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Cover Relations on Categories

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Abstract A *cover relation* on a category $\mathbb{C}$ is a binary relation $\sqsubset$ on the class of morphisms of $\mathbb{C}$, which is defined only for those pairs of morphisms which have the same codomain, and which has the following two properties: (i) if $f \sqsubset g$ and h is composable with f, then $hf \sqsubset hg$, (ii) if $f \sqsubset g$ and f is composable with e then $fe \sqsubset g$. We study cover relations arising from a special type of factorization systems, and cover relations arising from a special type of monoidal structures.

Keywords Cover relation · Factorization system · Monoidal category

Mathematics Subject Classifications (2000) 18A32 · 18D10

Introduction

As defined in [7], a *cover relation* on a category $\mathbb{C}$ is a binary relation, which we denote by the symbol $\sqsubset$ (which resembles the letter "c"), on the class of morphisms of $\mathbb{C}$, such that if $f \sqsubset g$ then f and g have the same codomain. When $f \sqsubset g$ we also write g $\sqsubset$*-covers* f, reading this as "g c-covers f". [The term "cover(s)" comes from the following example of the relation $\sqsubset$: in the category $\mathbb{C} = \mathbf{Set}$ of sets let $f \sqsubset g$ when f and g have the same codomain and the image of g *covers* the image of f, i.e. $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$. This term also appears in [11] in the "arrow conditions" defining a Grothendieck topology (however, there the term is used in a slightly more general situation).]

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The aim of the present article is to examine two different kinds of cover relations on a category $\mathbb{C}$ — those induced by a special type of factorization systems [6] on $\mathbb{C}$, and those induced by a special type of monoidal structures [10] on $\mathbb{C}$. In both cases the base structure can be fully recovered from the induced cover relation; in particular,

- the factorization system $(\mathcal{E}, \mathcal{M})$ that induces a cover relation $\sqsubset$ can be obtained back from $\sqsubset$ as follows: for a morphism $g : Y \to Z$, the morphism m in the $(\mathcal{E}, \mathcal{M})$-factorization $g = me$ of g is the morphism $m : C \to Z$ such that the diagram

$$C \xrightarrow{m} Z \xleftarrow{g} Y$$

  is the terminal object in the category of all diagrams

$$X \xrightarrow{f} Z \xleftarrow{g} Y, \qquad f \sqsubset g \tag{1}$$

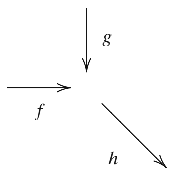
  with fixed g.
- The monoidal structure that induces a cover relation $\sqsubset$ can obtained back from $\sqsubset$ by taking $X \otimes Y$ (for any two objects X and Y) to be the object in the diagram

$$X \longrightarrow X \otimes Y \longleftarrow Y \tag{2}$$

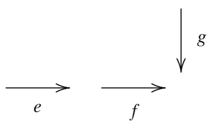
  which is the initial object in the category of all diagrams (1) with fixed X and Y.

Cover relations induced by factorization systems have in general very little common with cover relations induced by monoidal structures, but the following two properties are shared by both of these types of cover relations:

- *Left preservation property.* For any three morphisms f, g, h, as in the display



  if $f \sqsubset g$ then $hf \sqsubset hg$.
- *Right preservation property.* For any three morphisms e, f, g in $\mathbb{C}$, as in the display



  if $f \sqsubset g$ then $fe \sqsubset g$.

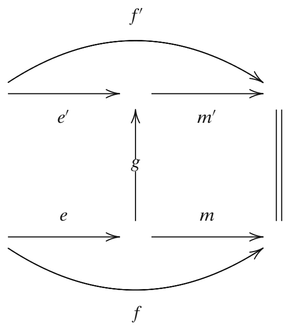
The aim of the paper is to give complete axiomatic descriptions, using conditions similar to the two conditions above, of cover relations induced by factorization systems and cover relations induced by monoidal structures.

Throughout the paper below we use the term "precover relation" instead of a "cover relation", and only when a precover relation has both left and right preservation properties we call it a cover relation.

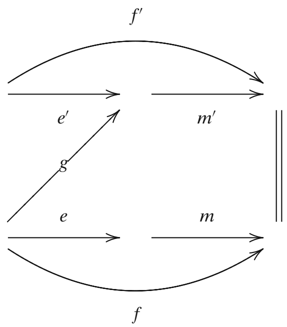
1 Cover Relations Corresponding to a Special Kind of (Right) Factorization Systems

1.1 Preliminaries

To a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$ we associate a cover relation $\sqsubset$ on $\mathbb{C}$, defined as follows: let $f \sqsubset f'$ when f and f' are morphisms that are part of a commutative diagram



for some morphisms $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$, and for some morphism g. Equivalently, instead of asking the arrow g to exist for *some* e, e', m, m', we could require its existence for *every* e, e', m, m'; further, we can also reposition g as follows:



and then we can of course also remove e and m. Yet another equivalent way to define $\sqsubset$ is to use just the class $\mathcal{M}$: let $f \sqsubset f'$ if and only if for any $m' \in \mathcal{M}$, if $f' = m'e'$ for some morphism e' (which does not necessarily belong to the class $\mathcal{E}$), then $f = m'g$

for some morphism g. This defines a precover relation $\sqsubset$ for an arbitrary class $\mathcal{M}$ of morphisms, and we will denote $\sqsubset$ so defined by $\sqsubset^{\mathcal{M}}$.

Lemma 1.1.1 *For any class $\mathcal{M}$ of morphisms, the precover relation $\sqsubset^{\mathcal{M}}$ is reflexive, transitive, and has right preservation property.*

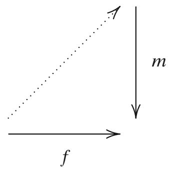
It may happen that two factorization systems give rise to the same cover relation. Indeed, consider the category **Set** and the following factorization systems on **Set**:

- $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ is the class of all epimorphisms and $\mathcal{M}$ is the class of all monomorphisms.
- $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ is the class of all isomorphisms and $\mathcal{M}$ is the class of all morphisms.

Both of the above factorization systems give rise to the following cover relation: $f \sqsubset f'$ if and only if $\mathrm{Im}(f) \subseteq \mathrm{Im}(f')$.

However, as we will show later on in this section, if a factorization system $(\mathcal{E}, \mathcal{M})$ is such that every morphism from the class $\mathcal{M}$ is a monomorphism, then the induced cover relation $\sqsubset^{\mathcal{M}}$ "remembers" the class $\mathcal{M}$ (and hence the whole factorization system). In particular, $\mathcal{M}$ is the class of all $\sqsubset^{\mathcal{M}}$-*images* in the following sense:

Definition 1.1.2 For a precover relation $\sqsubset$, a morphism m is said to be a $\sqsubset$-*image* of a morphism g, if $m \sqsubset g$ and m is universal with this property, i.e. whenever $f \sqsubset g$ there exists a unique dotted arrow making the triangle



commute (a morphism m is said to be *a* $\sqsubset$-*image* if m is a $\sqsubset$-image of some morphism).

