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Perfect *MV*-algebras and their Logic

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Abstract In this paper, after recounting the basic properties of perfect MV-algebras, we explore the role of such algebras in localization issues. Further, we analyze some logics that are based on Łukasiewicz connectives and are complete with respect to linearly ordered perfect MV-algebras.

Key words MV-algebras · Łukasiewicz logic · perfect MV-algebras · localization

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1 Introduction

The class of MV-algebras arises as algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras did with respect to the classical propositional logic. Due to the non-idempotency of the MV-algebraic conjunction, unlike Boolean algebras, MV-algebras can be non-archimedean and can contains elements x such that $x \odot \ldots \odot x$ (n times) is always greater than zero, for any $n > 0$ (here $\odot$ denotes the Łukasiewicz conjunction). In general, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are

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called *infinitesimals*. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical. Hence perfect MV-algebras can be seen as extreme examples of non-archimedean MV-algebras.

An important example of a perfect MV-algebra is the subalgebra S of the Lindenbaum algebra L of first order Łukasiewicz logic generated by the classes of formulas which are valid but non-provable. Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Łukasiewicz first order logic (see [3, 12]).

As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [11]. This makes the interest in MV-algebras relevant outside the realm of logic. Let us denote by Γ the functor implementing this equivalence. In particular each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit. But, more has been proved. Namely the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups, see ([9], Theorem 3.5, p.420). Let us denote by $\mathcal{D}$ the functor implementing this equivalence. Hence $\mathcal{D}$ maps functorially each perfect MV-algebra to an abelian ℓ-group and vice versa, without the help of a strong unit. Here a curious remark has to be made. Indeed the Γ functor maps a non-equational class, the category of abelian ℓ-groups with strong unit, to an equational class, the variety of MV-algebras. On the other hand, the functor $\mathcal{D}$ maps an equational class, the category of abelian ℓ-groups, to a non-equational class, the category of perfect MV-algebras. However, as a consequence of using the functor $\mathcal{D}$, a surprising result was proved showing the equivalence between the category of perfect MV-algebras with a distinguished generator of the radical, whose morphisms preserve the distinguished element, and the category of all MV-algebras, [4].

Perfect MV-algebras do not form a variety and contains non-simple subdirectly irreducible MV-algebras. It is worth stressing that the variety generated by all perfect MV-algebras is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [7]. The MV-algebra C is therefore a prototypical perfect MV-algebra, and we shall see it plays a crucial role in the theory of perfect MV-algebras, as well as in the variety it generates.

In a perfect MV-algebra all infinitesimals are localized in the unique maximal ideal. This property is common to a larger class of MV-algebras, namely to *local* MV-algebra, i.e., MV-algebras having just one maximal ideal. In this paper, after recounting the basic properties of perfect MV-algebras, we explore the role of these algebras in localization issues. We further analyze some logics based on Łukasiewicz connectives, which are complete with respect to linearly ordered perfect MV-algebras.

We summarize below the main results of the paper:

- We show that the class of perfect MV-algebras is a universal class;
- For a given MV-algebra A and a prime ideal P of A a local MV-algebra A_P can be canonically associated to A, actually A_P is perfect.
- We show that for any subalgebra A' of A, having P as maximal ideal, the spectrum of A'/O_P where O_P is the intersection of all prime ideals contained in P, is homeomorphic to a subspace of $Spec(A)$.
- We show that there exists a natural presheaf of perfect MV-algebras associated to each MV-algebra.

- We describe an equivalence between an enriched category of MV-algebras having as objects pairs (A, P) where A is an MV-algebra and P a prime ideal of A, and the category of perfect MV-algebras, a full subcategory of MV-algebras, via the canonical localization of A at P. This functor composed with the functor $\mathcal{D}$ sending any MV-algebra to an abelian ℓ-group gives an equivalence that arises via the described localization at a prime.
- We describe two logical systems, based on the sentential Łukasiewicz calculus, denoted by $\mathbf{Luk_p}$ and $\widetilde{\mathbf{Luk_p}}$, respectively, for which completeness theorems can be proved with respect to Chang's MV-algebra C and with respect to all perfect MV-chains, respectively.

2 Some Preliminary Notions

A structure $A = (A, 0, 1, \neg, \odot, \oplus)$ is an MV-algebra iff A satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $\neg 0 = 1$;
6. $\neg 1 = 0$;
7. $x \odot y = \neg(\neg x \oplus \neg y)$;
8. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \odot y) \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We shall write $x \leq y$ iff $x \wedge y = x$. We say that an MV-algebra A is an MV-chain when, as a lattice, A is linearly ordered. Boolean algebras are just the MV-algebras obeying the additional equation $x \odot x = x$. For any MV-algebra A we denote by $B(A) = \{x \in A \mid x \odot x = x\}$ the biggest Boolean algebra contained in A. We write nx instead of $x \oplus ... \oplus x$ (n-times) and x^n instead of $x \odot ... \odot x$ (n-times). The least integer for which $nx = 1$ is called the *order* of x. When such an integer exists, we denote it by $ord(x)$ and say that x has *finite order*, otherwise we say that x has *infinite order* and write $ord(x) = \infty$.

Example The unit interval of real numbers [0, 1] with operations defined by $x \oplus y = \min\{1, x + y\}$, $x \odot y = \max\{0, x + y - 1\}$, and $\neg x = 1 - x$ is an MV-algebra. We shall refer to this MV-algebra as [0, 1].

An *ideal* of an MV-algebra A is a non-empty subset I of A which is closed under $\oplus$ and such that if $x \leq y$ and $y \in I$ then $x \in I$. A *prime* ideal P of A is an ideal of A such that $x \wedge y$ implies $x \in P$ or $y \in P$. An ideal M of A is called *maximal* if $M \subseteq I$ implies $I = A$ or $I = M$, where I an ideal of A. Let M be a maximal ideal of A, then we say that M is *supermaximal* if $A/M \cong \{0, 1\}$. The set of all prime ideals of A shall be denoted by $Spec(A)$. For each element x of an MV-algebra A the set

$$id(x) = \{y \in A \mid y \leq nx, \text{ for some } n > 0\}$$

is the ideal of A generated by x. Each proper ideal is contained in a maximal ideal.

As MV-algebras form an equational class, the notions of MV-isomorphism, quotient, subalgebra, product, etc., are just the particular cases of the corresponding universal algebraic notions.

