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Quasi Right Factorization Structures as Presheaves

S. Sh. Mousavi · S. N. Hosseini

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Abstract In this article the notion of quasi right factorization structure in a category $\mathcal{X}$ is given. The main result is a one to one correspondence between certain classes of quasi right factorization structures and 2-reflective subobjects of a predefined object in $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$. Also a characterization of quasi right factorization structures in terms of images is given. As an application, the closure operators are discussed and it is shown that quasi closed members of certain collections are quasi right factorization structures. Finally several examples are furnished.

Keywords (Quasi) right factorization structure · Presheaf · 2-adjoint · Lax (functor) natural transformation · Closure operator

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1 Introduction

The notion of right factorization structure appears in [4] in connection with closure operators. In this paper, given a small category $\mathcal{X}$ with pullbacks and a class $\mathcal{E}$ of morphisms of $\mathcal{X}$ satisfying certain conditions, we first, in Section 2, set up a one to

S. Sh. Mousavi
Mathematics Department, Vali-e-Asr University of Rafsanjan,
Rafsanjan, Iran
e-mail: smousavi@mail.uk.ac.ir

S. Sh. Mousavi · S. N. Hosseini (✉)
Mathematics Department, Shahid Bahonar University of Kerman,
Kerman, Iran
e-mail: nhoseini@mail.uk.ac.ir

one correspondence between certain classes of morphisms of $\mathcal{X}$ ($\mathcal{E}$-domains) and subobjects of a certain object $\mathbf{P}_{\mathcal{E}}$ in $\boldsymbol{PrOrd}^{\mathcal{X}^{op}}$ ($\mathcal{E}$-subobjects). Then, in Section 3, we generalize the notion of right factorization structure to quasi right factorization structure and we set up a one to one correspondence between certain classes of quasi right factorization structures (QRF-domains) and certain $\mathcal{X}_1$-subobjects (lax 2-reflections). In Section 4, we discuss closure operators and we give the "quasi" version of some of the results given in [4], in particular, a characterization of quasi right factorization structures in terms of quasi images is given. Several supporting examples are given in Section 5.

To this end, we introduce some notations and state some basic facts about 2-adjoint. Let $PrOrd$ be the category of preordered sets and order preserving functions. For morphisms f and g, we write $f \leqslant g$ if for all x, $f(x) \leqslant g(x)$. We also consider $PrOrd$ as a 2-category with 2-cells $(X, \leqslant) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leqslant)$ with $\Downarrow \alpha$, where $g \leqslant f$.

Let $\mathcal{X}$ be a small category considered as a discrete 2-category, and $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$ be the category whose objects are lax functors and whose morphisms are lax natural transformations. For morphisms α and β, we say $\alpha \leqslant \beta$, if for all $X \in \mathcal{X}$, $\alpha_X \leqslant \beta_X$. We also consider $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$ as a 2-category with 2-cells $\mathbf{F} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbf{G}$ with $\Downarrow \Gamma$, where $\alpha \leqslant \beta$.

Finally for an object $\mathbf{F}$ in $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$ we consider the preordered set $\mathbf{F}(X)$ as a category.

Definition 1 For a diagram $\mathbf{F} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathbf{G}$ in $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$ we say α is a pointwise left adjoint of β if for all X in $\mathcal{X}$, α_X is a left adjoint of β_X. We also say the diagram is a pointwise adjoint pair.

If in addition β is a mono, we say F is a pointwise reflective subobject of G.

Remark 1 See [3]. The diagram $\mathbf{F} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathbf{G}$ in $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$ constitutes a 2-adjoint pair if and only if there are 2-cells $\Gamma : 1_{\mathbf{G}} \Longrightarrow \beta \circ \alpha$ and Λ: $\alpha \circ \beta \Longrightarrow 1_{\mathbf{F}}$ such that:

$$\begin{array}{ccc} \alpha & \overset{1_\alpha * \Gamma}{\Longrightarrow} & \alpha \circ \beta \circ \alpha \\ & {}_{1_\alpha}\searrow\!\!\searrow \quad /// & \Downarrow \Lambda * 1_\alpha \\ & & \alpha \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \beta & \overset{\Gamma * 1_\beta}{\Longrightarrow} & \beta \circ \alpha \circ \beta \\ & {}_{1_\beta}\searrow\!\!\searrow \quad /// & \Downarrow 1_\beta * \Lambda \\ & & \beta \end{array}$$

The above two triangles obviously commute, so the above diagram is a 2-adjoint pair if and only if $1_{\mathbf{G}} \leq \beta\alpha$ and $\alpha\beta \leq 1_{\mathbf{F}}$. Furthermore, $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$.

With β a mono, it follows that F is a 2-reflective subobject of G, i.e. β has a left 2-adjoint, if and only if there is a map $\alpha : \mathbf{G} \longrightarrow \mathbf{F}$ such that $1_{\mathbf{G}} \leq \beta\alpha$ and $\alpha\beta = 1_{\mathbf{F}}$.

Proposition 1 *The diagram* $\mathbf{F} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathbf{G}$ *in* $\boldsymbol{Lax}(\mathit{PrOrd}^{\mathcal{X}^{op}})$ *is a 2-adjoint pair if and only if it is a pointwise adjoint pair.*

Proof Follows easily. □

2 $\mathcal{E}$-domains and $\mathcal{E}$-subobjects

Assuming $\mathcal{X}$ is a small category with pullbacks, and $\mathcal{E}$ is a class of morphisms in $\mathcal{X}$ that has $\mathcal{X}$-pullbacks (i.e. $e \in \mathcal{E}$ and $f \in \mathcal{X}$ implies the pullback, $f^{-1}(e)$, of e along f is in $\mathcal{E}$), contains all the identities and is closed under composition, the main purpose of this section is to set up a one to one correspondence between $\mathcal{E}$-domains in $\mathcal{X}$, and $\mathcal{E}$-subobjects in $\boldsymbol{PrOrd}^{\mathcal{X}^{op}}$.

With $\langle g\rangle_{\mathcal{E}} = \{ge | ge \text{ is defined and } e \in \mathcal{E}\}$ (for $\mathcal{E} = \mathcal{X}_1$, $\langle g\rangle_{\mathcal{E}}$ is just the principal sieve $\langle g\rangle$, see [6]), we have:

Definition 2 An $\mathcal{E}$-domain is a class $\mathcal{M}$ of morphisms in $\mathcal{X}$ that has $\mathcal{X}$-pullbacks and $\langle m\rangle_{\mathcal{E}} = \langle f\rangle_{\mathcal{E}}$ with $m \in \mathcal{M}$ implies $f \in \mathcal{M}$.