Observation 1.1.3 Let $(\mathcal{E}, \mathcal{M})$ be a factorization system where $\mathcal{M}$ is a class of monomorphisms. Then a morphism m is a $\sqsubset^{\mathcal{M}}$-image of a morphism f if and only if m is part of an $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ of f.

As it turns out, we can characterize cover relations $\sqsubset$ which arise from factorization systems $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is a class of monomorphisms, via suitable "natural" properties of the relation $\sqsubset$. Before we can state this characterization, we need the following

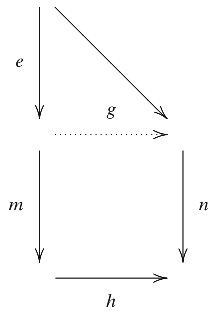
Definition 1.1.4 For a precover relation $\sqsubset$, a morphism m is said to be $\sqsubset$-*reflecting* if for every two morphisms f and f' such that mf and mf' are defined, we have: $mf \sqsubset mf' \;\Rightarrow\; f \sqsubset f'$.

As this follows from Theorem 1.2.2 below, a cover relation is induced by a factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is a class of monomorphisms, if and only if it is a reflexive and transitive relation and every morphism has a $\sqsubset$-reflecting $\sqsubset$-image. If we omit here "$\sqsubset$-reflecting", then we get a characterization of cover relations induced by *right factorization systems* [4] (see below).

1.2 The Main Result (Formulation)

Henceforth we will work in a fixed category $\mathbb{C}$. Recall from [4] that a class $\mathcal{M}$ of morphisms in $\mathbb{C}$ is part of a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{C}$ (and then, recall also that the class $\mathcal{E}$ is uniquely determined by $\mathcal{M}$), if and only if the following conditions are satisfied:

(i) $\mathcal{M}$ is closed under composition and contains isomorphisms.
(ii) Every morphism f in $\mathbb{C}$ has a so called "right $\mathcal{M}$-factorization", i.e. f factors as $f = me$, where $m \in \mathcal{M}$ and m, e are such that for any commutative diagram of solid arrows



in $\mathbb{C}$, if $n \in \mathcal{M}$ then there exists unique dotted arrow g such that the newly formed triangle and rectangle commute.

Of course, when every morphism from $\mathcal{M}$ is a monomorphism, the uniqueness of g in (ii) follows from the existence; then also commutativity of the square in the diagram (ii) implies commutativity of the triangle. If instead of requiring (i) above we require the weaker condition (i′) (see below), then, such $\mathcal{M}$'s are in one-to-one correspondence with (isomorphism classes of) the so called "right factorization systems" (see the Proposition in Section 5.3 in [4]).

(i′) $\mathcal{M}$ is closed under composition with isomorphisms.

For a factorization system $(\mathcal{E}, \mathcal{M})$ we have: a factorization $f = me$ of a morphism f is a right $\mathcal{M}$-factorization if and only if $f = me$ is an $(\mathcal{E}, \mathcal{M})$-factorization of f. Thus, Observation 1.1.3 is a particular instance of the following more general observation:

Observation 1.2.1 Let $\mathcal{M}$ be a class of monomorphisms such that (i′) and (ii) are satisfied. Then a morphism m is a $\sqsubset^{\mathcal{M}}$-image of a morphism f if and only if m is part of a right $\mathcal{M}$-factorization $f = me$ of f.

The main result of Section 1 is the following

Theorem 1.2.2 *There is a one-to-one correspondence between classes $\mathcal{M}$ of monomorphisms satisfying* (i′) *and* (ii)*, and cover relations which are reflexive, transitive and admit images. In particular,*

- *this correspondence associates to such class $\mathcal{M}$ the precover relation $\sqsubset^{\mathcal{M}}$, which turns out to be a cover relation,*
- *and it associates to each reflexive and transitive cover relation $\sqsubset$ admitting images, the class $\mathcal{M}$ of all $\sqsubset$-images.*

Further, for any class $\mathcal{M}$ of monomorphisms satisfying (i′) *and* (ii)*, the following conditions are equivalent:*

(a) *$\mathcal{M}$ satisfies* (i).
(b) *Every morphism in $\mathcal{M}$ is $\sqsubset^{\mathcal{M}}$-reflecting.*
(c) *Every morphism has a $\sqsubset^{\mathcal{M}}$-reflecting $\sqsubset^{\mathcal{M}}$-image.*

1.3 Coverings Relative to a Cover Relation

The class $\mathcal{E}$ in a factorization system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{M}$ is a class of monomorphisms, is precisely the class of all $\sqsubset^{\mathcal{M}}$-*coverings* in the following sense (see Theorem 1.3.4 below):

Definition 1.3.1 Let $\sqsubset$ be a precover relation. A morphism g is said to be a $\sqsubset$-*covering* if it $\sqsubset$-covers every morphism f (having the same codomain as g).

There is a nice characterization of $\sqsubset$-coverings when $\sqsubset$ is a precover relation having right preservation property, and in particular, when $\sqsubset$ is a cover relation:

Lemma 1.3.2 *Consider the following conditions on a morphism $f : X \to Y$:*

(a) *f is a $\sqsubset$-covering.*
(b) $1_Y \sqsubset f$.
(c) *1_Y is a $\sqsubset$-image of f.*

We have (a)⇒(b)⇔(c) *and if $\sqsubset$ has right preservation property then* (a)⇔(b)⇔(c).

Example 1.3.3 In a category $\mathbb{C}$ consider the following precover relation: $f \sqsubset g$ if and only if f and g have the same codomain, and whenever we have $h_1 f = h_2 f$ for some morphisms h_1 and h_2, we necessarily also have $h_1 g = h_2 g$. It is easy to see that $\sqsubset$ is in fact a cover relation. By (a)⇔(b) in Lemma 1.3.2, f is a $\sqsubset$-covering if and only if it is an epimorphism. Observe that if the square

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[h]{} & \bullet \end{array}$$

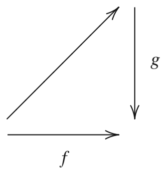
is a pushout, then a morphism e is a $\sqsubset$-image of f if and only if it is an equalizer of g and h. The equivalence of (b) and (c) in Lemma 1.3.2 becomes the following well-known result: f is an epimorphism if and only if in the above pushout $g = h$.

