mathbf{D}_2$ and its alter ego $\underset{\sim}{\mathbf{D}}_1 = \langle \{0, d, 1\}; \vee, \wedge, 0, 1, \mathcal{T}\rangle$ give rise to a dual equivalence between the category $\boldsymbol{\mathfrak{D}}_{\mathcal{T}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{D}}_1)$ of Boolean topological distributive lattices and the category $\boldsymbol{\mathfrak{X}} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{D}_2)$.

It should be noted that Davey et al. [12] have shown that the dualities on $\boldsymbol{\mathfrak{D}}$ that are full at the finite level form a lattice of cardinality the continuum with the duality given by $\underset{\sim}{\mathbf{D}}_2$ as it bottom element.

Example 4.4 We turn now to an example of a finite lattice-based algebra whose underlying lattice in non-distributive. Fix $k \geq 2$ and let $\mathbf{M}_1 := \langle M; \vee, \wedge, ', 0, 1\rangle$ be the orthomodular lattice of height 2 with $2k$ atoms, where $k \geqslant 2$. The underlying lattice of $\mathbf{M}_1$ is a non-distributive, modular lattice. Haviar et al. [28] exhibited the Pixley term for the variety $\boldsymbol{\mathfrak{MO}}_k := \mathbb{ISP}(\mathbf{M}_1)$ and applied the Arithmetic Strong Duality Theorem [3, 3.3.11] to show that $\underset{\sim}{\mathbf{M}}_2 := \langle M; \mathrm{Aut}(\mathbf{M}_1), h, \mathcal{T}\rangle$ fully dualises $\mathbf{M}_1$, where h is the partial endomorphism of $\mathbf{M}_1$ given by $0 \mapsto 0$, $a \mapsto 0$, $a' \mapsto 1$ and $1 \mapsto 1$, for some fixed $a \in M \setminus \{0, 1\}$. Once again, Theorem 2.1 and 2.3 and part (2) of the TopSwap Theorem yield paired full dualities and a description of the canonical extension of an orthomodular lattice $\mathbf{A}$ in $\boldsymbol{\mathfrak{MO}}_k$.

Of course, the TopSwap Theorem may be applied to algebras that are not lattice based to yield descriptions of the natural extension and therefore, by Theorem 2.1, of the profinite completion. The following example will be familiar to all students of linear algebra.

Example 4.5 Let $\mathbf{F} = \langle F; +, 0, \{\lambda_a \mid a \in F\}\rangle$ be a finite field $\mathbb{F} = \langle F; +, \cdot, 0, 1\rangle$ regarded as a one-dimensional vector space over itself; here λ_a is left multiplication

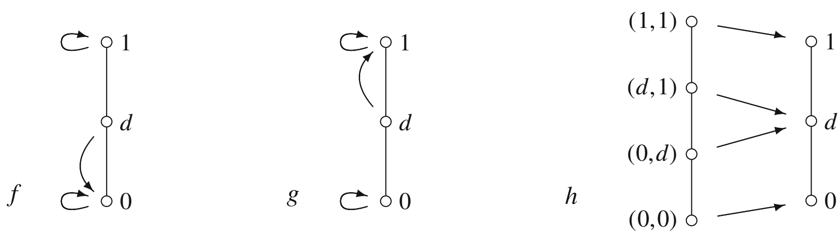


Fig. 2 The operations f, g and h

by a. Then $\mathcal{A} := \mathbb{ISP}(\mathbf{F})$ consists of all vector spaces over $\mathbb{F}$, the corresponding topological category $\mathcal{A}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{F}})$ consists of all Boolean topological vector spaces over $\mathbb{F}$, and $\underset{\sim}{\mathbf{F}}$ yields a full duality between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$; see the Clark–Davey text [3, Theorem 4.4.4] for details. In this case, as in the case of every self-dualising finite algebra, the functors F and G are identical to the functors E and D, respectively. Theorem 2.3 now reveals that the conventional double dual $\mathcal{A}(\mathcal{A}(\mathbf{A}, \mathbf{F}), \mathbf{F})$ of a vector space $\mathbf{A}$ over $\mathbb{F}$ is precisely the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of $\mathbf{A}$.