The intersection of all maximal ideals, the *radical* of A, will be denoted by $Rad(A)$.

An MV-algebra A such that $Rad(A) = 0$ is called *semisimple*. An MV-algebra A is called *simple* if and only if A is non trivial and $\{0\}$ is its only proper ideal. Every simple MV-algebra is isomorphic to a subalgebra of $[0, 1]$, (see, e.g., [8], Theorem 3.5.1, p. 70). Every non-zero element of a non trivial MV-algebra A has finite order if and only if A is simple.

If for every element x of the MV-algebra A there is an integer n such that nx is idempotent then A will be called *hyperarchimedean*. For all unexplained MV-algebraic notions we refer the reader to [8].

Let X be a non-empty set. Then the set $B = [0, 1]^X$ of all $[0, 1]$-valued functions over X, equipped with pointwise operations, is an MV-algebra. Up to isomorphism, subalgebras of B provide the most general possible examples of semisimple MV-algebras, (see, e.g., [1], Theorem 4.9, p. 486).

Let A be an MV-algebra, $P \in Spec(A)$ and I an ideal of A. We shall use the following notation:

- $\omega_P(A)$ is the set of prime ideal of A contained in P, i.e. $\omega_P(A) = \{Q \in Spec(A) \mid Q \subseteq P\}$.
- $\mathfrak{U}(I) = \{Q \in Spec(A) \mid I \subseteq Q\}$.
- $0_P(A) = \bigcap\{Q \mid Q \in \omega_P(A)\}$ so $0_P(A)$ is an ideal in A
- If $X \subseteq A$ by $alg(X)$ we mean the subalgebra of A generated by X.

Recall that an MV-algebra is *hypernormal* iff $\omega_M(A)$ is a chain for each maximal ideal M of A. Equivalently, if each maximal ideal contains a unique minimal prime.

Definition 1 An MV-algebra A is *local* if A has a unique maximal ideal. The class of all local MV-algebras will be denoted by **Local**.

It is also well known that for each $x \in A$, x is a member of a proper ideal, hence a maximal ideal, if and only if the order of x is ∞.

It turns out that an MV-algebra A is local if and only if for every $x \in A$ $ord(x) < \infty$ or $ord(\neg x) < \infty$.

Definition 2 An MV-algebra A is called *perfect* if for every nonzero element $x \in A$ $ord(x) = \infty$ if and only if $ord(\neg x) < \infty$. The class of all perfect MV-algebras will be denoted by **Perfect**.

It is clear that the class of all MV-algebras is a variety, here denoted by MV. For any subclass K of elements from MV, $V(K)$ shall denote the subvariety of MV generated by K. If K has just one element A then we also write $V(A)$ for $V(K)$.

Definition 3 Chang's MV-algebra is defined on the set

$$C = \{0, c, \ldots, nc, \ldots, 1 - nc, \ldots, 1 - c, 1\}$$

by the following operations (consider $0 = 0c$):

$$x \oplus y = \begin{cases} (m-n)c & \text{if } x = nc \text{ and } y = mc \\ 1-(m-n)c & \text{if } x = 1-nc \text{ and } y = mc \text{ and } 0 < n < m \\ 1-(n-m)c & \text{if } x = nc \text{ and } y = 1-mc \text{ and } 0 < m < n \\ 1 & \text{otherwise;} \end{cases}$$

$$\neg x = \begin{cases} 1-nc & \text{if } x = nc \\ nc & \text{if } x = 1-nc \end{cases}$$

C is a linearly ordered MV-algebra, $ord(nc) = \infty$ and $ord(1-nc) < \infty$ for every n. So C is a perfect MV-algebra.

3 Some Pertinent Facts about Perfect MV-algebras

In this section we will recall some results concerning perfect MV-algebras which are relevant to understand the role of perfect MV-algebras within the theory of MV-algebras. Also we shall establish some new results about perfect MV-algebras that we will use throughout the paper.

Definition 4 A proper ideal P of an MV-algebra A is called *perfect* if and only if for every $a \in A$, $a^n \in P$ for some $n \in \omega$ if and only if $(\neg a)^m \notin P$ for all $m \in \omega$.

P is a perfect ideal if and only if A/P is a perfect MV-algebra.

Proposition 5 *The following hold:*

(1) *The only finite perfect MV-algebra is* $\{0, 1\}$*;*
(2) *Every nonzero element in a perfect MV-algebra* $A \neq B(A)$ *generates a subalgebra isomorphic to the Chang MV-algebra C.*
(3) *Sudirectly irreducible MV-algebras in* $V(C)$ *are all perfect MV-chains;*
(4) $V(\textbf{Perfect}) = V(C)$*,*
(5) $\textbf{Perfect} = V(C) \cap \textbf{Local}$*;*
(6) $A \in V(C)$ *iff for every* $x \in A$*,* $2x^2 = (2x)^2$*;*
(7) A *is perfect iff* $A = alg(Rad(A))$*;*
(8) A *is perfect iff* $A = Rad(A) \cup \neg(Rad(A))$*; further* $x \in Rad(A)$ *iff* $ord(x) = \infty$*;*
(9) **Perfect** *is closed under homomorphic images and subalgebras;*
(10) A *is perfect iff any proper ideal of* A *is perfect iff* $\{0\}$ *is a perfect ideal.*

Proof Properties (1) and (2) are immediate consequences of order of elements in a perfect MV-algebra.

The proof of properties (3), (4), (5), (6) and of (7), (8), (9), (10) can be found in [9] and in [6], respectively. □

Now we are going to show that the class of perfect MV-algebras is first order definable.

Consider the following well formed formulas in the first order language of MV-algebras containing the equality relation as predicate symbol, operations of MV-algebras as functional symbols and 0 and 1 as constant symbols. Further denote by &, OR and $\Rightarrow$ the classical propositional connectives. Let σ be the wff $(\forall x)(x^2 \oplus x^2 = (x \oplus x)^2)$ and τ be the wff $(\forall x)(x^2 = x \Rightarrow (x = 0 \text{ OR } x = 1))$. Then,

Proposition 6 *Let A be an MV-algebra. Then the following are equivalent:*

(1) *A is perfect;*
(2) *A satisfies σ & τ.*

Proof (1) $\Rightarrow$ (2). Let A be a perfect MV-algebra. Then by Proposition 5(4), $A \in V(\mathbf{C})$ hence (2) holds since Chang's MV-algebra satisfies σ & τ.