Theorem 1.3.4 *Let $\mathcal{M}$ be a class of monomorphisms. If $\mathcal{M}$ is part of a factorization system $(\mathcal{E}, \mathcal{M})$ then necessarily $\mathcal{E}$ is the class of all $\sqsubset^{\mathcal{M}}$-coverings.*

Proof By Lemma 1.3.2, a morphism $f : X \to Y$ is a $\sqsubset^{\mathcal{M}}$-covering if and only if 1_Y is a $\sqsubset^{\mathcal{M}}$-image of f. By Observation 1.1.3, the latter is the case if and only if there is an $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ of f, where $m = 1_Y$, i.e. if and only if $f \in \mathcal{E}$. □

1.4 The Main Result (Proof)

Notation 1.4.1 By $<$ we denote the precover relation on $\mathbb{C}$ defined as follows: $f < g$ if f and g are part of a commutative triangle



The precover relation $<$ is a reflexive and transitive cover relation. Using $<$ we can express the right preservation property of a precover relation $\sqsubset$ as follows:

$$(f' < f) \wedge (f \sqsubset g) \;\Rightarrow\; f' \sqsubset g.$$

And, using $<$ we can define $\sqsubset^{\mathcal{M}}$, where $\mathcal{M}$ is a class of morphisms, as follows:

$$f \sqsubset^{\mathcal{M}} g \;\;\Leftrightarrow\;\; \forall_{m \in \mathcal{M}} (g < m \Rightarrow f < m).$$

Lemma 1.4.2 *Note that if every morphism in $\mathcal{M}$ is a monomorphism, then a morphism f has a right $\mathcal{M}$-factorization if and only if there exists $m \in \mathcal{M}$ such that $f < m$ and $hm \sqsubset^{\mathcal{M}} hf$ for any morphism h such that the composite hf is defined.*

The precover relation $\sqsubset^{\mathcal{M}}$ can be described also in following way, when every morphism has a right $\mathcal{M}$-factorization: $f \sqsubset^{\mathcal{M}} g$ if and only if $f < m$, where $m \in \mathcal{M}$ is part of a right $\mathcal{M}$-factorization of g.

Let $\sqsubset$ be a precover relation on $\mathbb{C}$.

Lemma 1.4.3 *Suppose $\sqsubset$ has right preservation property. Then:*

(a) *Every $\sqsubset$-image is a monomorphism.*
(b) *A morphism m is a $\sqsubset$-image of itself if and only if m is a monomorphism and for any morphism f we have $f < m \;\Leftrightarrow\; f \sqsubset m$.*

Lemma 1.4.4 *If m is an isomorphism then m is a $\sqsubset$-image of itself if and only if $m \sqsubset m$.*

Lemma 1.4.5 *The class of $\sqsubset$-reflecting morphisms is closed under composition. If $\sqsubset$ has left preservation property then any split monomorphism is $\sqsubset$-reflecting.*

Lemma 1.4.6 *If $\sqsubset$ is reflexive and has right preservation property then $<$ is a subrelation of $\sqsubset$.*

Lemma 1.4.7 *If $\sqsubset$ is transitive and $<$ is a subrelation of $\sqsubset$, then $\sqsubset$ has right preservation property.*

Lemma 1.4.8 *If $\sqsubset$ is transitive and $<$ is a subrelation of $\sqsubset$, then every $\sqsubset$-image of some morphism is at the same time a $\sqsubset$-image of itself.*

Remark 1.4.9 Suppose $\sqsubset$ is transitive and $<$ is a subrelation of $\sqsubset$. Then $\sqsubset$ has right preservation property (Lemma 1.4.7), and by combining Lemmas 1.4.8 and 1.4.3 we get that in this case the following conditions on a morphism m are equivalent to each other:

- m is a $\sqsubset$-image,
- m is a $\sqsubset$-image of itself,
- m is a monomorphism and for any morphism f we have $f < m \Leftrightarrow f \sqsubset m$.

Let $\mathcal{M}$ be a class of morphisms in $\mathbb{C}$. Note that $<$ is a subrelation of $\sqsubset^{\mathcal{M}}$ (this is straightforward but it can be also inferred from Lemmas 1.4.6 and 1.1.1).

Lemma 1.4.10 *For any morphism f and $m \in \mathcal{M}$ we have: $f \sqsubset^{\mathcal{M}} m \Leftrightarrow f < m$.*

Lemma 1.4.11 *For any morphism f and $m \in \mathcal{M}$, the following conditions are equivalent to each other:*

(a) *m is a $\sqsubset^{\mathcal{M}}$-image of f.*
(b) *m is a monomorphism, $f \sqsubset^{\mathcal{M}} m$ and $m \sqsubset^{\mathcal{M}} f$.*
(c) *m is a monomorphism, $f < m$ and $m \sqsubset^{\mathcal{M}} f$.*

Lemma 1.4.12 *For any morphism f and monomorphism $m \in \mathcal{M}$, the following conditions are equivalent to each other:*

(a) *f is a $\sqsubset^{\mathcal{M}}$-image of m.*
(b) *$f = mg$ for some isomorphism g.*

Lemma 1.4.13 *If $\sqsubset$ is transitive and $<$ is a subrelation of $\sqsubset$, then $\sqsubset$ is a subrelation of $\sqsubset^{\mathcal{M}}$, where $\mathcal{M}$ is the class of all $\sqsubset$-images.*

Lemma 1.4.14 *If $\mathcal{M}$ is closed under composition, then every monomorphism from $\mathcal{M}$ is $\sqsubset^{\mathcal{M}}$-reflecting.*

Lemma 1.4.15 *Consider a composite m_1m_2. If m_2 is a $\sqsubset^{\mathcal{M}}$-image of itself, and m_1 is a $\sqsubset^{\mathcal{M}}$-image of itself, and further, m_1 belongs to $\mathcal{M}$ and is $\sqsubset^{\mathcal{M}}$-reflecting, then the composite m_1m_2 is a $\sqsubset^{\mathcal{M}}$-image of itself.*

Lemma 1.4.16 *If $\sqsubset$ is reflexive, has right preservation property, and admits images, then $\sqsubset^{\mathcal{M}}$ is a subrelation of $\sqsubset$, where $\mathcal{M}$ is the class of all $\sqsubset$-images.*

Lemma 1.4.17 *If $\sqsubset^{\mathcal{M}}$ admits images then the following conditions are equivalent to each other:*

(a) *$\sqsubset^{\mathcal{M}}$ has left preservation property.*
(b) *For every morphism f and its $\sqsubset^{\mathcal{M}}$-image m we have $hm \sqsubset^{\mathcal{M}} hf$ for any morphism h such that hf is defined.*
(c) *Every morphism f has a $\sqsubset^{\mathcal{M}}$-image m such that $hm \sqsubset^{\mathcal{M}} hf$ for any morphism h such that hf is defined.*

Theorem 1.4.18 *For a precover relation $\sqsubset$ admitting images the following conditions are equivalent to each other:*

(a) *$\sqsubset$ is a reflexive and transitive precover relation and has right preservation property.*
(b) *$\sqsubset$ coincides with $\sqsubset^{\mathcal{M}}$, where $\mathcal{M}$ is the class of all $\sqsubset$-images.*
(c) *$\sqsubset$ is equal to $\sqsubset^{\mathcal{M}}$ for some class $\mathcal{M}$ of morphisms.*

Proof

(a)$\Rightarrow$(b) follows from Lemmas 1.4.16 and 1.4.13 (and Lemma 1.4.6).
(b)$\Rightarrow$(c) is obvious.
(c)$\Rightarrow$(a) follows from Lemma 1.1.1. □

Proposition 1.4.19 *For a class $\mathcal{M}$ of morphisms the following conditions are equivalent to each other:*

(a) *$\mathcal{M}$ is the class of all $\sqsubset$-images for a reflexive and transitive precover relation $\sqsubset$ admitting images and having right preservation property.*
(b) *$\sqsubset^{\mathcal{M}}$ admits images and $\mathcal{M}$ is the class of all $\sqsubset^{\mathcal{M}}$-images.*
(c) *Every morphism from $\mathcal{M}$ is a monomorphism, for any morphism f there exists $m \in \mathcal{M}$ such that $f < m$ and $m \sqsubset^{\mathcal{M}} f$, and whenever $m' \in \mathcal{M}$ also $m'g \in \mathcal{M}$ when $m'g$ is defined and g is an isomorphism.*