We close this section with a brief excursion into the world of multisorted dualities. Multisorted piggyback dualities arise naturally when studying the variety $\mathcal{V} := \mathbb{HSP}(\mathbf{M})$ generated by a finite distributive-lattice-based algebra $\mathbf{M}$. If $\mathbb{ISP}(\mathbf{M})$ fails to be a variety, then we cannot get a single-sorted duality for the variety based on $\mathbf{M}$. Thanks to Jònsson's Lemma, we can write $\mathcal{V}$ as $\mathbb{ISP}(\mathcal{M})$, for some finite set of (subdirectly irreducible) algebras $\mathcal{M}$. Then the Piggyback Duality Theorem of Davey and Priestley [17] (see [3, Theorem 7.2.1]) guarantees that there is a multisorted duality for $\mathcal{V}$ (based on $\mathcal{M}$) given by binary multisorted relations. The multisorted versions of Theorem 2.3 and part (1) of the TopSwap Theorem can be applied to varieties for which multisorted piggyback dualities have been worked out: for example, finitely generated varieties of Ockham algebras [17], and double MS-algebras [39], and more generally, Cornish algebras [38], and distributive lattices with a quantifier [37].

We shall sketch the ideas behind multisorted piggyback dualities as developed by Davey and Priestley [17] and refer to [3, Chapter 7], [17] and [11] for the missing details. Since the examples we wish to discuss, namely varieties of Ockham algebras generated by a finite subdirectly irreducible algebra, require a dual structure which is k-sorted, for $k \leqslant 2$, we shall restrict our discussion of multisorted dualities to the 2-sorted case.

So let us consider a class $\mathcal{A} := \mathbb{ISP}(\mathcal{M}_1)$, where $\mathcal{M}_1 = \{\mathbf{M}_0, \mathbf{M}_1\}$ with $\mathbf{M}_0$ and $\mathbf{M}_1$ finite algebras of common type each having a bounded distributive lattice reduct. The Multisorted Piggyback Duality Theorem (see [3, 7.2.1]) tells us that there is a duality between the quasivariety $\mathcal{A} := \mathbb{ISP}(\mathcal{M}_1)$ and a category $\mathcal{X} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathcal{M}}_2)$. The structure $\underset{\sim}{\mathcal{M}}_2$ is a discretely topologised 2-sorted structure

$$\underset{\sim}{\mathcal{M}}_2 = \langle M_0 \cup M_1; G, R, \mathcal{T}\rangle,$$

where $G = \bigcup\{\mathcal{A}(\mathbf{M}_i, \mathbf{M}_j) \mid i, j \in \{0, 1\}\}$ (so G is a 2-sorted version of an endomorphism monoid $\mathrm{End}(\mathbf{M})$) and R is a certain set of binary relations with each $r \in R$ forming a subalgebra of $\mathbf{M}_i \times \mathbf{M}_j$, for some $i, j \in \{0, 1\}$. Thus, the 2-sorted structure $\mathcal{M}_2 = \langle M_0 \cup M_1; G, R\rangle$ is compatible with $\mathcal{M}_1$ in an obvious sense. The proof of the cited theorem exploits Priestley duality for $\mathcal{D}$—recall that, by assumption, the algebras in $\mathcal{A}$ have a bounded distributive lattice reduct.

The multisorted version of Theorem 2.3 immediately yields a description of the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A} \in \mathcal{A}$ as the set of all 2-sorted maps from $\mathcal{A}(\mathbf{A}, \mathbf{M}_0) \cup \mathcal{A}(\mathbf{A}, \mathbf{M}_1)$ to $M_0 \cup M_1$ that preserve the maps in G and the relations in R. Likewise, part (1) of the multisorted version of the TopSwap Theorem tells us that $\mathcal{M}_2$ yields a multisorted duality between $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\{\underset{\sim}{\mathbf{M}}_0, \underset{\sim}{\mathbf{M}}_1\})$ and $\mathcal{X} := \mathbb{IS}^0\mathbb{P}^+(\mathcal{M}_2)$. (We remark that a proof quite different from that presented for the multisorted version of the TopSwap Theorem is also available. This exploits the fact that topology-swapping is known to be valid for Priestley duality, yielding the Banaschewski duality. The argument used to prove the Multisorted Piggyback Duality Theorem can then easily be adapted so that piggybacking is carried out over the latter duality; this gives the same result as the specialisation of the TopSwap Theorem.)