(2) $\Rightarrow$ (1). Assume A to be an MV-algebra satisfying the formula σ & τ, so $x^2 = x$ implies $x = 0$ or $x = 1$. Hence $B(A) = \{0, 1\}$ and $A \in V(C)$. Hence, by Chang Theorem (see [8, Theorem 1.3.3, p.20]), Theorem 5.1 in ([9] p.424) and Corollary 5.2 in ([9] p.425) (cfr Proposition 5(3)):

$$A \hookrightarrow \prod_{P \in Spec(A)} A/P$$

where, for every $P \in Spec(A)$, A/P is a perfect MV-chain. Assume A is not perfect. Then there is $z \in A$ such that $z \notin Rad(A) \cup \neg(Rad(A))$. Therefore, there are $P, Q \in Spec(A)$ such that

$$z/P \in Rad(A/P) \qquad \textit{and} \qquad z/Q \in \neg Rad(A/Q).$$

So we get $2(z^2/P) = 0/P$ and $2(z^2/Q) = 1/Q$, that is $2z^2 \in B(A) \setminus \{0, 1\}$, contradicting $B(A) = \{0, 1\}$. □

Proposition 7 *Let A be an MV-algebra and set*

$$per(A) = \bigcap\{J \mid J \textit{ is a perfect ideal of } A\}.$$

Then $per(A/per(A)) = 0$.

Proof If A has no perfect ideals then $per(A) = A$. Let $I/per(A)$ be a perfect ideal in $A/per(A)$. Consider the map

$$(A/per(A))/(I/per(A)) \to A/I, \quad \textit{with} \quad per(A) \subseteq I$$

given by $(x/per(A))/(I/per(A)) \to x/I$. In order to prove that the above map is well-defined suppose that $(x/per(A))/(I/per(A)) = (y/per(A)) / (I/per(A))$ so that $d((x/per(A)), (y/per(A))) \in I/per(A)$ where $d(a, b) = (a\neg \odot b) \oplus (\neg a \odot b)$. Then $d(x, y)/per(A) \in I/per(A)$ and since $per(A) \subseteq I$ we have $d(x, y) \in I$ so the map is well-defined. It is easy to check that the map is an epimorphism. As epimorphic image of perfect MV-algebras are perfect we see that if $I/per(A)$ is perfect in $A/per(A)$, then I is perfect in A. □

Note that we are only interested in the case where A contains at least one perfect ideal which we shall always assume. Call an MV-algebra A *semi-perfect* if $per(A) = \{0\}$. Thus if $per(A) = \{0\}$, then A is a subdirect product of perfect MV-algebras, hence, by Proposition 5(3), $A \in V(C)$. Hence $per(A)$ is a kind of "radical" for perfect ideals and we shall characterize it.

Proposition 8 *Let A be an MV-algebra. Then*

$$per(A) = id\{2x^2 \odot 2(\neg x)^2, (2x)^2 \odot (2\neg x)^2 \mid x \in A\}.$$

Proof If $2x^2 = (2x)^2$ we obtain $2x^2 \odot 2(\neg x)^2 = 0$ and $(2x)^2 \odot (2\neg x)^2 = 0$. Let

$$R = id\{2x^2 \odot 2(\neg x)^2, \quad (2x)^2 \odot (2\neg x)^2 \mid x \in A\}.$$

If $J \subseteq A$ is perfect, then since σ is satisfied on A/J, we see that $R \subseteq J$. Hence $R \subseteq per(A)$. Now let P be a prime ideal of A with $R \subseteq P$. Then A/P is linearly ordered and satisfies σ. Thus by a Proposition 5, A/P is perfect, and so P is a perfect ideal, hence $per(A) \subseteq P$. Hence we have the closed set $\mathfrak{U}(R) \subseteq \mathfrak{U}(per(A))$. But from $R \subseteq per(A)$ we have $\mathfrak{U}(per(A)) \subseteq \mathfrak{U}(R)$. Therefore $\mathfrak{U}(R) = \mathfrak{U}(per(A))$ and we may infer the claim of the proposition. □

4 The Category of Perfect MV-algebras

Let $\mathcal{P}$ be the full subcategory of the category of MV-algebras whose objects are perfect MV-algebras. Following [9], we can associate each perfect MV-algebra A with an abelian ℓ-group $G = \mathcal{D}(A)$ by the following construction: Let $\theta \subseteq Rad(A) \times Rad(A)$ be defined by $(x, y)\theta(x', y')$ if and only if $x \oplus y' = x' \oplus y$. θ can be shown to be a congruence and we set $[x, y] \oplus [x', y'] = [x \oplus x', y \oplus y']$, where $[x, y]$ denote the congruence class of (x, y). Then $\mathcal{D}(A) = (Rad(A) \times Rad(A)/\theta, +, \leq, [0, 0])$ is an abelian ℓ-group (where $-[x, y] = [y, x]$). It can be shown that this construction can be extended to a functor between **Perfect** and the category of abelian ℓ-groups, and that the two categories are equivalent.

Proposition 9 *The following statements hold:*

(1) $\{0, 1\}$ *is a terminal and an initial object of* $\mathcal{P}$*;*
(2) $\mathcal{P}$ *has pull-backs;*
(3) $\mathcal{P}$ *has arbitrary products.*

Proof

(1). Follows from the equivalence with the category of abelian ℓ-groups.
(2). Suppose now we have morphisms in $\mathcal{P}$, $f : A \to X \leftarrow B : g$. Let $\langle A, B\rangle = \{(a, b) \in A \times B \mid f(a) = g(b)\}$. It is easy to see that this set is a perfect MV-subalgebra of $A \times B$. Suppose for some perfect MV-algebra Y we have maps $\alpha : Y \to A, \beta : Y \to B$ such that $f\alpha = g\beta$. Define $h : Y \to \langle A, B\rangle$ by $h(y) = (\alpha(y), \beta(y))$. Then $\pi_1 h = f, \pi_2 h = g$. It follows that $\langle A, B\rangle$ is the pull-back of f along g.
(3). Follows from the equivalence with the category of abelian ℓ-groups. We give here a direct construction. Obviously the direct product of two or more perfect MV-algebras need not be perfect, but they always contain a perfect subalgebra. Let $\{A_i\}_{i\in I}$ be a family of MV-algebras, and let $A = \Pi_{i\in I} A_i$ be the direct product. By the *pseudo-diagonal* of A is meant the set of all $a \in A$ such that $ord(a_i) = ord(a_j)$ for all $i, j \in I$. Note that in a perfect MV-algebra