Note any collection, $\mathcal{M}$, that has $\mathcal{X}$-pullbacks is closed under composition with isomorphisms on both sides (since any pullback of $m \in \mathcal{M}$ along the identity is also in $\mathcal{M}$).

Proposition 2 *Let* $\mathcal{M}$ *be a collection of morphisms that has* $\mathcal{X}$*-pullbacks and* $\overline{\mathcal{M}}^{\mathcal{E}} = \{h | \exists m \in \mathcal{M} : \langle h\rangle_{\mathcal{E}} = \langle m\rangle_{\mathcal{E}}\}$*. Then* $\overline{\mathcal{M}}^{\mathcal{E}}$ *is an* $\mathcal{E}$*-domain containing* $\mathcal{M}$.

Proof The fact that $\overline{\mathcal{M}}^{\mathcal{E}}$ has $\mathcal{X}$-pullbacks follows from the fact that $\langle h\rangle_{\mathcal{E}} \subseteq \langle g\rangle_{\mathcal{E}}$ implies $\langle f^{-1}(h)\rangle_{\mathcal{E}} \subseteq \langle f^{-1}(g)\rangle_{\mathcal{E}}$. The rest can be easily verified. □

Definition 3 An $\mathcal{E}$-subobject is a subpresheaf of the presheaf $\mathbf{P}_{\mathcal{E}}$ in $\boldsymbol{PrOrd}^{\mathcal{X}^{op}}$, where $\mathbf{P}_{\mathcal{E}} : \quad \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ is defined as follows:

$$\begin{array}{ccc} X & \longmapsto & \mathbf{P}_{\mathcal{E}}(X) \overset{\text{def}}{:=} (\{\langle g\rangle_{\mathcal{E}} | g \in \mathcal{X}/X\}, \subseteq) \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \mathbf{P}_{\mathcal{E}}(f)} \\ Y & \longmapsto & \mathbf{P}_{\mathcal{E}}(Y) \overset{\text{def}}{:=} (\{\langle h\rangle_{\mathcal{E}} | h \in \mathcal{X}/Y\}, \subseteq) \end{array}$$

and $\mathbf{P}_{\mathcal{E}}(f)(\langle h\rangle_{\mathcal{E}}) = \langle f^{-1}(h)\rangle_{\mathcal{E}}$.

It can be easily verified that if $\langle h\rangle_{\mathcal{E}} \subseteq \langle h'\rangle_{\mathcal{E}}$, then $\langle f^{-1}(h)\rangle_{\mathcal{E}} \subseteq \langle f^{-1}(h')\rangle_{\mathcal{E}}$, rendering the well-definedness of $\mathbf{P}_{\mathcal{E}}(f)$.

Lemma 1

(a) *A class $\mathcal{M}$ of morphisms in $\mathcal{X}$ that has $\mathcal{X}$-pullbacks yields an $\mathcal{E}$-subobject* $\mathcal{M}_{\mathcal{E}} \overset{j_{\mathcal{E}}}{\hookrightarrow} \mathbf{P}_{\mathcal{E}}$.

(b) *An $\mathcal{E}$-subobject* $\mathbf{M} \overset{j_{\mathcal{E}}}{\hookrightarrow} \mathbf{P}_{\mathcal{E}}$ *yields an $\mathcal{E}$-domain $\mathbf{M}_{\mathcal{E}}$.*

Proof

(a) Given $\mathcal{M}$ that has $\mathcal{X}$-pullbacks, define $\mathcal{M}_{\mathcal{E}} : \ \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ by:

$$\begin{array}{ccc} X & \longmapsto & \mathcal{M}_{\mathcal{E}}(X) \overset{\text{def}}{:=} (\{\langle m \rangle_{\mathcal{E}} | m \in \mathcal{M}/X\}, \subseteq) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \widehat{\mathcal{M}(f)}} \\ Y & \longmapsto & \mathcal{M}_{\mathcal{E}}(Y) \overset{\text{def}}{:=} (\{\langle n \rangle_{\mathcal{E}} | n \in \mathcal{M}/Y\}, \subseteq) \end{array}$$

where $\mathcal{M}_{\mathcal{E}}(f)(\langle n \rangle_{\mathcal{E}}) = \langle f^{-1}(n) \rangle_{\mathcal{E}}$.
It is easy to see that $\mathcal{M}_{\mathcal{E}}$ is a subfunctor of $\mathbf{P}_{\mathcal{E}}$.

(b) Given an $\mathcal{E}$-subobject $\mathbf{M} \overset{j_{\mathcal{E}}}{\hookrightarrow} \mathbf{P}_{\mathcal{E}}$, define the class of morphisms $\mathbf{M}_{\mathcal{E}}$ in $\mathcal{X}$ by $\mathbf{M}_{\mathcal{E}} \overset{\text{def}}{:=} \bigcup_{X \in \mathcal{X}} \mathbf{M}_{\mathcal{E}}/X$ where $\mathbf{M}_{\mathcal{E}}/X \overset{\text{def}}{:=} \{m \mid \langle m \rangle_{\mathcal{E}} \in \mathbf{M}(X)\}$.
It is easy to see that $\mathbf{M}_{\mathcal{E}}$ is an $\mathcal{E}$-domain. □

Theorem 1

(a) *For any collection $\mathcal{M}$, $(\mathcal{M}_{\mathcal{E}})_{\mathcal{E}} = \overline{\mathcal{M}}^{\mathcal{E}}$. In particular if $\mathcal{M}$ is an $\mathcal{E}$-domain, then $(\mathcal{M}_{\mathcal{E}})_{\mathcal{E}} = \mathcal{M}$.*

(b) *For any $\mathcal{E}$-subobject* $\mathbf{M} \overset{j_{\mathcal{E}}}{\hookrightarrow} \mathbf{P}_{\mathcal{E}}$ *we have $(\mathbf{M}_{\mathcal{E}})_{\mathcal{E}} = \mathbf{M}$.*

(c) *There is a one-to-one correspondence between $\mathcal{E}$-domains $\mathcal{M}$ in $\mathcal{X}$ and $\mathcal{E}$-subobjects* $\mathbf{M} \overset{j_{\mathcal{E}}}{\hookrightarrow} \mathbf{P}_{\mathcal{E}}$ *in $\boldsymbol{PrOrd}^{\mathcal{X}^{op}}$.*

Proof

(a) $m' \in (\mathcal{M}_{\mathcal{E}})_{\mathcal{E}}$ if and only if there exists $X \in \mathcal{X}$ such that $\langle m' \rangle_{\mathcal{E}} \in \mathcal{M}_{\mathcal{E}}(X)$ if and only if there exists $m \in \mathcal{M}$ such that $\langle m' \rangle_{\mathcal{E}} = \langle m \rangle_{\mathcal{E}}$ if and only if $m' \in \overline{\mathcal{M}}^{\mathcal{E}}$. The last assertion is obvious.