In general, the duality given by the Multisorted Piggyback Duality Theorem is not full and hence part (2) of the TopSwap Theorem does not apply. The duality can always be upgraded to a strong (and therefore full) duality; see Clark and Davey [3, Theorem 7.1.2] and Davey and Talukder [20, Section 4]. Perhaps, however, of most interest are instances in which no upgrading, by the addition of suitable partial operations, is necessary to achieve fullness. Hence, in the example below, we shall concentrate on the situation where the piggyback duality is already full.

Example 4.6 We recall that the variety $\mathcal{O}$ of Ockham algebras consists of algebras $\mathbf{A} = \langle A; \vee, \wedge, 0, 1, \sim\rangle$, where $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ is a dual endomorphism that interchanges 0 and 1. (See Blyth and Varlet [2] for background on Ockham algebras.) Under this umbrella come in particular the varieties $\mathcal{M}$ (De Morgan algebras), $\mathcal{K}$ (Kleene algebras) and $\mathcal{S}$ (Stone algebras) which have provided important test case examples for natural duality theory from its earliest days. Dualities for these, and subsequently for other finitely generated subvarieties and subquasivarieties of $\mathcal{O}$, were profitably used, for example, to describe free algebras and coproducts. A number of these dualities are in fact full, though the issue of fullness was not addressed when they first appeared in the literature; at the time the necessary methodology of strong dualities was not available. We shall concentrate here on dualities which are full.

Let $\mathbf{M}$ be a finite subdirectly irreducible algebra in $\mathcal{O}$ and let $\mathcal{V} = \mathbb{HSP}(\mathbf{M})$ be the variety generated by $\mathbf{M}$. It is well known that $\mathcal{V} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M} = \{\mathbf{M}, \mathbf{M}_1\}$ with $\mathbf{M}_1$ a particular homomorphic image of $\mathbf{M}$. In this case, the set R of relations required by the Piggyback Duality Theorem has size 4 (see [17, Theorem 3.7] and [3, Theorem 7.5.5]). The conditions under which the piggyback duality for $\mathcal{V}$ is already full are most conveniently expressed in terms of the restricted Priestley duality for $\mathcal{O}$, as it applies in particular to $\mathbf{M}$ and $\mathbf{M}_1$. See Clark and Davey [3, Chapter 7], Davey and Priestley [17] and Goldberg [24] for proofs of all claims made below.

The dual of a finite Ockham algebra under restricted Birkhoff–Priestley duality is a finite ordered set (which can be taken to be the set of join-irreducible elements of the algebra), equipped with an order-reversing map g and the discrete topology $\mathcal{T}$; such a structure is a (finite) *Ockham space*. Goldberg's characterisation of the finite subdirectly irreducible Ockham algebras [24, Proposition 2.5] tells us that the Ockham space dual to the generating algebra $\mathbf{M}$ takes the form $\mathbf{X} = \langle X; \leqslant, g, \mathcal{T}\rangle$, where $X = \{\, g^k(e) \mid k \geqslant 0 \,\}$, for some $e \in X$. Let $|X| = m$ and let n be the least k such that $g^k(e) = g^m(e)$. Then $\{g^n(e), g^{n+1}(e), \dots, g^{m-1}(e)\}$ is called the *loop* of $\mathbf{X}$. The dual of the algebra $\mathbf{M}_1$ is the substructure $\mathbf{X}_1$ of $\mathbf{X}$ generated by $g(e)$. Thus, either $\mathbf{X}$ is a loop, in which case $\mathbf{X} = \mathbf{X}_1$, or $X_1 = X \setminus \{e\}$.