Proof

(a)$\Leftrightarrow$(b) follows from Theorem 1.4.18.
(b)$\Rightarrow$(c) follows from Lemmas 1.4.11 and 1.4.12.
(c)$\Rightarrow$(b): The fact that $\sqsubset^{\mathcal{M}}$ admits images follows from Lemma 1.4.11. By Lemma 1.4.11 again, every morphism from $\mathcal{M}$ is a $\sqsubset^{\mathcal{M}}$-image (of itself). Let f be a $\sqsubset^{\mathcal{M}}$-image of some morphism, and hence of itself (Lemma 1.4.8). Take $m \in \mathcal{M}$ such that $f < m$ and $m \sqsubset^{\mathcal{M}} f$. Then $m < f$. Since f is a monomorphism (Lemma 1.4.3) and m is a monomorphism, $f < m$ and $m < f$ give that $f = mg$ for some isomorphism g. This implies $f \in \mathcal{M}$.
□

Proposition 1.4.20 *Let $\sqsubset$ be a reflexive and transitive precover relation admitting images and having right preservation property. The following conditions are equivalent to each other:*

(a) *The class of $\sqsubset$-images is closed under composition.*
(b) *Every $\sqsubset$-image is $\sqsubset$-reflecting.*

Proof Let $\mathcal{M}$ denote the class of $\sqsubset$-images. By Theorem 1.4.18, $\sqsubset$ coincides with $\sqsubset^{\mathcal{M}}$. By Proposition 1.4.19, every morphism from $\mathcal{M}$ is a monomorphism. Applying Lemma 1.4.14 we get (a)⇒(b). (b)⇒(a) follows from Lemmas 1.4.8 and 1.4.15. □

Theorem 1.4.21 *For a class $\mathcal{M}$ of morphisms the following conditions are equivalent to each other:*

(a) *$\mathcal{M}$ is the class of all $\sqsubset$-images where $\sqsubset$ is a reflexive and transitive cover relation admitting images.*
(b) *$\sqsubset^{\mathcal{M}}$ is a cover relation admitting images, and $\mathcal{M}$ is the class of all $\sqsubset^{\mathcal{M}}$-images.*
(c) *Every morphism from $\mathcal{M}$ is a monomorphism, $\mathcal{M}$ is closed under composition with isomorphisms, and every morphism has a right $\mathcal{M}$-factorization.*

Proof

(a)⇔(b) follows from Theorem 1.4.18 (and Lemma 1.1.1).
(b)⇔(c): Apply the equivalence (b)⇔(c) in Proposition 1.4.19 and Lemmas 1.4.2, 1.4.17 and 1.4.11; after this it only remains to show that if (b) is satisfied, then for any morphism $m \in \mathcal{M}$ and isomorphism g, we have $gm \in \mathcal{M}$. This can be shown by using Lemmas 1.4.8, 1.4.4 and 1.4.5 while applying Lemma 1.4.15. □

Proof of Theorem 1.2.2 Let $\mathfrak{M}$ denote the collection of all classes $\mathcal{M}$ of monomorphisms in $\mathbb{C}$ satisfying (i′) and (ii), and let $\mathfrak{C}$ denote the collection of all reflexive and transitive cover relations on $\mathbb{C}$ admitting images. By the implication (c)⇒(b) in Theorem 1.4.21, and by Lemma 1.1.1, for any $\mathcal{M} \in \mathfrak{M}$ the precover relation $\sqsubset^{\mathcal{M}}$ is indeed a cover relation and it belongs to $\mathfrak{C}$. Thus, the assignment $\mathcal{M} \mapsto \sqsubset^{\mathcal{M}}$ defines a map $F : \mathfrak{M} \to \mathfrak{C}$. By the implication (a)⇒(c) in Theorem 1.4.21, we get a map backwards $G : \mathfrak{C} \to \mathfrak{M}$ which sends each cover relation $\sqsubset$ to the class of all $\sqsubset$-images. Using the implication (c)⇒(b) in Theorem 1.4.21 again, we get $GF = 1$. From the implication (a)⇒(b) in Theorem 1.4.18, we get $FG = 1$. Next, we prove the equivalence of (a), (b), and (c) in Theorem 1.2.2 for any $\mathcal{M} \in \mathfrak{M}$. Note that by Lemma 1.4.4, $\mathcal{M}$ contains all isomorphisms. After this, applying Proposition 1.4.20 we get the equivalence of (a) and (b). The implication (b)⇒(c) is obvious. We show (c)⇒(b). Suppose $m \in \mathcal{M}$. Let m' be a $\sqsubset^{\mathcal{M}}$-reflecting $\sqsubset^{\mathcal{M}}$-image of m. Then, by Lemma 1.4.12, $m = m' f$ where f is an isomorphism. By Lemma 1.4.5, m is $\sqsubset^{\mathcal{M}}$-reflecting. □

2 Cover Relations Corresponding to a Special Kind of Monoidal Structures

2.1 Bicover Relations

If a precover relation $\sqsubset$ has left preservation property, then its inverse relation $\sqsubset^{\circ}$ also has left preservation property. The similar statement is not true for the right preservation property.

Definition 2.1.1 A precover relation $\sqsubset$ is said to be a *bicover relation* if both $\sqsubset$ and $\sqsubset^{\circ}$ are cover relations.

If a bicover relation $\sqsubset$ is reflexive, then it is in fact the indiscete precover relation, i.e. $f \sqsubset g$ for any two morphisms f and g having the same codomain. So, bicover relations are in essence very different from the cover relations induced by factorization systems.

2.2 Wedges

Let $\sqsubset$ be a precover relation on a category $\mathbb{C}$ and let X and Y be objects in $\mathbb{C}$. A diagram

$$X \xrightarrow{f} C \xleftarrow{g} Y \tag{3}$$

where $f \sqsubset g$ will be called a $\sqsubset$-*wedge* on the pair (X, Y). The *category of* $\sqsubset$-*wedges on* (X, Y) is the category where

- objects are $\sqsubset$-wedges on (X, Y),
- and a morphism from a $\sqsubset$-wedge (3) to another $\sqsubset$-wedge

$$X \xrightarrow{f'} C' \xleftarrow{g'} Y$$

  is a morphism $c : C \to C'$ such that $f' = cf$ and $g' = cg$.

In the present section we will work with precover relations $\sqsubset$ which are bicover relations and satisfy the following condition:

(I) For any two objects X and Y, the category of $\sqsubset$-wedges on (X, Y) has an initial object.

If $\sqsubset$ is the indiscrete precover relation (which is a bicover relation), then a diagram (3) is an initial $\sqsubset$-wedge if and only if it is a coproduct diagram.