each element has order 2 or order infinite. In this case the pseudo-diagonal is a perfect MV-algebra. Another way to look at the pseudo-diagonal is as follows. In the product $\Pi_{i\in I}$ consider the ideal $R = \{a \mid a_i \in Rad(A_i), i \in I\}$. Then $alg(R) = R \cup \neg R$ is a perfect subalgebra of the product $\Pi_{i\in I} A_i$. When all the A_i are perfect this subalgebra is the pseudo-diagonal. In the case that $I = \emptyset$ then $\Pi_{i\in I} A_i = \{\emptyset\}$ and we set $alg\{\emptyset\} = \{0, 1\}$. It is straightforward to show that the pseudo-diagonal of a family of perfect MV-algebras is indeed the product in the category $\mathcal{P}$. □

For notation we shall write $p\delta_{i\in I} A_i$for the pseudo-diagonal of $\Pi_{i\in I} A_i$. We shall show later that the pseudo-diagonal will allow us to define a presheaf of perfect algebras on $Spec(A)$ for any MV-algebra A.

5 Localization

In some cases all the elements of infinite order can be "localized" into one maximal ideal. Evidently, in such an MV-algebra there can only be one maximal ideal. Such an MV-algebra is then local.

In [2] a type of localization at a prime ideal of an MV-algebra was introduced. The motive for that was related to the Going Up and Going Down Theorems for MV-algebras. In this work we wish to examine and expand this idea of localization to study how the localized algebras relate to the original algebra.

We also wish our notion of localization to parallel as closely as possible the situation in commutative rings.

In our case the localizations will be homomorphic images of subalgebras. Moreover, unlike the ring situation, there will be more than one localization at a given prime. We want, therefore, our localizations at a prime P of an MV-algebra A to satisfy the following conditions:

(1) P, or the image of P, is the unique maximal ideal in the localization,
(2) elements in the localization not in the image of P generate improper ideals,
(3) the prime ideal space of the localization be homeomorphic to the subspace of prime ideals of A contained in P.

Proposition 10 *Let $P \in Spec(A)$ and let I be an ideal of A such that $0_P(A) \subseteq I$. Then $I \subseteq P$ or $P \subseteq I$.*

Proof Suppose not. Then there are $x, y \in A$ such that $x \in I \setminus P$, $y \in P \setminus I$. Then $x \odot \neg y \in I \setminus P$; for if $x \odot \neg y \in P$, then $(x \odot \neg y) \oplus y \in P$ and so $x \vee y \in P$ which implies $x \in P$. Similarly $\neg x \odot y \in P \setminus I$. Now let Q be a prime ideal contained in P. Then $x \odot \neg y \notin Q$. Hence $\neg x \odot y \in Q$. Since Q is arbitrary in $\omega_P(A)$ we may infer that $\neg x \odot y \in 0_P$, thus $\neg x \odot y \in I$ which is impossible. □

Let A be an MV-algebra and P a prime ideal of A. Let

$$\mathcal{L}(P) = \{A' \mid A' \text{ is a subalgebra of } A \text{ and } P \text{ is maximal in } A'\}.$$

Then we have:

Proposition 11 *The following statements hold:*

(1) $\mathcal{L}(P) \neq \emptyset$;
(2) $alg(P)$ *is the only element in* $\mathcal{L}(P)$ *in which* P *is supermaximal.*

Proof Given a prime ideal $P \subseteq A$, consider the subalgebra A_P generated by P. We know that $alg(P) = P \cup \neg P$ where $\neg P = \{\neg x \mid x \in P\}$. Clearly P is an ideal in $alg(P)$ and if $x \in alg(P),\ x \notin P$, then $\neg x \in P$. Therefore we see that P is a maximal ideal in $alg(P)$, in fact, a supermaximal ideal. Hence $\mathcal{L}(P)$ is non-empty.

If P is supermaximal in $A' \in \mathcal{L}(P)$, we must have $A' = alg(P)$. For otherwise there is an $x \in A'$ with $x \notin P$ and $x \notin \neg P$. Hence $x \wedge \neg x \notin P$ which is impossible since P is supermaximal in A'. Therefore $alg(P)$ is the only element of $\mathcal{L}(P)$ in which P is supermaximal. □

We must point out, however, that $alg(P)$, indeed any $A' \in \mathcal{L}(P)$, may have maximal ideals other than P. To obtain a local algebra we must filter out any other maximal ideals.

That $\mathcal{L}(P)$ can have one or infinitely many members is illustrated in the following example. Let ${}^\star[0,\ 1]$ be a non-trivial ultrapower of the MV-algebra $[0,\ 1]$, and let C be the Chang MV-algebra. Set $A = {}^\star[0,\ 1] \times C$. Let $P = 0 \times C$; then P is prime in $A,\ alg(P) = \{0,\ 1\} \times C$. But for any subalgebra $S \subseteq [0,\ 1] \subseteq {}^\star[0,\ 1]$, P is maximal in $S \times C$. On the other hand, if $Q = {}^\star[0,\ 1] \times 0$, then Q is prime in A but the only subalgebra of A in which Q is maximal is $alg(Q) = {}^\star[0,\ 1] \times \{0,\ 1\}$.

Proposition 12 *Let* $A' \in \mathcal{L}(P)$. *Then* $A'/0_P(A)$ *is a local MV-algebra with maximal ideal* $P/0_P(A)$.

Proof Let $x/0_P(A) \notin P/0_P(A),\ x \in A'$. Then $x \notin P$; since P is maximal in A' there is an n with $(\neg x)^n \in P$. Hence $(\neg x)^n/0_P(A) \in P/0_P(A)$ and so $P/0_P(A)$ is a maximal ideal of $A'/0_P(A)$. Now let $I/0_P(A)$ be any ideal in $A'/0_P(A)$. We can assume without loss of generality that $I/0_P(A)$ is prime. Thus I is prime in A'. By the [2, Proposition 2, p.94] there is a prime ideal Q of A such that $I = Q \cap A'$. Hence $0_P(A) \subseteq Q$. By Proposition 10, $P \subseteq Q$ or $Q \subseteq P$. The former implies that $P \subseteq I$, the latter that $I \subseteq P$. Consequently we have either $P/0_P(A) = I/0_P(A)$ or $I/0_P(A) \subseteq P/0_P(A)$. □

Proposition 13 *The MV-algebra* $A_P = alg(P)/0_P(A)$ *is the unique perfect MV-algebra with maximal ideal* $P/0_P(A)$.