If $\mathcal{V} = \mathbb{ISP}(\mathbf{M})$, then we may choose to ignore the algebra $\mathbf{M}_1$ and use a (single-sorted) duality based on $\mathbf{M}$. Building on work of Goldberg [24], Davey and Priestley [17, Corollary 3.11] show that $\mathcal{V} = \mathbb{ISP}(\mathbf{M})$ if and only if one of the following conditions holds:

(a) $\mathbf{X}$ is an antichain;

(b) $m \geqslant 2$ and $\mathbf{X}$ is isomorphic to the Ockham space $\mathbf{Y}_1$ in Fig. 3 or its order-theoretic dual;
(c) $m \geqslant 4$, $n \geqslant 1$, $m \geqslant n+3$ and $\mathbf{X}$ is isomorphic to the Ockham space $\mathbf{Y}_2$ in Fig. 3 or its order-theoretic dual;
(d) $\mathbf{X}_1$ is a loop (including the case that $\mathbf{X} = \mathbf{X}_1$ is a loop).

The characterisation of finite subdirectly irreducible Ockham algebras implies that every non-trivial subalgebra of $\mathbf{M}$ is subdirectly irreducible. In addition, we note that $\mathbf{M}$ has no one-element subalgebras. The NU Strong Duality Theorem [3, 3.3.9] implies that a (single-sorted) piggyback duality for $\mathbb{ISP}(\mathbf{M})$ can be converted to a full (in fact strong) duality by the addition of a set of partial endomorphisms of $\mathbf{M}$. Likewise, the Multisorted NU Strong Duality Theorem [3, 7.1.2] implies that a 2-sorted piggyback duality for $\mathbb{HSP}(\mathbf{M})$ can be converted to a full duality by the addition of a set of partial maps of the form $h\colon \mathbf{A} \to \mathbf{M}_j$, where $\mathbf{A}$ is a subalgebra of $\mathbf{M}_i$ for some $\mathbf{M}_i, \mathbf{M}_j \in \{\mathbf{M}, \mathbf{M}_1\}$. If every such partial operation extends to a total operation, then the additional partial operations are not required, as their total extensions are already in the set G, and the original piggyback duality is already strong and therefore full. The Unary Total Structure Theorem [3, 6.2.4], and its multisorted variant, tell us that this is equivalent to the injectivity of $\mathbf{M}$ in the single-sorted case and to the injectivity of both $\mathbf{M}$ and $\mathbf{M}_1$ in the 2-sorted case. The following characterisation of when this happens is due to Goldberg [24]—see the discussion of injectivity in [24, Section 4], and [24, Theorem 4.17] in particular.

(1) Assume that $\mathcal{V} = \mathbb{ISP}(\mathbf{M})$. Then $\mathbf{M}$ is injective in $\mathcal{V}$ and consequently the single-sorted piggyback duality for $\mathcal{V}$ is strong and therefore full.
(2) Assume that $\mathcal{V} \supsetneqq \mathbb{ISP}(\mathbf{M})$. Then the following are equivalent:
   (i) the 2-sorted piggyback duality for $\mathcal{V}$ is strong and therefore full,
   (ii) both $\mathbf{M}$ and $\mathbf{M}_1$ are injective in $\mathcal{V}$,
   (iii) (a) e is comparable with $g^k(e)$, for some $k > 0$, or
   (b) e is not comparable with $g^i(e)$, for all $i > 0$ and there exists $k, \ell \in \mathbb{N}$ such that $g^k(e) < g(e) < g^\ell(e)$.

These conditions on the Ockham space $\mathbf{X}$ dual to $\mathbf{M}$ are quite restrictive. Nevertheless (1) is satisfied in the case of $\mathcal{M}$, $\mathcal{S}$, $\mathcal{K}$ and also the much studied enveloping variety of $\mathcal{MS}$ algebras (see Blyth and Varlet [2]), and conditions (1) and (2) indicate that there is a large collection of varieties $\mathcal{V}$ of Ockham algebras for which the