2.3 Sum Structures

We will now show that there is a duality between

- the ordered set $\mathfrak{B}$ of bicover relations on $\mathbb{C}$ satisfying (I),

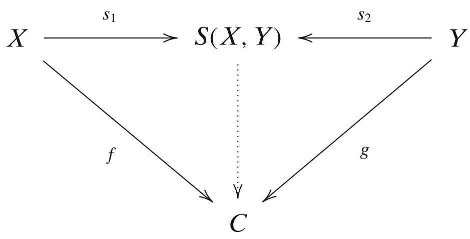
- and the category $\mathfrak{S}$ of E-wedges

$$P_1 \xrightarrow{s_1} S \xleftarrow{s_2} P_2 \tag{4}$$

 on (P_1, P_2), where P_1 and P_2 are the product projections $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ regarded as objects in the functor category $\mathbb{C}^{\mathbb{C}\times\mathbb{C}}$, and E is a precover relation on this category defined as follows: a morphism $g : Y \to Z$ in $\mathbb{C}^{\mathbb{C}\times\mathbb{C}}$ covers a morphism $f : X \to Z$ in $\mathbb{C}^{\mathbb{C}\times\mathbb{C}}$, when for any two objects C_1 and C_2 in $\mathbb{C}$, the (C_1, C_2)-components f_{C_1,C_2} and g_{C_1,C_2} of the natural transformations f and g, respectively, are jointly epimorphic.

An object (4) in $\mathfrak{S}$ will be called a *sum structure* on $\mathbb{C}$, and we will often write it as the triple (S, s_1, s_2). We chose the name "sum structure" simply because for any two objects X and Y in $\mathbb{C}$, the object $S(X, Y)$ is always a quotient of the coproduct $X + Y$ (when the latter exists).

To show the above duality, we begin by constructing a functor $\sqsubset_{(-)}: \mathfrak{S}^{\mathrm{op}} \to \mathfrak{B}$, which we will show to be a category equivalence. For an object $S = (S, s_1, s_2)$ in $\mathfrak{S}$, define $\sqsubset_S$ as follows: for any two morphisms $f : X \to C$ and $g : Y \to C$ in $\mathbb{C}$, $f \sqsubset_S g$ if and only if there exists the dotted arrow in the diagram



making it commutative. Since s_1, s_2 are jointly epimorphic, when that dotted arrow exists, it is necessarily unique. It is easy to see that $\sqsubset_S$ is a bicover relation. Further, $\sqsubset_S$ satisfies (I) since the wedge

$$X \xrightarrow{s_1} S(X, Y) \xleftarrow{s_2} Y \tag{5}$$

is the is an initial $\sqsubset_S$-wedge on (X,Y). So each $\sqsubset_S$ is an object in $\mathfrak{B}$. That $\sqsubset_{(-)}$ defines a functor $\mathfrak{S}^{\mathrm{op}} \to \mathfrak{B}$ (in a unique way, sine $\mathfrak{S}^{\mathrm{op}}$ is in fact a preorder) can be easily seen.

Since $\mathfrak{S}^{\mathrm{op}}$ is a preorder and $\mathfrak{B}$ is an ordered set, to show that $\sqsubset_{(-)}$ is an equivalence of categories we just have to show that

- $\sqsubset_{(-)}$ is full,
- $\sqsubset_{(-)}$ is surjective on objects.

The first can be checked directly and so we omit the proof. The second is also easy to prove, but nevertheless we give the proof below:

Theorem 2.3.1 *A precover relation is of the form $\sqsubset_S$ for some object S in $\mathfrak{S}$ if and only if it is a bicover relation and satisfies* (I).

Proof We already know that a precover relation of the form $\sqsubset_S$ is a bicover relation satisfying (I). So we only prove now the "if part". Let $\sqsubset$ be a bicover relation satisfying (I). For any two objects X and Y, let (5) be the initial $\sqsubset$-wedge on (X, Y). In particular, this implies that s_1 and s_2 in (5) are jointly epimorphic (because of the left preservation property of $\sqsubset$). For any two morphisms $f : X \to X'$ and $g : Y \to Y'$, define $S(f, g)$ as the unique dotted arrow making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{(s_1)_{X,Y}} & S(X,Y) & \xleftarrow{(s_2)_{X,Y}} & Y \\ {\scriptstyle f}\downarrow & & \vdots & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[(s_1)_{X',Y'}]{} & S(X',Y') & \xleftarrow[(s_2)_{X',Y'}]{} & Y' \end{array}$$

commute (note: we use here the fact that both $\sqsubset$ and $\sqsubset^\circ$ have right preservation property). Left preservation property of $\sqsubset$ gives that $\sqsubset$ coincides with $\sqsubset_{(S,s_1,s_2)}$. □

Below, we will gradually impose conditions on a bicover relation satisfying (I) which will make the S in the corresponding sum structure (S, s_1, s_2) into a monoidal product $\otimes$ of some (unique) monoidal structure $(\otimes, I, \alpha, \varrho, \lambda)$ on $\mathbb{C}$. Monoidal structures $(\otimes, I, \alpha, \varrho, \lambda)$ arising in this way are those in which

(M1) I is an initial object in $\mathbb{C}$,
(M2) For any two objects X and Y in $\mathbb{C}$, the morphisms

$$X \otimes I \xrightarrow{1_X \otimes i_Y} X \otimes Y \xleftarrow{i_X \otimes 1_Y} I \otimes Y$$

where i_X and i_Y denote the unique morphisms $I \to X$ and $I \to Y$, respectively, are jointly epimorphic.

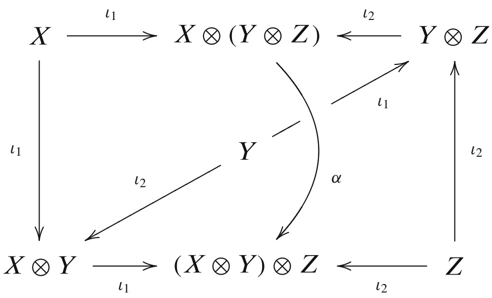
2.4 Associative Sum Structures

Henceforth the S in a sum structure (S, s_1, s_2) will be denoted by the symbol $\otimes$, and $S(X, Y)$ will be written as $X \otimes Y$. Also, the s_1 and s_2 in (S, s_1, s_2) will be denoted by ι_1 and ι_2, respectively.

Definition 2.4.1 A sum structure $(\otimes, \iota_1, \iota_2)$ on $\mathbb{C}$ is said to be *preassociative* if for any three objects X, Y, Z in $\mathbb{C}$ there exists a morphism

$$X \otimes (Y \otimes Z) \xrightarrow{\alpha} (X \otimes Y) \otimes Z$$

called the *associativity morphism at* X, Y, Z, such that the diagram



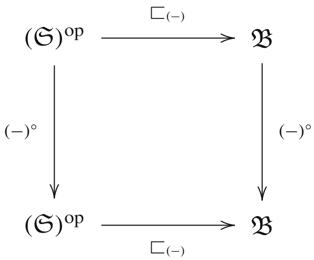
(6)

commutes. This determines α uniquely, and, moreover, it forces α's to be natural in all three arguments. The resulting natural transformation will be called the *associativity natural transformation*. We say that $(\otimes, \iota_1, \iota_2)$ is *associative* if it is preassociative with the associativity natural transformation being a natural isomorphism.