Proof Let $a \in alg(P)/0_P(A)$ and assume $a/0_P(A)$ has infinite order. Since $alg(P)/0_P(A)$ is local we must have $ord(\neg a) < \infty$. Conversely, suppose that $ord(\neg a) < \infty$. Then $\neg a \notin P$. As P is supermaximal in $alg(P)$ (Proposition 11), it follows that $a \in P$ and so $a/0_P(A) \in P/0_P(A)$. Therefore $ord(a) = \infty$. So $alg(P)/0_P(A)$ is perfect.

It is clear that we have a natural injection $A_P = alg(P)/0_P(A) \to A'/0_P(A)$ for each $A' \in \mathcal{L}$, in general if $A',\ A'' \in \mathcal{L}(P),\ A' \subseteq A''$, then there is a natural injection $A'/0_P(A) \to A''/0_P(A)$.

Suppose now that for some $A' \in \mathcal{L}(P)$, $A'/0_P(A)$ is perfect with maximal ideal $P/0_P(A)$. We have that $alg(P)/0_P(A)$ embeds into $A'/0_P(A)$. Let $x \in A'$. If $x/0_P(A)$ has infinite order, then $x/0_P(A) \in P/0_P(A)$. Thus for some $y \in P$, $y/0_P(A) = x/0_P(A)$ which implies that $x \in P$, hence $x \in alg(P)$. Since $alg(P) \subseteq A'$ we may infer that $A' = alg(P)$. □

We shall call A_P the *canonical localization* of A at the prime ideal P. We shall call the local MV-algebras $A'/0_P(A)$, with $A' \in \mathcal{L}(P)$, the *localization* of A at P *relative to* A'.

The following lemma is evident.

Lemma 14 *Let $I \subseteq P \in Spec(A)$. Then*

i) *for all prime ideals $Q \in \omega_P(A)$ we have $I \subseteq Q$ iff $(I + 0_P(A))/0_P(A) \subseteq Q/0_P(A)$;*
ii) *if $I \subseteq P$ is an ideal of $A' \in \mathcal{L}(P)$, then I is an ideal of A.*

Theorem 15 (Localization Theorem) *Let A be an MV-algebra and P a prime ideal of A. Then for any $A' \in \mathcal{L}(P)$ there is a natural bijection between $\omega_P(A) = \{Q \in Spec(A) \mid Q \subseteq P\}$ and $Spec(A'/0_P(A))$.*

Proof For any $Q \in \omega_P(A)$, $0_P(A) \subseteq Q$. Thus we can define a map $Q \to Q/0_P(A)$ that is evidently injective. If $Q/0_P(A) \in Spec(A'/0_P(A))$ then $Q \in Spec(A')$ and $Q \subseteq P$ or $P \subseteq Q$ by Proposition 10. Now $P \subseteq Q$ implies $P = Q$ since P is maximal in A'; thus in either case we have $Q \in Spec(A')$, $Q \subseteq P$. By ([2], Corollary to Proposition 2, p. 94) there is a prime ideal $Q_o \subseteq A$ with $Q = Q_o \cap A'$. Hence $0_P(A) \subseteq Q_o$ and so $Q_o \subseteq P$ or $P \subseteq Q_o$. The former gives $Q = Q_o \cap P \cap A' = Q_o$. The latter gives $P \subseteq Q_o \cap A' = Q$. In either cases we see that $Q \in Spec(A)$. Thus the map $Q \to Q/0_P(A)$ is onto. □

Consider $\omega_P(A)$ as a subspace of $Spec(A)$, where the open sets are all of the form $U(J) = \{Q \in Spec(A) \mid J \nsubseteq Q\}$, where J any ideal of A.

Proposition 16 *$\omega_P(A)$ is homeomorphic to $Spec(A'/0_P(A))$.*

Proof Let $O \subseteq \omega_P(A)$, and let $O' = \{Q/0_P(A) \mid Q \in O\}$; then $O = \{Q \mid Q/0_P(A) \in O'\}$. Thus we need only show that O is open in $\omega_P(A)$ iff O' is open in $Spec(A'/0_P(A))$. Clearly $O = \omega_P(A)$ iff $O' = Spec(A'/0_P(A))$. Thus we can assume O, O' are proper subsets of $\omega_P(A)$, $Spec(A_P)$, respectively. Suppose then O is open in $\omega_P(A)$. Thus, $O = U(I) \cap \omega_P(A)$ for some ideal I of A. O proper implies that $I \subseteq P$. By Lemma 14, $O' = U(I')$, where $I' = (I + 0_P(A)(A))/0_P(A)$. Hence O' is open. Conversely, suppose O' is open so that $O' = U(I')$ for some proper ideal I' of $Spec(A'/0_P(A))$. Then $I' = I/0_P(A)$ for some proper ideal I of A' with $0_P(A) \subseteq I$. As P is maximal in A', we have $I \subseteq P$. By Lemma 14 above, item ii), we see that I is an ideal of A. Moreover, $I = I + 0_P(A)$, so by the lemma above, item i) we have $O = U(I) \cap \omega_P(A)$ and is open in $\omega_P(A)$. Therefore the bijection $Q \leftrightarrow Q/0_P(A)$ between $\omega_P(A)$ and $Spec(A'/0_P(A))$ is a homeomorphism. □

Let us examine the special case where the localizations are with respect to a maximal ideal $M \subseteq A$. In this case $\mathcal{L}(M)$ also has a unique maximal element, namely A. Thus we have a maximal localization at M, namely $A/0_M$. We also have the minimal localization A_M. We shall eventually show there are localizations at M intermediate between these two extremes.

First let us examine some properties of the localizations A_P and how they relate to the MV-algebra A.