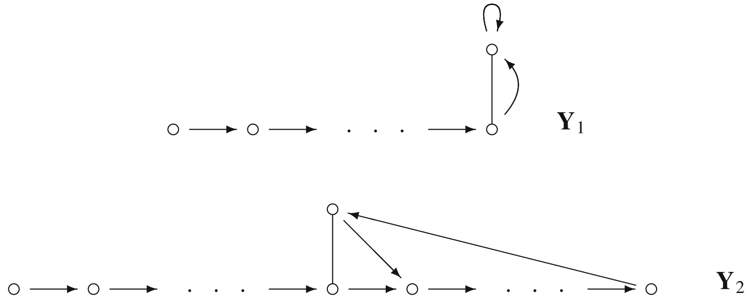


Fig. 3 The Ockham spaces $\mathbf{Y}_1$ and $\mathbf{Y}_2$

single-sorted or 2-sorted piggyback duality is full. In each such case, we may apply part (2) of the TopSwap Theorem to yield a dual equivalence between $\mathcal{V}_{\mathcal{T}}$ and a class of multisorted structures. We do not discuss further the varieties meeting the conditions for a full piggyback duality and nor, in the next section, do we consider the axiomatisation of the dual category. To address these issues we would need to venture into specialised aspects of Ockham algebra dualities inappropriate for this paper.

5 The Top Level: Axiomatised Dual Categories

Assume that we have paired full dualities arising from a finite algebra $\mathbf{M}_1$ and a compatible structure $\mathbf{M}_2$ via the TopSwap Theorem. The duality between $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$ and $\mathcal{X} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_2)$ is likely to be of maximum use when studying and applying natural/canonical extensions if we have an axiomatisation, preferably first order, for both categories.

The class $\mathcal{X}$ certainly has a first-order axiomatisation: as it is closed under the class operators $\mathbb{I}$, $\mathbb{S}^0$ and $\mathbb{P}^+$, it is a universal Horn class and so is of the form $\mathcal{X} = \mathrm{Mod}^0(\Sigma)$, for some set Σ of universal Horn sentences. (Here $\mathrm{Mod}^0(\Sigma)$ denotes the class of possibly empty models of Σ.) The topological class $\mathcal{A}_{\mathcal{T}}$ is more of a problem—it need not equal the class $\mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ of Boolean topological models of Σ for any set Σ of first-order sentences. This bad behaviour occurs, for example, if we choose $\mathbf{M}_1 = \langle\{0, d, 1\}; f, g\rangle$, where f and g are the unary operations in Example 4.3 [6, Theorems 8.1, 8.8 and 8.10]. Much effort has been expended in finding first-order axiomatisations of classes of the form $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}})$, where $\mathbf{M} = \langle M; G, H, R\rangle$ is a finite structure, or in proving that no such axiomatisation exists: see, for example, [4–6, 21, 41–43].

We would like to start from a full duality between the quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M}_1)$ of algebras and the category $\mathcal{X}_{\mathcal{T}} = \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ of Boolean topological structures, with a known first-order axiomatisation of $\mathcal{A}$ and a known (not necessarily first-order) axiomatisation of $\mathcal{X}_{\mathcal{T}}$. We then apply the TopSwap Theorem to yield the dual equivalence $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$ and, in an ideal situation, would like to convert the axiomatisations of $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$ into axiomatisations of $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$, respectively. This is not always possible. Nevertheless, as observed in Davey [10], earlier results on transferring axiomatisations ([4, Theorem 4.3 and Remark 6.9] and more generally [6, Theorem 2.13]) can often be applied to achieve the desired outcome. We state the first result in the case that $\mathbf{M}_1$ is a finite algebra, but note that it holds whenever $\mathbf{M}_1$ is a finite total structure, that is, $H = \varnothing$.

Proposition 5.1 ([10, Theorem 3.2]) *Let $\mathbf{M}_1$ be a finite algebra and define $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ and $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$. Assume that $\mathcal{A}$ is closed under forming homomorphic images (whence $\mathcal{A}$ is the variety generated by $\mathbf{M}_1$) and that $\mathcal{A}$ is congruence distributive. If Σ is a set of quasi-equations such that $\mathcal{A} = \mathrm{Mod}(\Sigma)$, then $\mathcal{A}_{\mathcal{T}} = \mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$.*

This result applies, for example, whenever $\mathbf{M}_1$ is a lattice-based algebra such that $\mathbb{HSP}(\mathbf{M}_1) = \mathbb{ISP}(\mathbf{M}_1)$. When applied to the two-element lattice, **2**, it tells us that $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}})$ is the class consisting of all Boolean topological distributive lattices, a result first proved by Numakura [34]. We turn now to the problem of transferring a possibly non-first-order axiomatisation of $\mathfrak{X}_{\mathcal{T}}$ to a first-order axiomatisation of $\mathfrak{X}$. The following two observations come from Davey [10].