(b) Let $\langle m \rangle_{\mathcal{E}} \in \mathbf{M}(X)$ be given. So $m \in \mathbf{M}_{\mathcal{E}}$ and hence $\langle m \rangle_{\mathcal{E}} \in (\mathbf{M}_{\mathcal{E}})_{\mathcal{E}}(X)$. It follows that $\mathbf{M}(X) \subseteq (\mathbf{M}_{\mathcal{E}})_{\mathcal{E}}(X)$. Now let $\langle m' \rangle_{\mathcal{E}} \in (\mathbf{M}_{\mathcal{E}})_{\mathcal{E}}(X)$ be given. Thus there exists $m \in \mathbf{M}_{\mathcal{E}}$ such that $\langle m' \rangle_{\mathcal{E}} = \langle m \rangle_{\mathcal{E}}$. Since $\langle m \rangle_{\mathcal{E}} \in \mathbf{M}(X)$, $\langle m' \rangle_{\mathcal{E}} \in \mathbf{M}(X)$. Therefore $(\mathbf{M}_{\mathcal{E}})_{\mathcal{E}}(X) \subseteq \mathbf{M}(X)$. It follows that $\mathbf{M} = (\mathbf{M}_{\mathcal{E}})_{\mathcal{E}}$.

(c) Follows from (a) and (b). □

3 Quasi Right Factorization Structures and Lax 2-Reflections

Definition 4 Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$. We say that $\mathcal{X}$ has quasi right $\mathcal{M}$- factorizations or $\mathcal{M}$ is a quasi right factorization structure in $\mathcal{X}$, whenever for all morphisms $Y \xrightarrow{f} X$ in $\mathcal{X}$, there exists $M \xrightarrow{m_f} X \in \mathcal{M}/X$ such that:

(a) $f = m_f g$ for some g;
(b) if there exists $m \in \mathcal{M}/X$ such that $f = mg$ for some g, then $m_f = mh$ for some h.

m_f is called a quasi right part of f.

Note that (a) is equivalent to:

(a′) $\langle f \rangle \subseteq \langle m_f \rangle$;

and (b) is equivalent to:

(b′) if there exists $m \in \mathcal{M}/X$ such that $\langle f \rangle \subseteq \langle m \rangle$, then $\langle m_f \rangle \subseteq \langle m \rangle$.

Note that right $\mathcal{M}$-factorizations as defined in [4] are quasi right $\mathcal{M}$-factorizations.

Proposition 3 *Suppose $\mathcal{X}$ has quasi right $\mathcal{M}$- factorizations. Let f be a morphism in $\mathcal{X}$ and m_f be a quasi right part of f.*

(a) *If $f \in \mathcal{M}$, then $\langle m_f \rangle = \langle f \rangle$.*
(b) *m is a quasi right part of f if and only if $m \in \mathcal{M}$ and $\langle m \rangle = \langle m_f \rangle$.*
(c) *If $\langle g \rangle = \langle f \rangle$, then m_f is a quasi right part of g.*

Proof Straightforward. □

With $\mathcal{E}$ as in the previous section, we have:

Proposition 4 *$\mathcal{M}$ is a quasi right factorization structure if and only if $\overline{\mathcal{M}}^{\mathcal{E}}$ is.*

Proof Suppose $\mathcal{M}$ is a quasi right factorization structure. Given f, there is $m_f \in \mathcal{M} \subseteq \overline{\mathcal{M}}^{\mathcal{E}}$ such that $\langle f \rangle \subseteq \langle m_f \rangle$. Now if $\langle f \rangle \subseteq \langle h \rangle$, for $h \in \overline{\mathcal{M}}^{\mathcal{E}}$, then since $\langle h \rangle_{\mathcal{E}} = \langle m \rangle_{\mathcal{E}}$, for some $m \in \mathcal{M}$, $\langle h \rangle = \langle m \rangle$. Therefore $\langle f \rangle \subseteq \langle m \rangle$. It follows that $\langle m_f \rangle \subseteq \langle m \rangle = \langle h \rangle$, as desired. The converse can be verified similarly. □

Let $\mathbf{P} \overset{\text{def}}{:=} \mathbf{P}_{\mathcal{X}_1}$, see [5], where $\mathcal{X}_1$ is the class of all morphisms in $\mathcal{X}$. Calling a 2-reflection in $\boldsymbol{Lax}(PrOrd^{\boldsymbol{\mathcal{X}}^{op}})$ a lax 2-reflection, we have:

Lemma 2

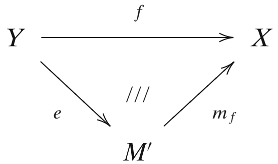
(a) *Let $\mathcal{M} \subseteq \mathcal{X}_1$ have $\mathcal{X}$-pullbacks and $\mathbf{M} \overset{\boldsymbol{j}}{\hookrightarrow} \mathbf{P}$ be the associated $\mathcal{X}_1$-subobject. If $\mathcal{M}$ is a quasi right factorization structure, then $\boldsymbol{j}$ is a lax 2-reflection.*

(b) *Let* $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$ *be an* $\mathcal{X}_1$*-subobject and* $\mathcal{M}$ *be the associated* $\mathcal{X}_1$*-domain. If* $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$ *is a lax 2-reflection, then* $\mathcal{M}$ *is a quasi right factorization structure.*

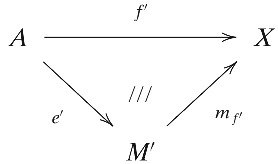
Proof

(a) Define the lax natural transformation $\boldsymbol{m} : \mathbf{P} \stackrel{\cdot}{\longrightarrow} \mathbf{M}$ as follows:

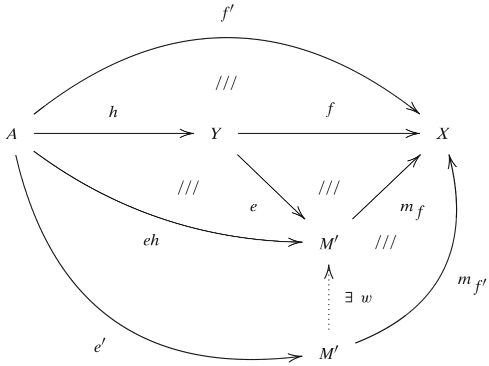
❶ For all $X \in \mathcal{X}$, define $\boldsymbol{m}_X : \mathbf{P}(X) \longrightarrow \mathbf{M}(X)$ by $\langle f \rangle \overset{\boldsymbol{m}_X}{\longmapsto} \langle \boldsymbol{m}_f \rangle$ .To show that $\boldsymbol{m}_X$ is a morphism in $PrOrd$, let $\langle f' \rangle \subseteq \langle f \rangle$ and so $f' = fh$. Suppose that:



and



are quasi right $\mathcal{M}$-factorizations of f and f' respectively. So we have the following diagram:



therefore $\langle m_{f'} \rangle \subseteq \langle m_f \rangle$, as desired.

❷ For all objects X and Y and morphisms $f : Y \longrightarrow X$ in $\mathcal{X}$, let the mapping $\mathbf{P}_{XY} : \mathcal{X}^{op}(X, Y) \longrightarrow PrOrd(\mathbf{P}(X), \mathbf{P}(Y))$ be $\mathbf{P}_{XY}(f^{op}) \stackrel{\text{def}}{:=} \mathbf{P}(f)$ and the mapping $\mathbf{M}_{XY} : \mathcal{X}^{op}(X, Y) \longrightarrow PrOrd(\mathbf{M}(X), \mathbf{M}(Y))$ be $\mathbf{M}_{XY}(f^{op}) \stackrel{\text{def}}{:=} \mathbf{M}(f)$.
Define the natural transformation

$$\boldsymbol{\tau}_{XY} : PrOrd(\boldsymbol{m}'_X, 1) \circ \mathbf{M}_{XY} \stackrel{\cdot}{\longrightarrow} PrOrd(1, \boldsymbol{m}_Y) \circ \mathbf{P}_{XY}$$

by $(\boldsymbol{\tau}_{XY})_{f^{op}} : \mathbf{M}(f)\boldsymbol{m}_X \longrightarrow \boldsymbol{m}_Y\mathbf{P}(f)$, where $(\boldsymbol{\tau}_{XY})_{f^{op}}(\langle g \rangle)$ is the inclusion $\boldsymbol{m}_Y\mathbf{P}(f)(\langle g \rangle) = \langle m_{f^{-1}(g)} \rangle \subseteq \mathbf{M}(f)\boldsymbol{m}_X(\langle g \rangle) = \langle f^{-1}(m_g) \rangle$.
So $\boldsymbol{m} : \mathbf{P} \stackrel{\cdot}{\longrightarrow} \mathbf{M}$ is a morphism in $\boldsymbol{Lax}(PrOrd^{\mathcal{X}^{op}})$. Since $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations, it is easy to see that $\boldsymbol{j}$ is a right pointwise adjoint of $\boldsymbol{m}'$. Therefore $\boldsymbol{j}$ is a 2-reflection.

(b) let $\boldsymbol{m}$ be a pointwise left adjoint of $\boldsymbol{j}$ and $f : Y \longrightarrow X$ be an arbitrary morphism in $\mathcal{X}$. Define $\langle m_f \rangle \overset{\text{def}}{:=} \boldsymbol{m}_X(\langle f \rangle)$. Since $\boldsymbol{j}$ is a pointwise right adjoint of $\boldsymbol{m}$, $\langle f \rangle \subseteq \boldsymbol{j}_X(\langle m_f \rangle) = \langle m_f \rangle$. Suppose that there exists $m \in \mathcal{M}$ such that $\langle f \rangle \subseteq \langle m \rangle$. So $\langle f \rangle \subseteq \boldsymbol{j}_X(\langle m \rangle)$ and since $\boldsymbol{m}$ is a pointwise left adjoint of $\boldsymbol{j}$, $\boldsymbol{m}_X(\langle f \rangle) \subseteq \langle m \rangle$, and hence $\langle m_f \rangle \subseteq \langle m \rangle$. □

The following theorem states that the converse of both parts of Lemma 5 are also valid.

Theorem 2 *Let the class $\mathcal{M}$ of morphisms in $\mathcal{X}$, with $\mathcal{X}$-pullbacks, correspond to the $\mathcal{X}_1$-subobject $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$. Then $\mathcal{M}$ is a quasi right factorization structure if and only if $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$ is a lax 2-reflection.*

Proof First let $\mathcal{M}$ be a collection of morphisms and $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$ be the associated $\mathcal{X}_1$-subobject. We only need to prove the converse of Lemma 2(a). So assume j is a lax 2-reflection. By Lemma 2(b), j gives an $\mathcal{M}'$ that is a quasi right factorization structure. However $\mathcal{M}' = \overline{\mathcal{M}}^{\mathcal{X}_1}$. So by Proposition 4, $\mathcal{M}$ is a quasi right factorization structure.

Next let $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$ be an $\mathcal{X}_1$-subobject and $\mathcal{M}$ be the associated $\mathcal{X}_1$-domain. We only need to prove the converse of Lemma 2(b). So assume $\mathcal{M}$ is a quasi right factorization structure. By Lemma 2(a), $\mathcal{M}$ gives a lax 2-reflection j'. By Proposition 4, $j'=j$. The result follows. □

Definition 5 Let $\mathcal{M}$ be a quasi right factorization structure. We say quasi right $\mathcal{M}$-factorization is pullback stable if for all $f, g \in \mathcal{X}$, $\langle f^{-1}(m_g) \rangle = \langle m_{f^{-1}(g)} \rangle$.