Let F denote the symmetry automorphism of $\mathbb{C} \times \mathbb{C}$. It gives rise to the following automorphism of $\mathfrak{S}$:

$$(-)^\circ : \mathfrak{S} \to \mathfrak{S}, \quad S = (\otimes, \iota_1, \iota_2) \mapsto S^\circ = (\otimes F, \iota_2 F, \iota_1 F).$$

Lemma 2.4.2 *The following diagram commutes:*

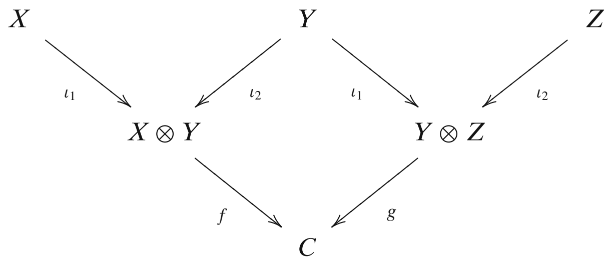


Proposition 2.4.3 *For a sum structure S the following conditions are equivalent to each other:*

(a) *Both S and S° are preassociative, and for any three objects X, Y, Z, the associativity morphism for S at X, Y, Z is the inverse to the associativity morphism for S° at Z, Y, X.*
(b) *S is associative.*
(c) *Both S and S° are preassociative.*

Theorem 2.4.4 *A sum structure* $S = (\otimes, \iota_1, \iota_2)$ *is preassociative if and only if the corresponding bicover relation* $\sqsubset$ *satisfies the following condition:*

(A) *For any commutative diagram*



(7)

if $f \sqsubset g\iota_2$ *then* $f\iota_1 \sqsubset g$.

Proof For the proof of the "if part" take $C = (X \otimes Y) \otimes Z$, $f = \iota_1$ and $g = \iota_2 \otimes 1_Z$ in the above diagram. The proof of the "only if part" is straightforward. □

From Proposition 2.4.3 and Theorem 2.4.4 we get:

Corollary 2.4.5 *A sum structure* $S = (\otimes, \iota_1, \iota_2)$ *is associative if and only if the corresponding bicover relation* $\sqsubset$ *satisfies the following condition:*

(A*) *For any commutative diagram* (7), $f \sqsubset g\iota_2$ *if and only if* $f\iota_1 \sqsubset g$.

2.5 The Unit of a Sum Structure

Definition 2.5.1 An object I in $\mathbb{C}$ is said to be a *left unit* of a sum structure $S = (\otimes, \iota_1, \iota_2)$ if for any object X in $\mathbb{C}$ the morphism $\iota_2 : X \to I \otimes X$ is an isomorphism, and a *right unit* if for any object X in $\mathbb{C}$ the morphism $\iota_1 : X \to X \otimes I$ is an isomorphism (equivalently, I is a right unit if it is a left unit of S°). We say that I is a *unit* of S if I is both left and right unit of S.

Proposition 2.5.2 *Let* S *be a sum structure on a category* $\mathbb{C}$, *and let* $\sqsubset$ *be the corresponding bicover relation. For an object* I *in* $\mathbb{C}$ *the following conditions are equivalent to each other:*

(a) *I is a left unit of S.*
(b) *For every morphism* $f : X \to Y$ *there exists a unique morphism* $i_Y : I \to Y$ *with the property* $i_Y \sqsubset f$.
(c) *The same as (b) with the additional requirement that for each* Y, *the morphism* i_Y *in (b) is the same for all* X *and* $f : X \to Y$, *i.e. for any object* Y *there exists* $i_Y : I \to Y$ *such that for any morphism* $f : X \to Y$, *we have* $i_Y \sqsubset f$ *and, further, for each* f, *the morphism* i_Y *is the unique morphism* $I \to Y$ *with the property* $i_Y \sqsubset f$.

Proof The proof of (a)⇔(b) is straightforward.

(b)⇒(c): Choose i_Y with the property $i_Y \sqsubset 1_Y$. Then $i_Y \sqsubset f$ by the right preservation property of $\sqsubset^\circ$.

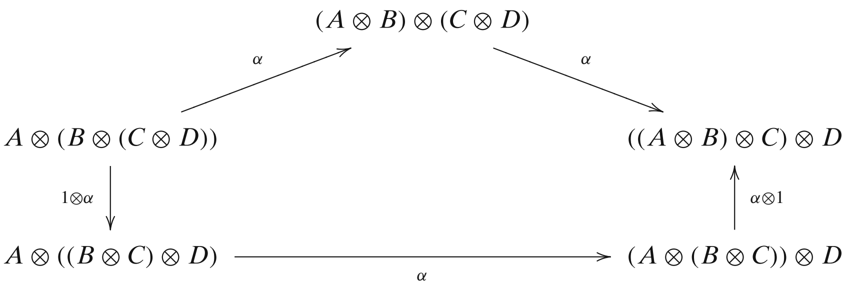
(c)⇒(b) is trivial. □

2.6 Monoidal Sum Structures

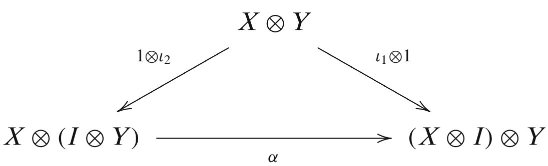
Here we construct a bijection between monoidal structures satisfying (M1) and (M2) and *monoidal sum structures* in the following sense:

Definition 2.6.1 A *monoidal sum structure* on a category $\mathbb{C}$ is a system $(\otimes, \iota_1, \iota_2, I)$ where $(\otimes, \iota_1, \iota_2)$ is an associative sum structure and I is its unit.

It is easy to check that for a preassociative sum structure S, the associativity morphisms make the Mac Lane pentagon



commute, where 1's denote the appropriate identity morphisms. Further, if I is a unit of S, then the triangle



commutes. We arrive to the following

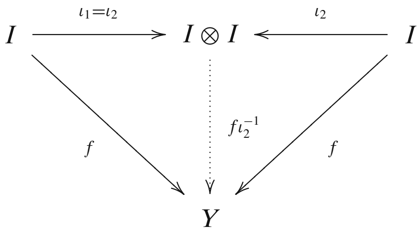
Theorem 2.6.2 *Let $(\otimes, \iota_1, \iota_2, I)$ be a monoidal sum structure on a category $\mathbb{C}$. Then the system $(\otimes, I, \alpha, \lambda, \varrho)$, where*

- *α is the associativity natural transformation for S,*
- *λ is the natural transformation $\lambda = \iota_2 : X \to I \otimes X$,*
- *ϱ is the natural transformation $\varrho = \iota_1 : X \to X \otimes I$,*

is a monoidal category structure on $\mathbb{C}$.