Proposition 5.2 ([10, 3.4]) *Let* $\mathbf{M}_2 = \langle M; G, H, R\rangle$ *be a finite structure and define* $\mathfrak{X} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_2)$ *and* $\mathfrak{X}_{\mathcal{T}} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *Let* Σ *be a set of universal Horn sentences of type* $\langle G, H, R\rangle$. *Assume that every finitely generated model of* Σ *is finite and that* $[\mathfrak{X}_{\mathcal{T}}]_{\text{fin}} = [\mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)]_{\text{fin}}$. *Then* $\mathfrak{X} = \mathrm{Mod}^0(\Sigma)$.

Proposition 5.3 ([10, 3.5]) *Let* $\mathbf{M}_2 = \langle M; G, H, R\rangle$ *be a finite structure and define* $\mathfrak{X} := \mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_2)$ *and* $\mathfrak{X}_{\mathcal{T}} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *Let* Σ_0 *and* Σ_1 *be sets of universal Horn sentences of type* $\langle G, H, R\rangle$ *and let* Φ *be some possibly topological condition. Assume that the finite models of* $\Sigma_0 \cup \Sigma_1$ *are precisely the finite models of* $\Sigma_0 \cup \Phi$ *and that every finitely generated model of* $\Sigma_0 \cup \Sigma_1$ *is finite. Then* $\mathfrak{X} = \mathrm{Mod}^0(\Sigma)$*, where* $\Sigma := \Sigma_0 \cup \Sigma_1$.

In many examples, $\mathbf{X} \models \Phi$ is the statement that $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, in which case the natural choice for Σ_1 is the axioms for an ordered set.

Example 5.4 Propositions 5.1 and either 5.2 or 5.3 may be applied in tandem to many of the examples presented in Section 4 to yield axiomatisations of the classes $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}$ from known axiomatisations of $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$. Their application to distributive lattices, Stone algebras, double Stone algebras, Kleene algebras and De Morgan algebras is discussed in Davey [10, 7.2–7.4].

Example 5.5 As mentioned in Example 3.2, the variety $\mathfrak{L}^{(n)}$ of relative Stone algebras generated by the n-element chain has a full duality given by the endomorphisms and partial endomorphisms of $\mathbf{n}$. An axiomatisation of the dual category is known only for $n = 2, 3, 4$. As $\mathfrak{L}^{(2)}$ is term equivalent to Boolean algebras, its dual category is simply Boolean spaces and no axioms are required. Hecht and Katriňák [29] proved that a Heyting algebra belongs to $\mathfrak{L}^{(3)}$ if and only if it satisfies the identity

$$(x_0 \to x_1) \vee (x_1 \to x_2) \vee (x_2 \to x_3) = 1. \tag{2}$$

A full duality for $\mathfrak{L}^{(3)} = \mathbb{ISP}(\mathbf{3}_1)$ is given by $\underset{\sim}{\mathbf{3}}_2 := \langle\{0, d, 1\}; g, \mathcal{T}\rangle$, where $g\colon 0 \mapsto 0$, $d \mapsto 1$, $1 \mapsto 1$ is the map from Example 4.3 above [3, Theorem 4.2.3]. It is easily seen that $\mathfrak{X}^{(3)}_{\mathcal{T}} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{3}}_2)$ is precisely the class of Boolean topological structures $\langle X; g, \mathcal{T}\rangle$ such that g is a retraction, and so is axiomatised by the equation $g(g(x)) = g(x)$. It follows at once from Propositions 5.1 and 5.2 that $\mathfrak{L}^{(3)}_{\mathcal{T}}$ is the class of Boolean topological Heyting algebras satisfying the identity (2) and $\mathfrak{X}^{(3)}$ is the class of all unars $\langle X; g\rangle$ satisfying the identity $g(g(x)) = g(x)$.

A full duality for $\mathfrak{L}^{(4)}$ necessarily involves partial operations. Such a duality with an axiomatisation of the dual category was given by Davey and Talukder [21]. The application of Propositions 5.1 and 5.2 to this duality is discussed in Davey [10, 7.4–7.5].