Proposition 5 *Let $\mathcal{M}$ be a quasi right factorization structure that has $\mathcal{X}$-pullbacks. Then quasi right $\mathcal{M}$-factorization is pullback stable if and only if the lax natural transformation $\boldsymbol{m} : \mathbf{P} \overset{\cdot}{\longrightarrow} \mathbf{M}$ is a natural transformation.*

Proof Quasi right $\mathcal{M}$-factorization is pullback stable if and only if for all $f, g \in \mathcal{X}$, $\langle f^{-1}(m_g) \rangle = \langle m_{f^{-1}(g)} \rangle$ if and only if for all $f : X \longrightarrow Y$, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{P}(X) & \xrightarrow{\boldsymbol{m}_X} & \mathbf{M}(X) \\ {\scriptstyle \mathbf{P}(f)} \downarrow & /// & \downarrow {\scriptstyle \mathbf{M}(f)} \\ \mathbf{P}(Y) & \xrightarrow[\boldsymbol{m}_Y]{} & \mathbf{M}(Y) \end{array}$$

if and only if $\boldsymbol{m}$ is a natural transformation. □

Definition 6 A class $\mathcal{M}$ of morphisms of $\mathcal{X}$ is called a QRF-domain if it is a quasi right factorization structure and an $\mathcal{X}_1$-domain.

Proposition 6 *Any quasi right factorization structure $\mathcal{M}$ that has $\mathcal{X}$-pullbacks can be extended to a QRF-domain.*

Proof By Propositions 2 and 4, $\overline{\mathcal{M}}^{\mathcal{X}_1}$ is the required extension. □

Theorem 3 *There is a one-to-one correspondence between QRF-domains $\mathcal{M}$ in $\mathcal{X}$ and lax 2-reflections* $\mathbf{M} \overset{j}{\hookrightarrow} \mathbf{P}$.

Proof The proof follows from Theorems 1 and 2. □

4 Quasi Right Factorization Structures and Closure Operators

Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$. Define a preorder on $\mathcal{M}/X$ by $f \leq g$ if there is h such that $f = gh$ and define the equivalence relation on $\mathcal{M}/X$ by $f \sim g$ if $f \leq g$ and $g \leq f$. Assuming $\mathcal{M}$ has $\mathcal{X}$-pullbacks, we have:

Definition 7 (see [4, 5]) A closure operator $\boldsymbol{C}$ on the category $\mathcal{X}$ with respect to $\mathcal{M}$ is given by $\boldsymbol{C} = (\boldsymbol{c}_X)_{X\in\mathcal{X}}$, where $\boldsymbol{c}_X : \mathcal{M}/X \longrightarrow \mathcal{M}/X$ is a map satisfying:

(i) the extension property: for all $X \in \mathcal{X}$ and for all $m \in \mathcal{M}/X$, $m \leq \boldsymbol{c}_X(m)$;
(ii) the monotonicity property: for all $X \in \mathcal{X}$ and for all m, m' in $\mathcal{M}/X$ whenever $m \leq m'$ in $\mathcal{M}/X$, then $\boldsymbol{c}_X(m) \leq \boldsymbol{c}_X(m')$;
(iii) the continuity property: for all morphisms $f : X \longrightarrow Y$ in $\mathcal{X}$ and for all $m \in \mathcal{M}/Y$, $\boldsymbol{c}_X(f^{-1}(m)) \leq f^{-1}(\boldsymbol{c}_Y(m))$.

A closure operator is said to be (quasi) idempotent, if for each $X \in \mathcal{X}$ and each $m \in \mathcal{M}/X$, $(\boldsymbol{c}_X(\boldsymbol{c}_X(m)) \sim \boldsymbol{c}_X(m))$ $\boldsymbol{c}_X(\boldsymbol{c}_X(m)) \cong \boldsymbol{c}_X(m)$.

Definition 8 Suppose that $\boldsymbol{C}$ is a closure operator and $m \in \mathcal{M}/X$. We say m is (quasi) $\boldsymbol{C}$-closed in X, if $(\boldsymbol{c}_X(m) \sim m)$ $\boldsymbol{c}_X(m) \cong m$.

Remark 2 If $\mathcal{M}$ is a class of monomorphisms, then m is quasi $\boldsymbol{C}$-closed if and only if m is $\boldsymbol{C}$-closed.

Proposition 7 *Let $f : X \longrightarrow Y$ be a morphism. If n is quasi $\boldsymbol{C}$-closed in Y, then $f^{-1}(n)$ is quasi $\boldsymbol{C}$-closed in X.*

Proof If $\boldsymbol{c}_Y(n) \sim n$, then one can easily verify that $f^{-1}(\boldsymbol{c}_Y(n)) \sim f^{-1}(n)$. But $\boldsymbol{c}_X(f^{-1}(n)) \leq f^{-1}(\boldsymbol{c}_Y(n))$, so $\boldsymbol{c}_X(f^{-1}(n)) \leq f^{-1}(n)$. Hence $\boldsymbol{c}_X(f^{-1}(n)) \sim f^{-1}(n)$. □

Remark 3 Let $\mathcal{M}^{QC}$ be the class of quasi $\boldsymbol{C}$-closed members of $\mathcal{M}$. Proposition 7 asserts that $\mathcal{M}^{QC}$ has $\mathcal{X}$-pullbacks.

Proposition 8 *If $n, m \in \mathcal{M}/X$, n is a monomorphism and $n \circ m$ is quasi $\boldsymbol{C}$-closed, then m is quasi $\boldsymbol{C}$-closed.*

Proof The proof follows from the Proposition 7 and the fact that pullback of $n \circ m$ along the monomorphism n is m. □

For every object X of $\mathcal{X}$, $\langle \mathcal{M}/X \rangle = \{\langle m \rangle : m \in \mathcal{M}/X\}$ is partially ordered under set inclusion. For $f : X \longrightarrow Y$ in $\mathcal{X}$, define $\langle f^{-1}(-) \rangle : \langle \mathcal{M}/Y \rangle \longrightarrow \langle \mathcal{M}/X \rangle$ by

$$\langle f^{-1}(-) \rangle(\langle n \rangle) = \langle f^{-1}(n) \rangle$$

We can easily see that $\langle f^{-1}(-) \rangle$ is an order preserving map, hence also well-defined.

With $\mathcal{M}$, a quasi right factorization structure, let $\langle f(-) \rangle : \langle \mathcal{M}/X \rangle \longrightarrow \langle \mathcal{M}/Y \rangle$ be defined by

$$\langle f(-) \rangle(\langle m \rangle) = \langle f(m) \rangle$$

where $f(m) : f(M) \longrightarrow Y$ is an $\mathcal{M}$-part of a quasi right $\mathcal{M}$-factorization of the composite $f \circ m$. Property (b) of Definition 4 implies that $\langle f(-) \rangle$ is an order preserving map, hence also well-defined.