Corollary 2.6.3 *The unit in a monoidal sum structure on a category $\mathbb{C}$ is necessarily an initial object in $\mathbb{C}$.*

Proof Suppose I is a unit of an associative sum structure $S = (\otimes, \iota_1, \iota_2)$. Let $(\otimes, I, \alpha, \lambda, \varrho)$ be the monoidal structure corresponding to S (see Theorem 2.6.2). Then, by coherence, the two isomorphisms $\lambda = \iota_2 : I \to I \otimes I$ and $\varrho = \iota_1 : I \to I \otimes I$ coincide. Take any object Y and let $i_Y : I \to Y$ be as in 2.5.2(c). To show that I is an initial object, we must show that every morphism $f : I \to Y$ coincides with i_Y. Indeed, for any morphism $f : I \to Y$ the diagram



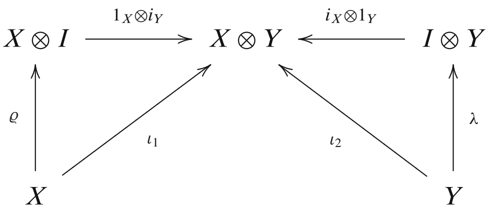
commutes. So $f \sqsubset_S f$ and hence $f = i_Y$. □

Combining Proposition 2.5.2 and Corollary 2.6.3, we get:

Theorem 2.6.4 *An associative sum structure on a category* $\mathbb{C}$ *has a unit if and only if the corresponding bicover relation* $\sqsubset$ *on* $\mathbb{C}$ *satisfies the following condition:*

(U) *For every morphism* $f : X \to Y$ *in* $\mathbb{C}$ *we have* $i_Y \sqsubset f$ *and* $f \sqsubset i_Y$, *where* I *is the initial object in* $\mathbb{C}$ *and* i_Y *denotes the unique morphism* $i_Y : I \to Y$.

Theorem 2.6.2 gives a passage from monoidal sum structures $(\otimes, \iota_1, \iota_2, I)$ on a category $\mathbb{C}$ to monoidal structures $(\otimes, I, \alpha, \lambda, \varrho)$ on $\mathbb{C}$. From Corollary 2.6.3 we get that for a monoidal structure $(\otimes, I, \alpha, \lambda, \varrho)$ obtained in this way, the unit I is an initial object in $\mathbb{C}$. Further, when λ and ϱ are defined as in Theorem 2.6.2, then for any two objects X and Y the diagram

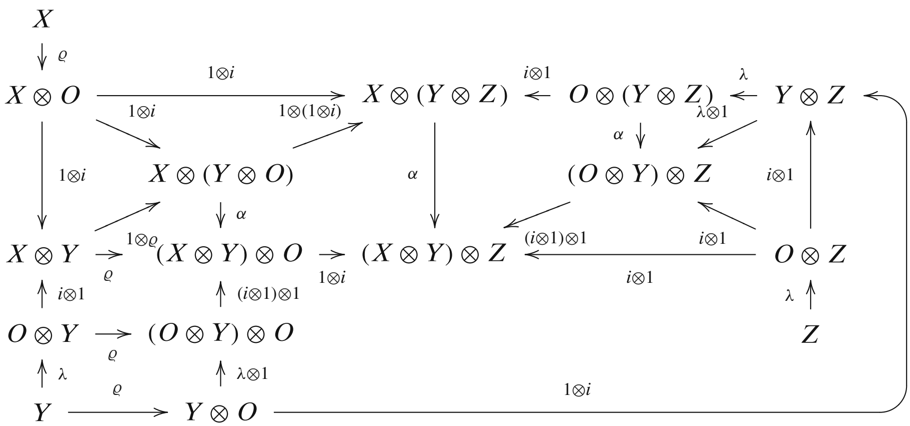


(8)

commutes and the fact that ι_1 and ι_2 are jointly epimorphic gives that $1_X \otimes i_Y$ and $i_X \otimes 1_Y$ are jointly epimorphic. So the monoidal structures coming from monoidal sum structures satisfy (M1) and (M2). Note further that commutativity of the diagram (8) also implies that ι_1 and ι_2 are uniquely determined by ϱ and λ, and so there exists at most one monoidal sum structure which gives rise to a given monoidal structure $(\otimes, I, \alpha, \lambda, \varrho)$. In fact, when the monoidal category satisfies (M1) and (M2) then such monoidal sum structure does always exist, as we will show below (Theorem 2.6.5), and so the passage from monoidal sum structures to monoidal structures described in Theorem 2.6.2 is a bijection between monoidal sum structures and monoidal structures satisfying (M1) and (M2).

Theorem 2.6.5 *Let* $(\otimes, I, \alpha, \lambda, \varrho)$ *be a monoidal structure satisfying* (*M1*) *and* (*M2*). *Then the system* $(\otimes, \iota_1, \iota_2, I)$ *where* ι_1 *and* ι_2 *are defined using the diagram* (8), *is a monoidal sum structure. Further,* α *is the associativity natural transformation of the sum structure* $S = (\otimes, \iota_1, \iota_2)$.

Proof Commutativity of (6) can be checked using the following display (below we have omitted subscripts for 1's and i's):



That I is a unit of S follows from commutativity of the diagram (8) both for $X = I$ and for $Y = I$, and the fact that λ and ϱ are isomorphisms. □

2.7 The Main Result

We have thus constructed bijections between the following structures on a category $\mathbb{C}$, which was the main goal of this section:

- monoidal structures on $\mathbb{C}$ satisfying (M1) and (M2),
- monoidal sum structures,
- pairs $(\sqsubset, I)$ where I is an initial object in $\mathbb{C}$ and $\sqsubset$ is a bicover relation on $\mathbb{C}$ satisfying (I), (A*) and (U).

2.8 Monoids

Here we give a characterization of internal monoids in a monoidal category, whose monoidal structure is obtained from a monoidal sum structure, in terms of the corresponding bicover relation.

Definition 2.8.1 Let $\sqsubset$ be a precover relation. An object X is said to be $\sqsubset$-*indiscrete* if for any two morphisms f and g with codomain X we have $f \sqsubset g$, i.e. X is $\sqsubset$-indiscrete if every morphism with codomain X is a $\sqsubset$-covering.

Note that if $\sqsubset$ is a bicover relation, then an object X is $\sqsubset$-indiscrete if and only if $1_X \sqsubset 1_X$.

Theorem 2.8.2 *Let $\mathbb{C}$ be a monoidal category whose monoidal structure satisfies (M1) and (M2) and let $\sqsubset$ be the corresponding bicover relation. Let* $\mathbf{Mon}(\mathbb{C})$ *denote the category of internal monoids in the monoidal category $\mathbb{C}$. The forgetful functor*

$$\mathbf{Mon}(\mathbb{C}) \to \mathbb{C}$$

is a full inclusion, and further, the objects in the image of this functor are precisely the $\sqsubset$-indiscrete objects in $\mathbb{C}$.