Example 5.6 Recall that in Example 4.3 we represented the class of bounded distributive lattices as $\mathfrak{D} := \mathbb{ISP}(\mathbf{D}_1)$, where $\mathbf{D}_1 = \langle\{0, d, 1\}; \vee, \wedge, 0, 1\rangle$ is the three-element bounded lattice, and that $\mathfrak{X}_{\mathcal{T}} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}}_2)$ is generated by the alter ego $\underset{\sim}{\mathbf{D}}_2 := \langle\{0, d, 1\}; f, g, h, \mathcal{T}\rangle$, with f, g and h given in Fig. 2. The application of

Propositions 5.1 to $\mathcal{D}$ tells us the now familiar fact that $\mathcal{D}_\mathcal{T} := \mathbb{IS}_c\mathbb{P}(\mathbf{D}_1)$ is the class of Boolean topological bounded distributive lattices. Davey et al. [14] axiomatised the class $\mathcal{X}_\mathcal{T}$ in the following non-first-order way. Assume that $\mathbf{X}$ is a Boolean topological structure of the same type as $\underset{\sim}{\mathbf{D}}_2$ and define $\mathbf{P_X} := \langle P_\mathbf{X}; \preccurlyeq, \mathcal{T}\rangle$, where $P_\mathbf{X} := \mathrm{fix}(f) \subseteq X$, the relation $\preccurlyeq$ is defined on $P_\mathbf{X}$ by

$$u \preccurlyeq v \iff (\exists x \in X)\ f(x) = u\ \ \&\ \ g(x) = v,$$

and $\mathcal{T}$ is the relative topology from $\mathbf{X}$. They first prove that $\langle P_\mathbf{X}; \preccurlyeq\rangle$ is an ordered set provided $\mathbf{X}$ satisfies the following quasi-equations:

(1) $g(f(x)) = f(f(x)) = f(x)$ and $f(g(x)) = g(g(x)) = g(x)$,
(2) $f(x) = x \iff g(x) = x$, that is, $\mathrm{fix}(f) = \mathrm{fix}(g)$,
(3) $[f(x) = g(y)\ \ \&\ \ f(y) = g(x)] \implies x = y$,
(4) $[f(x) = f(y)\ \ \&\ \ g(x) = g(y)] \implies x = y$,
(5) $(x, y) \in \mathrm{dom}(h) \iff g(x) = f(y)$,
(6) $(x, y) \in \mathrm{dom}(h) \implies f(h(x, y)) = f(x)\ \ \&\ \ g(h(x, y)) = g(y)$.

They then prove that $\mathbf{X}$ belongs to $\mathcal{X}_\mathcal{T}$ if and only if it satisfies quasi-equations (1)–(6) as well as the following topological condition:

(7) for each pair x, y of distinct points in P_X there exists a clopen down-set U of $\mathbf{P_X}$ containing exactly one of x and y.

Since (7) always holds at the finite level, we can now apply Proposition 5.3 with $\Sigma_0 = \{(1), (2), (3), (4), (5), (6)\}$, $\Phi = (7)$ and $\Sigma_1 = \varnothing$. Hence $\mathcal{X} = \mathrm{Mod}^0(\Sigma_0)$.

Example 5.7 Johansen [32] establishes three interesting dualities. We will concentrate on two of them. In both cases, a full (in fact strong) duality is established between a class $\mathcal{A}$ of distributive-lattice-based algebras and a very natural class $\mathcal{X}_\mathcal{T}$ of Boolean topological relational structures. Then a topology swap is performed to yield a full duality between $\mathcal{X}$ and $\mathcal{A}_\mathcal{T}$. An axiomatisation of the class $\mathcal{A}_\mathcal{T}$ is also given. As the quasivariety $\mathcal{A}$ is not a variety, Proposition 5.1 does not apply and the axiomatisation of $\mathcal{A}_\mathcal{T}$ requires topological methods. (To be consistent with our earlier examples, we will adopt a different notation from that used in [32].)