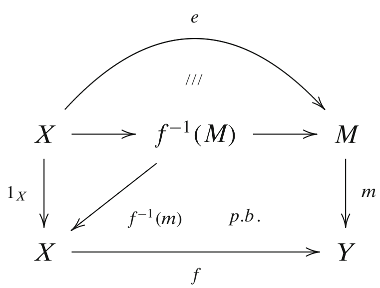
$\langle f^{-1}(m) \rangle$, respectively $\langle f(m) \rangle$ is called the quasi inverse image, respectively the quasi image of m under f.

Also recall, see [4], that with P and Q preordered classes, $\phi : P \longrightarrow Q$ is left adjoint to $\psi : Q \longrightarrow P$, if for all $m \in P$ and $n \in Q$, $m \leqslant \psi(n) \Leftrightarrow \phi(m) \leqslant n$.

Theorem 4 *Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$ that has $\mathcal{X}$-pullbacks. Then $\mathcal{M}$ is a quasi right factorization structure in $\mathcal{X}$ if and only if for each $f : X \longrightarrow Y$, $\langle f^{-1}(-) \rangle$ has a left adjoint.*

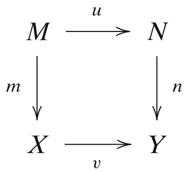
Proof Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. For $f : X \longrightarrow Y$ in $\mathcal{X}$, it easily follows that $\langle f(-) \rangle$ is a left adjoint to $\langle f^{-1}(-) \rangle$.

Conversely suppose for each $f : X \longrightarrow Y$, $\langle f^{-1}(-) \rangle$ has a left adjoint, which we denote by $\langle f(-) \rangle$. Given $f : X \longrightarrow Y$ in $\mathcal{X}$, $\langle f(1_X) \rangle = \langle m \rangle$ for some $m \in \mathcal{M}/Y$. Since $\langle f^{-1}(-) \rangle$ is the right adjoint of $\langle f(-) \rangle$, $\langle 1_X \rangle \subseteq \langle f^{-1}(m) \rangle$. Let e be as in the following diagram, which is commutative.



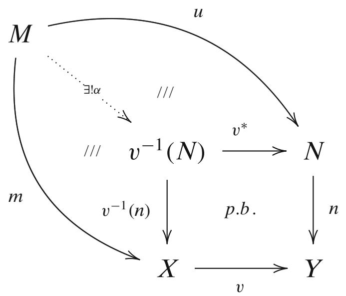
Then $f = m \circ e$. Now let $n \in \mathcal{M}$ and $\langle f \rangle \subseteq \langle n \rangle$. So there is t such that $f = n \circ t$. The pullback property yields a morphism s such that $1_X = f^{-1}(n) \circ s$, i.e., $\langle 1_X \rangle \subseteq \langle f^{-1}(n) \rangle$. Therefore $\langle f(1_X) \rangle \subseteq \langle n \rangle$, i.e., $\langle m \rangle \subseteq \langle n \rangle$. □

Lemma 3 *Suppose $\mathcal{M}$ is a quasi right factorization structure for $\mathcal{X}$ and $\boldsymbol{C}$ is a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$. For every commutative diagram*



with $m, n \in \mathcal{M}$, if n is quasi $\boldsymbol{C}$-closed, then there exists $w : \boldsymbol{c}_X(M) \longrightarrow N$ such that $n \circ w = v \circ \boldsymbol{c}_X(m)$.

Proof The commutativity of the above square yields a unique morphism α such that the triangles in the following diagram commute.

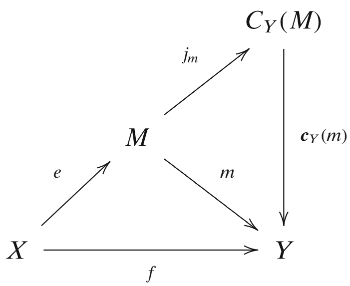


So $\langle m\rangle \subseteq \langle v^{-1}(n)\rangle$. By Theorem 4, $\langle v(m)\rangle \subseteq \langle n\rangle$. Since $v(\boldsymbol{c}_X(m)) \leqslant \boldsymbol{c}_X(v(m))$, we obtain $\langle v(\boldsymbol{c}_X(m))\rangle \subseteq \langle \boldsymbol{c}_X(n)\rangle = \langle n\rangle$, as n is quasi $\boldsymbol{C}$-closed. Therefore by Theorem 4, $\langle \boldsymbol{c}_X(m)\rangle \subseteq \langle v^{-1}(n)\rangle$ and so there exists $\beta : \boldsymbol{c}_X(M) \longrightarrow v^{-1}(N)$ such that $v^{-1}(n) \circ \beta = \boldsymbol{c}_X(m)$. Setting $w := v^* \circ \beta$, we have $n \circ w = n \circ v^* \circ \beta = v \circ v^{-1}(n) \circ \beta = v \circ \boldsymbol{c}_X(m)$. □

With $(\mathcal{E}, \mathcal{M})$-factorization structure as defined in [2], we have:

Theorem 5 *Suppose that $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorization structure and $\boldsymbol{C}$ is a quasi idempotent closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$. Then $\mathcal{M}^{QC}$ is a quasi right factorization structure for $\mathcal{X}$.*

Proof Every morphism $f : X \longrightarrow Y$ has an $(\mathcal{E}, \mathcal{M})$-factorization $f = m \circ e$. Since $\boldsymbol{C}$ is quasi idempotent, $\boldsymbol{c}_Y(m) \in \mathcal{M}^{QC}$. We have the following commutative diagram.



Thus $f = \boldsymbol{c}_Y(m) \circ (j_m \circ e)$. Suppose that the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad v \quad} & Z \\ {\scriptstyle j_m \circ e}\downarrow & & \downarrow{\scriptstyle n} \\ C_Y(m) & \xrightarrow[\boldsymbol{c}_Y(m)]{} & Y \end{array}$$

where $n \in \mathcal{M}^{QC}$ is given. So $\langle m \circ e \rangle \subseteq \langle n \rangle$ and since $(\mathcal{E}, \mathcal{M})$ is a factorization structure, $\langle m \rangle \subseteq \langle n \rangle$. Hence there exists $w : M \longrightarrow Z$ such that $m = n \circ w$. So we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{w} & Z \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ Y & = & Y \end{array}$$

Since $n \in \mathcal{M}^{QC}$ and $\mathcal{M}$ is a right factorization structure and therefore a quasi right factorization structure, by Lemma 3 there exists $d : C_Y(M) \longrightarrow Z$ such that $n \circ d = \boldsymbol{c}_Y(m)$ as desired. □

The following summarizes the main theorems of Sections 2 and 3.