Proposition 17 *Let A be an MV-algebra and P a prime ideal of A. Then the following are equivalent:*

(1) *A_P is MV-chain;*
(2) *$0_P(A)$ is a minimal prime ideal of $alg(P)$.*

Proof Let us assume that A_P is an MV-chain. Then $0_P(A)$ is prime in $alg(P)$. Hence the prime ideals of A under P form a chain and so $0_P(A)$ is prime in A, in fact a minimal prime. Conversely, assume that $0_P(A)$ is a minimal prime ideal of A, then $\omega_P(A)$ is a chain and P is totally ordered too. Hence A_P is a perfect MV-chain. □

Proposition 18 *A_P is an MV-chain for all P iff A is hypernormal.*

Proof Suppose that A is hypernormal. Then clearly for each maximal ideal M and prime $P \subseteq M$, $0_M = 0_P(A)$ is a minimal prime. Thus $0_P(A)$ is prime in $alg(P)$, hence A_P is linearly ordered. Conversely, if A_P is an MV-chain for all $P \in Spec(A)$, then in particular, A_M is linearly ordered. By Proposition 17, 0_M is a minimal prime of A. It follows that $\omega_M(A)$ is a chain and so A is hypernormal. □

We can ask if the same proposition is true for other members of $\mathcal{L}(P)$.

Proposition 19 *For $A' \in \mathcal{L}(P)$, the localization at P relative to A' is linear iff A is hypernormal.*

Proof Let $A' \in \mathcal{L}(P)$ and suppose that $A'/0_P(A)$ is linear. Then $0_P(A)$ is a prime ideal in A'. Thus the prime ideals of A under P form a chain, hence $0_P(A)$ is prime in A. Clearly it is a minimal prime. □

Corollary 20 *If one localization at P is linear, then all localizations at P are linear.*

Let us now consider the case where the localization A_P is simple.

Proposition 21 *$A_P = \{0, 1\}$ iff P is a minimal prime.*

Proof Let $A_P = \{0, 1\}$. Then $0_P(A)$ must be a maximal ideal of $alg(P)$. Since $0_P(A) \subseteq P \subseteq alg(P)$, we see that $0_P(A) = P$ so that $0_P(A)$ is a maximal ideal in $alg(P)$. But P is supermaximal in $alg(P)$, hence $A_P = \{0, 1\}$.

Now $0_P(A) = P$ implies that P is a minimal prime of A. Conversely, suppose that P is a minimal prime. Then $\omega_P(A) = \{P\}$ and so $0_P(A) = P$. It now follows that $A_P = \{0, 1\}$. □

Corollary 22 *A is hyperarchimedean iff $A_P = \{0, 1\}$ for all primes P.*

We see from the preceding corollary that in general it is not possible to recover A from the localizations A_P.

Suppose now we have a localization $A'/0_P(A)$, $A' \in \mathcal{L}(P)$, such that $A'/0_P(A)$ is simple. We must have that $0_P(A)$ is a maximal ideal in A'. Since $0_P(A) \subseteq P$ we see that $0_P(A) = P$. Again we have that P is a minimal prime of A. Even though P is a maximal ideal of A', it can be not super-maximal, thus we cannot conclude in general that $A'/0_P(A) = \{0, 1\}$. We evidently have,

Proposition 23 *For $A' \in \mathcal{L}(P)$, $A'/0_P(A)$ is simple iff P is a minimal prime. Also $A_P = \{0, 1\}$. Moreover, if $A'/0_P(A)$ is simple for one $A' \in \mathcal{L}(P)$, then all localizations at P are simple.*

6 Other Categories of Perfect MV-algebras

Let MV_* denote the category whose objects are pairs (A, P) where A is an MV-algebra, P a prime ideal of A, and whose morphisms $(A, P) \to (A', P')$ are MV-homomorphisms $f : A \to A'$ such that $f^{-1}(P') \subseteq P$. It is easy to verify that MV_* is a category.

Proposition 24 *The categories $\mathcal{P}$ and MV_* are equivalent.*

Proof Let A be an MV-algebra, P a prime ideal of A and A_P is the (perfect) localization of A at P. Let F be a map $F : MV_* \to \mathcal{P}$ defined as follows: by $F(A, P) = A_P$. If f is a morphism $(A, P) \to (A', P')$, f induces a morphism $f_* : A_P \to A'_{P'}$ by $x/0_P(A)(A) \to f(x)/0_{P'}(A')$. To see that this is well-defined suppose that $x/0_P(A) = y/0_P(A)$, $x, y \in A_P$. Let $Q' \subseteq P'$, Q' prime in A'. Then $F^{-1}(Q') \subseteq f^{-1}(P') \subseteq P$. But $f^{-1}(Q') \in Spec(A)$ and therefore $0_P(A)(A) \subseteq f^{-1}(Q')$. Now $d(x, y) \in 0_P(A)(A)$. Hence $d(x, y) \in f^{-1}(Q')$ and so $d(f(x), f(y)) \in Q'$. Since A' was an arbitrary prime ideal contained in P' we may infer that $d(f(x), f(y)) \in 0_{P'}(A')$ and so $f(x)/0_{P'}(A') = f(y)/0_P(A)(A')$. We also have an obvious functor $G : \mathcal{P} \to MV_*$ given by $A \to (A, M)$ where M is the unique maximal ideal of A. Clearly $F \circ G$ is the identity on $\mathcal{P}$. □

We can modify the preceding result to make it a bit more general. To this end let $\mathcal{L}oc$ denote the category of local MV-algebras and MV-homomorphisms. Let MV_{**} be the category whose objects are triples (A, A', P) where A is an MV-algebra, A' a subalgebra, $P \in Spec(A)$ and P a maximal ideal in A'. The morphism $(A, A', P) \to (A_1, A'_1, P_1)$ are MV-homomorphism $f : A \to A_1$ such that $f(A') \subseteq A'_1$ and $f^{-1}(P_1) \subseteq P$. It is easy to see this is a category.

Proposition 25 *The categories $\mathcal{L}oc$ and MV_{**} are equivalent.*

Proof We have a functor $H : MV^{**} \to \mathcal{L}oc$, $H(A, A', P) = A'/0_P(A)$ as in Proposition 24. Similarly we have a functor $J : \mathcal{L}oc \to MV_{**}$ by sending A to (A, A, M) where M is the unique maximal ideal in A. Again we have $H \circ J$ the identity on $\mathcal{L}oc$. □

That products exist in $\mathcal{P}$ gives rise to a natural pre-sheaf of perfect MV-algebras associated with each MV-algebra. To see this let A be an MV-algebra and let $X = Spec(A)$ endowed with the Zariski topology. Let $\mathcal{O}(X)$ be the category of open sets of X; we have an arrow $U \to V$ if and only if $U \subseteq V$. Thus in the opposite category, $\mathcal{O}(X)^{op}$, we have an arrow $V \to U$ if and only if $U \subseteq V$. For each $U \in \mathcal{O}(X)$ let $A_U = p\delta_{P \in U} A_P$. Thus we have a map $\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{P}$, $\mathcal{F}(U) = A_U$.