Let $\mathbf{Q}_1 := \langle\{0, d, 1\}; \vee, \wedge, k, 0, d, 1\rangle$, where $\langle\{0, d, 1\}; \vee, \wedge\rangle$ is a lattice with order $0 < d < 1$, and k is given by $0 \mapsto 0$, $d \mapsto 1$ and $1 \mapsto d$. Let $\mathbf{Q}_2 := \langle\{0, d, 1\}; \trianglelefteq\rangle$, where $\trianglelefteq := \{0, d, 1\}^2 \setminus \{(d, 0), (1, 0)\}$. See Fig. 4 for drawings of both $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Then $\mathbf{Q}_2$ is a quasi-ordered set and, as observed in [32], it is easy to see that $\mathcal{Q} := \mathbb{IS}^0\mathbb{P}(\mathbf{Q}_2)$ is the class of all quasi-ordered sets.

Johansen [32] shows that $\underset{\sim}{\mathbf{Q}}_2$ strongly and therefore fully dualises $\mathbf{Q}_1$ and then applies the Two-for-One Strong Duality Theorem [10, 6.9] to conclude that $\underset{\sim}{\mathbf{Q}}_1$ yields a dual equivalence between the class $\mathcal{Q}$ of quasi-ordered sets and the class $\mathcal{A}_\mathcal{T} :=$

Fig. 4 The algebra $\mathbf{Q}_1$ and the quasi-ordered set $\mathbf{Q}_2$

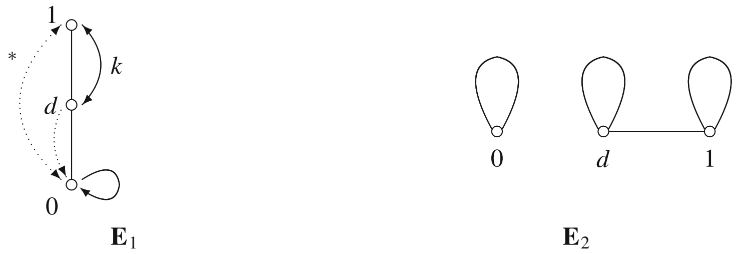


Fig. 5 The algebra $\mathbf{E}_1$ and equivalence-relationed set $\mathbf{E}_2$

$\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{Q}}_1)$. Of course $\mathcal{Q}$ is axiomatised by the usual axioms for a quasi-order, and Johansen [32, Lemma 3.11 and Remark 3.12] shows that $\mathcal{A}_{\mathcal{T}}$ and its non-topological cousin $\mathcal{A} := \mathbb{ISP}(\mathbf{Q}_1)$ are axiomatised by the equations for bounded distributive lattices plus six equations and one quasi-equation involving k. By Theorem 2.3, the natural extension and therefore the canonical extension of an algebra $\mathbf{A} \in \mathcal{A}$ is very simply described as the algebra consisting of all quasi-order-preserving maps from $\mathcal{A}(\mathbf{A}, \mathbf{Q}_1)$ to $\mathbf{Q}_2$.

Now let $\mathbf{E}_1 := \langle\{0, d, 1\}; \vee, \wedge, k, {}^*, 0, d, 1\rangle$ be the algebra obtained by adding the unary operation * of pseudocomplementation to the algebra $\mathbf{Q}_1$, and define $\mathbf{E}_2 := \langle\{0, d, 1\}; \equiv\rangle$, where $\equiv$ is the equivalence relation corresponding to the partition $\{0 \mid d, 1\}$. See Fig. 5 for drawings of both $\mathbf{E}_1$ and $\mathbf{E}_2$.

Using the same approach as for quasi-ordered sets, Johansen [32, Lemma 5.1] shows that $\underset{\sim}{\mathbf{E}}_1$ yields a dual equivalence between the category $\mathcal{E} := \mathbb{IS}^0\mathbb{P}(\mathbf{E}_2)$ of equivalence-relationed sets and the category $\mathcal{B}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{E}}_1)$, and then proves that both $\mathcal{B}_{\mathcal{T}}$ and $\mathcal{B} := \mathbb{ISP}(\mathbf{E}_1)$ are axiomatised by the quasi-equations that axiomatise $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{Q}$ along with two additional equations involving *. Again, Theorem 2.3 yields a very simple description of the natural extension and therefore the canonical extension of an algebra $\mathbf{A} \in \mathcal{B}$ as the algebra consisting of all equivalence-preserving maps from $\mathcal{B}(\mathbf{A}, \mathbf{E}_1)$ to $\mathbf{E}_2$.

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