Theorem 6 *Let $\mathcal{X}$ be a category with pullbacks and the class $\mathcal{M}$ of morphisms in $\mathcal{X}$, with $\mathcal{X}$-pullbacks, correspond to the $\mathcal{X}_1$-subobject $\mathbf{M} \overset{\boldsymbol{j}}{\hookrightarrow} \mathbf{P}$. Then $\mathcal{M}$ is a quasi right factorization structure if and only if in the following commutative diagram, for all X, the right arrow has a left adjoint in Prord if and only if for all $f : X \longrightarrow Y$, the top arrow has a left adjoint in Prord.*

$$\begin{array}{ccc} \langle \mathcal{M}/Y \rangle & \xrightarrow{\quad \langle f^{-1}(-) \rangle \quad} & \langle \mathcal{M}/X \rangle \\ {\scriptstyle j_Y}\big\downarrow & & \big\downarrow{\scriptstyle j_X} \\ \langle \mathcal{X}_1/Y \rangle & \xrightarrow[\quad \langle f^{-1}(-) \rangle \quad]{} & \langle \mathcal{X}_1/X \rangle \end{array}$$

Proof Follows from Theorems 2 and 4. □

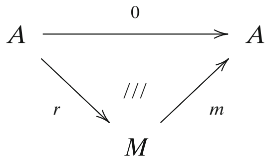
5 Examples

In the following *Id* denotes the collection of all the identities and *Ret* denotes the collection of all the retractions.

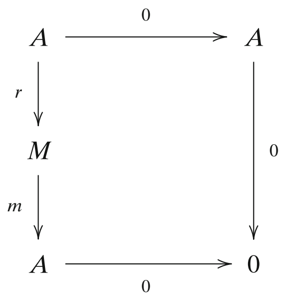
Example 1 If there is $\mathcal{L}$ such that $(\mathcal{L}, \mathcal{M})$ is a weak factorization structure in $\mathcal{X}$, see [1], then $\mathcal{M}$ is a quasi right factorization structure in $\mathcal{X}$.

Example 2 As a special case of Example 1, in an abelian category with enough injectives, the collection $\mathcal{M}$ of all epimorphisms whose kernels are injective, see [1], is a quasi right factorization system. A morphism $f : X \longrightarrow Y$ can be factored as $\pi_2 \circ \langle i, f \rangle$, where $i : X \longrightarrow E$ is the monomorphism from X to an injective object E and $\pi_2 : E \times X \longrightarrow Y$ is the projection to the second factor.

To show $\mathcal{M}$ is not a right factorization structure, consider the zero morphism $0 : A \longrightarrow A$,where A is a non-zero injective object. Let



be any factorization of 0 with $m \in \mathcal{M}$. The commutative square



has $w = 0$ and $w' = m$ as diagonals. But $w \neq w'$, because otherwise $m = 0$ and since m is epic, $A = 0$ and that is a contradiction.

Example 3 As a special case of Example 2, in the category, ***Ab***, of abelian groups, the collection $\mathcal{M}$ of all epimorphisms whose kernels are injective, is a quasi right factorization system.

Example 4 Let $\mathcal{M} \subseteq Ret$ have enough retractions, i.e. for each $Y \in \mathcal{X}$, there is a retraction, in $\mathcal{M}$, to Y. Then $\mathcal{M}$ is a quasi right factorization system. Each $f : X \longrightarrow Y$ can be factored as $m \circ (s \circ f)$, where s is any section of m.

In particular every $\mathcal{M}$ such that $Id \subseteq \mathcal{M} \subseteq Ret$ is a quasi right factorization system.

Example 5 As a special case of Example 4, in the category, ***Top***, of topological spaces, the collection $\mathcal{M}$ of all covering retractions is a quasi right factorization system, because $Id \subseteq \mathcal{M} \subseteq Ret$.

To show $\mathcal{M}$ is not a right factorization structure, let X be a space with more than one element and $f : X \longrightarrow X$ be the constant map with value x. Let

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle e}\searrow \quad /// \quad \nearrow{\scriptstyle m} & \\ & M & \end{array}$$

be any factorization of f, with $m \in \mathcal{M}$. Let $g, h : M \longrightarrow P$ be the cokernel pair of e. The commutative square

$$\begin{array}{ccc} X & \xrightarrow{\langle f, g\circ e\rangle} & X \times P \\ {\scriptstyle e}\downarrow & & \\ M & & \downarrow{\scriptstyle \pi_1} \\ {\scriptstyle m}\downarrow & & \\ X & \xrightarrow[\quad 1 \quad]{} & X \end{array}$$

has two diagonals, $w = \langle m, g\rangle$ and $w' = \langle m, h\rangle$. Since cardinality of X is bigger than 1 and m is epic, e is not epic and so $g \neq h$, implying $w \neq w'$.

Example 6 As another special case of Example 4, in the full subcategory ***ProjRMod*** of the category ***RMod***, consisting of all projective R-modules, the collection $\mathcal{M}$ of all epimorphisms with free domain is a quasi right factorization structure, because $\mathcal{M} \subseteq Ret$ and has enough retractions. Note that $Id \nsubseteq \mathcal{M}$.

For $R=\mathbb{Z}_6$ we show $\mathcal{M}$ is not a right factorization structure. Let $f : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_6$ be the multiplication by 2 morphism and

$$\begin{array}{ccc} \mathbb{Z}_3 & \xrightarrow{\quad f \quad} & \mathbb{Z}_6 \oplus \mathbb{Z}_6 \\ & {\scriptstyle e}\searrow \quad /// \quad \nearrow{\scriptstyle m} & \\ & M & \end{array}$$

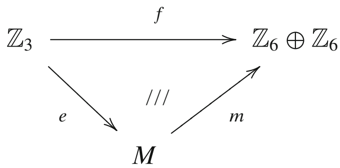
be any factorization of f, with $m \in \mathcal{M}$. The commutative square

$$\begin{array}{ccc} \mathbb{Z}_3 & \xrightarrow{\quad 0 \quad} & \mathbb{Z}_6 \oplus \mathbb{Z}_6 \\ {\scriptstyle e}\downarrow & & \\ M & & \downarrow{\scriptstyle \pi_1} \\ {\scriptstyle m}\downarrow & & \\ \mathbb{Z}_6 & \xrightarrow[\quad g \quad]{} & \mathbb{Z}_6 \end{array}$$

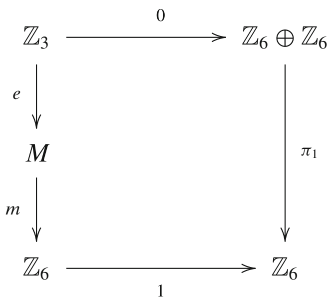
in which g is the multiplication by 3 morphism, has two diagonals, $w = \langle gm, 0\rangle$ and $w' = \langle gm, h\rangle$, where $h = g\pi_\alpha$ and $\pi_\alpha : M = \oplus\mathbb{Z}_6 \longrightarrow \mathbb{Z}_6$ is any projection map. Note that $he(1) = g\pi_\alpha e(1) = g\pi_\alpha(\sum x_i) = g(x_\alpha) = 3x_\alpha = 0$. The last equality holds, because 1 has order 3, so $he = 0$. Let x be any element of M whose α'th coordinate is 1. Then $h(x) = 3$, implying $h \neq 0$, so $w \neq w'$.