Proposition 26 *$\mathcal{F}$ is a pre-sheaf of perfect MV-algebras.*

Proof We note that $\mathcal{F}(\emptyset) = \{0, 1\}$. We claim this is a functor. Indeed suppose that $U \subseteq V$. Then in $\mathcal{O}(X)^{op}$ we have $V \to U$, hence a map $A_V \to A_U$ given by restriction. That is $\langle a_P \rangle_{P \in V} \to \langle a_P \rangle_{P \in U}$. All the necessary conditions are met to show that indeed $\mathcal{F}$ is a functor and therefore a pre-sheaf of perfect MV-algebras. □

7 Logics of Perfect MV-algebras

Let **Luk** denote the Łukasiewicz propositional logic with usual axioms and rule of inference, [10].

Consider the axiomatic extension $\mathbf{Luk}_p$ of propositional Łukasiewicz logic, obtained by adding to the system **Luk** the following axiom schema:

$$(2\alpha^2) \leftrightarrow (2\alpha)^2.$$

Evaluation of formulas of $\mathbf{Luk}_p$ is defined as usual, with Łukasiewicz connectives interpreted by MV-algebraic operations. We shall use the same symbols for denoting connectives and their interpretations.

Proposition 27 *A wff of* $\mathbf{Luk_p}$ *is valid on all perfect MV-chains iff it is provable in* $\mathbf{Luk_p}$.

Proof It is straightforward to show that if α is a theorem in this new system, then α is valid on all perfect MV-algebras. Indeed axioms of $\mathbf{Luk}_p$ are valid on all perfect MV-algebras and modus ponens keeps this validity.

Conversely, the Lindenbaum algebra, $Lind_p$, of $\mathbf{Luk_p}$, satisfies $[\alpha]^2 \oplus [\alpha]^2 = (2[\alpha])^2$, that is, it satisfies σ, hence $Lind_p \in MV(C)$. Now let α be a wff of $\mathbf{Luk_p}$ and suppose that α is valid on all perfect MV-chains. Suppose that α is not provable in $\mathbf{Luk_p}$; then $[\alpha] \neq 1$, and so $[\neg\alpha] \neq 0$. Since $Lind_p$ is semi-perfect (see Proposition 5) there is a prime ideal J with $[\neg\alpha] \notin J$. Moreover J is a perfect ideal. So in $Lind_p/J$ we have that $[\neg\alpha]/J \neq 0$, that is $[\alpha]/J \neq 1$. From this we may infer that α is not valid on the perfect MV-chain $Lind_p/J$ via the assignment $v \to [v]/J$ for each propositional variable v. □

Let us consider now first order theories. As we said in Proposition 6, the class of perfect MV-algebras is first order definable. We further have the following result, concerning semi-perfect MV-algebras:

Let $\mathcal{L}_{sp}$ be a classical first-order logic with inequality, whose language contains function symbols $\oplus, \neg$ and constant symbol 0. We take as axioms the axioms of first-order logic plus the axioms for MV-algebras, and finally the formula $\forall x(2(x)^2 = (2x)^2)$.

Theorem 28 *A wff is provable in $\mathcal{L}_{sp}$ iff it is valid on all semi-perfect MV-algebras iff is valid on C.*

Proof By proposition 5, since Chang's MV-algebra C satisfies equation $2(x)^2 = (2x)^2$ then semi-perfect MV-algebras, as members of the variety $V(\mathbf{Perfect}) = V(C)$, satisfy $2(x)^2 = (2x)^2$.
Claim 1 An MV-equation $\alpha = 1$ holds on C iff it holds on all perfect MV-chains.

Indeed any model of the language $\mathcal{L}_{sp}$ will be a semi-perfect MV-algebra.

Now take any MV-polynomial $\alpha(x_1, \ldots, \alpha_n)$. Consider the first order wff $\forall x_1 \cdots \forall x_n(\alpha(x_1, \ldots, \alpha_n) = \neg 0)$. Suppose that this formula holds on all models of $\mathcal{L}_{sp}$. Then it clearly holds on C. Conversely, suppose it holds on C. Then it holds on the universal class, generated by C. But if A is any semi-perfect algebra, we know $A \in V(C)$, (see [9]), and so the formula will hold on A. In particular, if the formula holds on C it will hold on all perfect MV-chains.

Now let α be a wff of $\mathcal{L}_{sp}$ that is valid on C. Then the corresponding formula $\forall\alpha$, the universal closure of α, will hold on C when C is considered a model of $\mathcal{L}_{sp}$. Thus the formula will hold on all perfect MV-chains considered as models of $\mathcal{L}_{sp}$. From this we see that the MV-polynomial α will hold on all perfect MV-chains. □

In the next result we consider first-order Łukasiewicz logic. Here, for sake of completeness we describe such a logic and provide a notion of *model* for it.

Definition 29 Let $\mathcal{L}$ be a language containing symbols of variables $v_0, v_1, \ldots$, logical symbols $\rightarrow, \neg$; predicate symbols $R_0, R_1, \ldots$; a quantifier symbol $\exists$; improper symbols (,); and a function $d : \mathbb{N} \rightarrow \mathbb{N}$, $\mathbb{N} = \{0, 1, 2, ...\}$.

The set of well-formed formulas of $\mathcal{L}$, WFF, is defined as usual, as follows: atomic formulas, $R_n(v_{i_1}, v_{i_2}, ..., v_{i_{d(n)}})$ are in WFF. If $\alpha, \beta \in WFF$ so are $(\alpha \rightarrow \beta)$ and $\neg\alpha$. If $\alpha \in WFF$ and x is a variable then $(\exists x)\alpha$ is in WFF.