Proof Since the unit I of the monoidal structure of $\mathbb{C}$ is an initial object, for any object M in $\mathbb{C}$ there is exactly one morphism $i_M : I \to M$. A monoid in $\mathbb{C}$ is a triple (M, m, i_Y) where M is an object in $\mathbb{C}$, and m is a morphism $m : M \otimes M \to M$ such that the diagrams

$$\begin{array}{ccccc} M & \xrightarrow{\iota_1} & M \otimes M & \xleftarrow{\iota_2} & M \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1} & \\ & & M & & \end{array} \tag{9}$$

$$\begin{array}{ccc} M \otimes (M \otimes M) & \xrightarrow{1\otimes m} & M \otimes M \\ \downarrow{\scriptstyle \alpha} & & \\ (M \otimes M) \otimes M & & \downarrow{\scriptstyle m} \\ \downarrow{\scriptstyle m\otimes 1} & & \\ M \otimes M & \xrightarrow[m]{} & M \end{array} \tag{10}$$

commute, where ι_1, ι_2 are the same as in the commutative diagram (8). Since in (9) the morphisms ι_1, ι_2 are jointly epimorphic, each object has at most one monoid structure. Further, commutativity of (9) yields $1_M \sqsubset 1_M$ and hence if a monoid structure on M exists, then M is $\sqsubset$-indiscrete. The converse is also true: if $1_M \sqsubset 1_M$ then we define m using (9) and then it can be easily shown that for this m (10) also commutes. It is also easy to show that if M and M' are $\sqsubset$-indiscrete, then any morphism $f : M \to M'$ is also a morphism of monoids. □

Theorem 2.8.3 *A monoidal structure satisfying (M1) and (M2) is a symmetric monoidal structure if and only if the corresponding bicover relation is symmetric. When this is the case, all monoids become automatically commutative monoids.*

2.9 Examples

We now consider several examples of bicover relations arising from monoidal (sum) structures.

First of all, let us note that in a category $\mathbb{C}$ with coproducts, the indiscrete precover relation is a bicover relation induced by the coproduct monoidal structure. This bicover relation is at the same time induced by the factorization system $(\mathcal{E}, \mathcal{M})$ in

$\mathbb{C}$ where $\mathcal{E}$ is the class of all morphisms and $\mathcal{M}$ is the class of all isomorphisms. In fact, the indiscrete precover relation is the only precover relation on $\mathbb{C}$ which arises, at the same time, from a factorization system and from a monoidal structure on $\mathbb{C}$.

Next, we consider an example of a monoidal sum structure, where $X \otimes Y$ is really the sum of X and Y: let $\mathbb{C}$ be take the simplicial category $\mathbb{C} = \Delta$, where objects are finite ordinals $\mathbf{n} = \{1, ..., n\}$ and morphisms $f : \mathbf{n} \to \mathbf{m}$ are order preserving maps $f : \{1, ..., n\} \to \{1, ..., m\}$. Define a precover relation $\sqsubset$ on Δ as follows: g covers f when every element of the image of g is greater than any element of the image of f. This precover relation is in fact a bicover relation induced by a monoidal sum structure where $\otimes$ is the ordinal addition (see [10]):

$$\mathbf{n} \otimes \mathbf{m} = \{1, ..., n + m\}.$$

Now consider the category $\mathbb{C}$ whose

- objects are pairs (A, R), where A is a set and R is a binary relation on A,
- and a morphism $f : (A, R) \to (B, S)$ between such pairs is a map $f : A \to B$ such that $aRa' \Rightarrow f(a)Sf(a')$ for all $a, a' \in A$.

Then Δ is the full subcategory of $\mathbb{C}$ consisting of those objects (A, R) where $A = \{1, ..., n\}$ for some natural number $n \in \{0, 1, 2, ...\}$, and R is the usual order on the ordinal A. The above monoidal sum structure on Δ extends to the whole $\mathbb{C}$. This monoidal sum structure on $\mathbb{C}$ corresponds to the bicover relation on $\mathbb{C}$ defined as follows: a morphism g with codomain (A, R) covers a morphism f with codomain (A, R), if aRa' for every element a in the image of f and every element a' in the image of g.

Let $\mathbb{C}$ be a unital category [1], i.e. a pointed category with finite limits in which for any two objects X and Y the product injections

$$X \xrightarrow{(1,0)} X \times Y \xleftarrow{(0,1)} Y$$

are jointly strongly epimorphic. Then the monoidal category $(\mathbb{C}, \times)$ is a symmetric monoidal category having the properties (M1) and (M2). The corresponding bicover relation $\sqsubset$ is precisely the relation "cooperates" defined in [2]. Coverings relative to $\sqsubset$ are what in [2] are called *central morphisms*, and $\sqsubset$-indiscrete objects are what in [2] are called *commutative objects*. In [3] is given the construction of $\sqsubset$-images (which are called *centralizers* there), when $\mathbb{C}$ belongs to a certain class of unital categories.

Consider the case $\mathbb{C} = \mathbf{Grp}$ in the above example. Then, the induced bicover relation $\sqsubset$ is the following one: for group homomorphisms f and g with codomain X, we have $f \sqsubset g$ if and only if every element x in the image of f commutes with every element y in the image of g, i.e. $xy = yx$ in X. In this case

- $\sqsubset$-coverings are the central homomorphisms between groups.
- For a group homomorphism $f : X \to Y$, a $\sqsubset$-image of f is a monomorphism $C \to Y$ whose image is the centralizer of the image of f. In particular, if $f = 1_Y$, then a $\sqsubset$-image of f is a monomorphism $C \to Y$ whose image is the center of Y.
- $\sqsubset$-indiscrete objects are the abelian groups.

Consider the category **Rng** of rings with unit, and the following bicover relation $\sqsubset$ on **Rng**: for two homomorphisms f and g of rings, with codomain X, we have $f \sqsubset g$ if and only if every element x in the image of f commutes with every element y in the image of g, i.e. $xy = yx$ in X. This bicover relation corresponds to a monoidal structure on **Rng**, where $X \otimes Y$ is the usual tensor product of rings X and Y, and not the cartesian product as in the case of groups, but similarly as in the case of groups, we have:

- $\sqsubset$-coverings are the central homomorphisms between rings.
- For a ring homomorphism $f : X \to Y$, a $\sqsubset$-image of f is a monomorphism $C \to Y$ whose image is the centralizer of the image of f. In particular, if $f = 1_Y$, then a $\sqsubset$-image of f is a monomorphism $C \to Y$ whose image is the center of Y.
- $\sqsubset$-indiscrete objects are the commutative rings.

The category **Rng** of rings with unit can be fully embedded in the category **AlgTh** of algebraic theories [8]. In particular, to each ring R this embedding assigns the algebraic theory of the variety of R-modules. The bicover relation on **Rng** considered above extends to the following bicover relation on **AlgTh**: for morphisms $f : S \to U$ and $g : T \to U$ of algebraic theories define $f \sqsubset g$ if every m-ary operator s in the image of f *commutes* with every n-ary operator t in the image of g, i.e. the identity

$$s(t(x_{11}, ..., x_{1n}), ..., t(x_{m1}, ..., x_{mn})) = t(s(x_{11}, ..., x_{m1}), ..., s(x_{1n}, ..., x_{mn}))$$

is a theorem in U. This bicover relation arises from a monoidal structure on **AlgTh**. In particular, for two theories S and T, $S \otimes T$ is the Kronecker product of S and T [5] ($S \otimes T$ is also called the tensor product of theories S and T). Here we have a similar description of $\sqsubset$-coverings, $\sqsubset$-images and $\sqsubset$-indiscrete objects as in the case of rings (see [9]).

Remark 2.9.1 Note that in all examples we considered above, apart from the example where $\mathbb{C}$ was a general unital category, the bicover relation $\sqsubset$ admits $\sqsubset$-reflecting $\sqsubset$-images.

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