Example 7 As another special case of Example 4, in the full subcategory ***SSimRing*** of the category ***Ring***, consisting of all semi-simple rings, the collection $\mathcal{M}$ of all retractions is a quasi right factorization structure.

To show $\mathcal{M}$ is not a right factorization structure, let $f : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_6$ be the zero morphism and



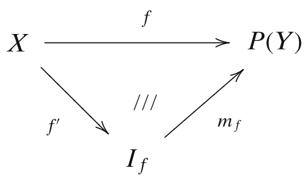
be any factorization of f, with $m \in \mathcal{M}$. The commutative square



has two diagonals, $w = \langle m, 0\rangle$ and $w' = \langle m, m\rangle$. Since $m \neq 0$, $w \neq w'$.

Example 8 Let $\mathcal{M}$ be a collection of morphisms such that for all $f \in \mathcal{X}$, $\mathcal{M}_f = \{m \in \mathcal{M} : \exists r \in Ret \ni: f = mr\} \neq \emptyset$. Then $\mathcal{M}$ is a quasi right factorization system. Here for each $f : X \longrightarrow Y$, every $m : Z \longrightarrow Y$ in $\mathcal{M}_f$ is a quasi right part of f.

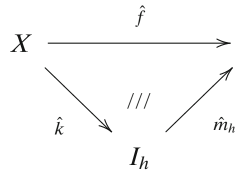
Example 9 As a special case of Example 8, consider the Kleisli category $Set_{\mathbb{P}}$ for the power set monad $\mathbb{P} = (P, \eta, \mu)$. For each $\hat{f} : X \longrightarrow Y$ in $Set_{\mathbb{P}}$, let $f : X \longrightarrow P(Y)$ be its associated morphism in *Set* and



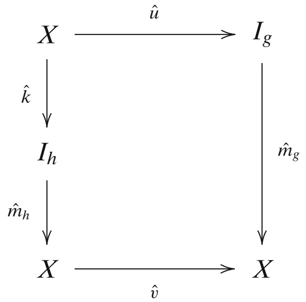
be the $(Epi, Mono)$ factorization of f. Define $e_f = \eta_{I_f} f' : X \longrightarrow P(I_f)$. One can easily verify that $\hat{f} = \hat{m}_f \hat{e_f}$. Now let $\mathcal{M} = \{\hat{m}_f : \hat{f} \in Set_{\mathbb{P}}\}$.

For each morphism $\hat{f} : X \longrightarrow Y$ in $Set_{\mathbb{P}}$, let s' be a section of f' and define $s = \eta_X s' : I_f \longrightarrow P(X)$. We have $\mu_{I_f} P(e_f) s = \mu_{I_f} P(\eta_{I_f}) P(f') \eta_X s' = P(f') \eta_X s' = \eta_{I_f} f' s' = \eta_{I_f}$, i.e., $\hat{e_f} \hat{s} = 1_{I_f}$. Hence $\hat{e_f}$ is a retraction and so, by Example 4.6, $\mathcal{M}$ is a quasi right factorization structure.

To show $\mathcal{M}$ is not a right factorization structure, let $X = \{x, x', x''\}$ and consider the map $f : X \longrightarrow P(X)$ taking all the points to $\{x\}$. Let



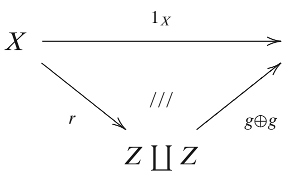
be any factorization of $\hat{f}$, with $\hat{m}_h$ obviously in $\mathcal{M}$. The commutativity of the triangle, after some computations, implies that $k(x)$, $k(x')$ and $k(x'')$ are equal to $\{\{x\}\}$ or $\{\emptyset, \{x\}\}$ with no specified order. The commutative square



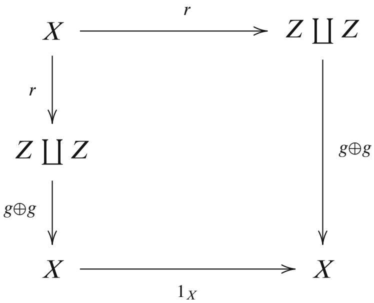
has no diagonal, since if w is a diagonal, $\hat{w} \circ \hat{k} = \hat{u}$, implying $w(\{x\}) = \{\{x\}, \{x'\}\}$ or $w(\{x\}) \cup w(\emptyset) = \{\{x\}, \{x'\}\}$ and also that $w(\{x\}) = \{\{x, x'\}\}$ or $w(\{x\}) \cup w(\emptyset) = \{\{x, x'\}\}$. One can now see that non of the four cases hold.

Example 10 In the category ***Top***, let $\mathcal{M} = \{\, h \oplus h : h \in Top\}$. Any $f : X \longrightarrow Y$ can be factored as $f = (f \oplus f)\nu_1$, where ν_1 is the injection of the coproduct. Now if $f = (h \oplus h) \circ g$, then $f \oplus f = ((h \oplus h) \circ g) \oplus ((h \oplus h) \circ g) = (h \oplus h) \circ (g \oplus g)$. So $\mathcal{M}$ is a quasi right factorization structure.

To show $\mathcal{M}$ is not a right factorization structure, consider the identity morphism $1_X : X \longrightarrow X$, where X is the one point space. Let



be any factorization of f, with $g \oplus g$ obviously in $\mathcal{M}$. The commutative square



has $w = 1$ and $w' = r(g \oplus g)$ as diagonals. But $w \neq w'$, because otherwise $Z \coprod Z$ is isomorphic to a singleton space and that is a contradiction.

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