Definition 30 Let A be an MV-algebra and X a nonempty set. An $\{A, X\} - model$ is a system $\langle A, X, (F_n)_{n\in\mathbb{N}}\rangle$ such that for each $n \in \mathbb{N}$ there is a function $F_n : X^{d(n)} \rightarrow A$.

An $\{A, X\} - model$ is *linear* if A is an MV-chain, is *canonical* if $A \subseteq [0, 1]$. Given an $\{A, X\} - model$ $\langle A, X, (F_n)_{n\in\mathbb{N}}\rangle$, an *assignment* is a function $f : Var \rightarrow X$, with

$Var = \{v_0, v_1, \ldots\}$, i.e., Var is the set of variables of $\mathcal{L}$. If f is an assignment, $v \in Var$, $x \in X$, then f_{vx} is the assignment

$$f_{vx}(v_i) = \begin{cases} f(v_i), & \text{if } v_i \neq v \\ x, & \text{if } v_i = v. \end{cases}$$

If $S \subseteq A$ we define

$$\sum S = \begin{cases} \text{least upper bound of } S \text{ in } A \text{ (if it exists)} \\ A, \quad \text{otherwise.} \end{cases}$$

Given $\mathfrak{M} = \langle A, X, (F_n)_{n\in\mathbb{N}}\rangle$ we assign values to each $\alpha \in WFF$. We define, therefore a function

$$Val(\alpha, \mathfrak{M}, f) : WFF \to A,$$

where f is an $\mathfrak{M}$-assignment inductively defined by the following conditions:

(1) $Val(R_n(v_{i_1}, v_{i_2}, ..., v_{i_{d(n)}}), \mathfrak{M}, f) = F_n(f(v_{i_1}), f(v_{i_2}), ..., f(v_{i_{d(n)}}));$
(2) Assuming Val defined for $\alpha \in WFF$, then

$$Val(\neg\alpha, \mathfrak{M}, f) = \begin{cases} \neg Val(\alpha, \mathfrak{M}, f) & \text{if } Val(\alpha, \mathfrak{M}, f) \in A \\ A, \quad \text{otherwise;} \end{cases}$$

(3) Assuming Val defined for $\alpha, \beta \in WFF$, then

$$Val(\alpha \to \beta, \mathfrak{M}, f) = \begin{cases} \neg Val(\alpha, \mathfrak{M}, f) \oplus Val(\beta, \mathfrak{M}, f), \\ \quad \text{provided both } Val(\alpha, \mathfrak{M}, f), Val(\beta, \mathfrak{M}, f) \in A \\ A, \quad \text{otherwise;} \end{cases}$$

(4) $Val((\exists v)\alpha, \mathfrak{M}, f) = \sum_{x\in X} Val(\alpha, \mathfrak{M}, f_{vx})$.

Call an $\mathfrak{M}$-assignment an *interpretation* if $Val(\alpha, \mathfrak{M}, f) \in A$ for all $\alpha \in WFF$. Let $\beta_0, \beta_1, ...$ be an enumeration of all $\beta \in WFF$ of the form $\beta_n = (\exists v_j)\alpha_n$. For each $n, m \geq 0$, let $\beta(n, m)$ be defined in the following way starting from α_n:

(1) let v be the first variable not occurring in α_n;
(2) replace all bounded occurrences of v_m in α_n by v;
(3) replace all free occurrences of v_j in α_n by v_m.

The axioms of $\mathcal{L}$ include all the axioms for Łukasiewicz propositional logic, and we have modus ponens, so we can form the Lindenbaum MV-algebra L. An element from L shall be denoted by $[\alpha]$, with α a formula from $\mathcal{L}$.

Theorem 31 *In L, $\sum_{m\in\mathbb{N}}[\beta(n, m)]$ exists and equals $[\beta_n]$.*

Proof See ([5, Proposition 8, p.13] and [3]). □

An ideal I in L is said to *preserve sums* if for any n, if $[\beta(n, m)] \in I$ for all m, then $\sum_{m\in\mathbb{N}}[\beta(n, m)] \in I$.

Theorem 32 *Let* $[\alpha], [\beta] \in L$, $[\alpha] \notin id([\beta])$. *Then there is an ideal* P *of* L *such that*

(1) $[\beta] \in P$;
(2) P *is maximal with respect to not containing* $[\alpha]$, *so* P *is prime;*
(3) P *preserves the sums* $[\beta_n]$ *for all* $n \in \mathbb{N}$.

Proof See ([5, Proposition 9, p. 13] and [3]). □

Theorem 33 *Suppose* $ord([\beta]) = \infty$. *Then there is a maximal ideal* M *of* L *such that* $[\beta] \in M$ *and* M *preserves the sums* $[\beta_n]$ *for all* $n \in \mathbb{N}$.

Proof See ([5, Corollary 10, p.14] and [3]). □

Let **L** be first order Łukasiewicz logic. Then we know what it means for a first order formula to be valid on the class of perfect MV-chains.

Let $ValForm_p$ denote the class of all wff valid on all perfect MV-chains. We can ask is this set axiomatizable?

Add to **L** the axiom schema, $2\alpha^2 \leftrightarrow (2\alpha)^2$ and denote the resulting system by $\widetilde{\mathbf{Luk_p}}$. Then we have,

Proposition 34 *A wff* α *of* $\widetilde{\mathbf{Luk_p}}$ *is provable, in* $\widetilde{\mathbf{Luk_p}}$, *iff* α *is valid on all perfect MV-chains.*

Proof In the Lindenbaum algebra of **L**, [**L**] we have a dense set of prime ideals P that preserve $\exists$. Since the axioms of $\widetilde{\mathbf{Luk_p}}$ include the axioms of **L**, the same is true of the Lindenbaum algebra $[\widetilde{\mathbf{Luk_p}}]$. But the latter algebra is now semi-perfect. Consequently each of the prime ideals of this algebra is perfect. From this it follows that if α is valid on all perfect MV-chains and not provable in $\widetilde{\mathbf{Luk_p}}$, then in $[\widetilde{\mathbf{Luk_p}}]$, $[\alpha] \neq 1$, hence $\neg[\alpha] \neq 0$. Thus for some prime ideal P of the dense set we have $\neg[\alpha] \notin P$ and so in $[\widetilde{\mathbf{Luk_p}}]/P$ we have $[\alpha]/P \neq 1$. But evaluation on $[\widetilde{\mathbf{Luk_p}}]/P$ of α will result in the value $[\alpha]/P$. □

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