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Cleft Extensions of Hopf Algebroids

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Abstract The notions of a cleft extension and a cross product with a Hopf algebroid are introduced and studied. In particular it is shown that an extension (with a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R)$) is cleft if and only if it is $\mathcal{H}_R$-Galois and has a normal basis property relative to the base ring L of $\mathcal{H}_L$. Cleft extensions are identified as crossed products with invertible cocycles. The relationship between the equivalence classes of crossed products and gauge transformations is established. Strong connections in cleft extensions are classified and sufficient conditions are derived for the Chern–Galois characters to be independent on the choice of strong connections. The results concerning cleft extensions and crossed product are then extended to the case of *weak* cleft extensions of Hopf algebroids hereby defined.

Key words Hopf algebroid · cleft extensions · cocycles · comodule algebra

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Dedicated to Stef Caenepeel on the occasion of his 50th birthday.

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1 Introduction

Cleft extensions of algebras by a Hopf algebra, or cleft Hopf comodule algebras, are one of the simplest and best known examples of Hopf–Galois extensions. Indeed, by [20, Theorem 9] a Hopf–Galois extension with the normal basis property is necessarily a cleft extension. With geometric interpretation of Hopf–Galois extensions over fields as non-commutative principal bundles, cleft extensions can be understood as such principal bundles that every associated bundle is trivial. Motivated by examples coming from non-commutative differential geometry, the notion of a Hopf–Galois extension was generalised to a coalgebra-Galois extension in [15, 16]. Subsequently, the notion of a *cleft coalgebra extension* was introduced in [12, p. 293], and most comprehensively studied in terms of *cleft entwining structures* in [1, 19]. The latter were extended further to weak entwining structures in [2, 3].

The aim of the present paper is to extend the theory of cleft extensions in a different direction, in the first instance motivated by recent developments in the theory of depth-2 and Frobenius ring extensions [21–23], in long term motivated by the increasing interest in Galois-type extensions with Hopf algebroid symmetries [4, 8]. Thus we introduce and study basic properties of Hopf algebroid cleft extensions. Very much as cleft extensions for Hopf algebras are an example and a testing ground for more general Hopf–Galois extensions, also Hopf algebroid cleft extensions provide one with a useful tool (or a toy model) for more general Hopf algebroid extensions. In particular, as announced in [9], cleft extensions for Hopf algebroids give a concrete illustration to the relative Chern–Galois theory. In fact the current paper can be considered as a sequel to [9] in which the ideas and results, announced in a few examples, are developed in detail and further extended. Specifically, in Section 5, sufficient conditions for the existence of (strong connection independent) relative Chern–Galois characters in Hopf algebroid cleft extensions are stated.

The construction of Hopf algebroid cleft extensions, although motivated by similar ideas, is significantly different from that of cleft Hopf algebra (or coalgebra) extensions. One should remember that a Hopf algebroid involves two different coring (and bialgebroid) structures on the same k-module. The interplay between these intricate structures is an immanent feature of Hopf algebroid extensions. This is already present in the notion of a convolution inverse (cf. Definition 3.1), which relates two coring structures on the same k-module, but is perhaps most significant in the characterisation of cleft extensions in terms of the Galois and normal basis properties (cf. Theorem 3.12): a cleft $\mathscr{H}$-extension is a Galois extension with respect to the *right* bialgebroid $\mathscr{H}_R$ but it has a normal basis property with respect to the *left* bialgebroid $\mathscr{H}_L$.

In the standard Hopf algebra theory, cleft extensions of Hopf algebras are examples of *crossed products* with Hopf algebras: indeed a cleft extension is the same as a crossed product with an invertible cocycle (cf. [20, Theorem 11], [6, Theorem 1.18]). Motivated by this correspondence, we also develop a general theory of crossed products with bialgebroids and Hopf algebroids. In particular this involves developing the notions of a *measuring* and a *2-cocycle*, while to relate different crossed products one needs to give meaning to *gauge transformations* and *equivalent crossed products*. In parallel to the bialgebra case, we show in Theorem 4.7 that two crossed products are equivalent if and only if one is a gauge transform of the other. We then identify cleft extensions of Hopf algebroids with crossed products with invertible cocycles (cf. Theorem 4.11 and Theorem 4.12). A generalisation of this theory to the case of *weak crossed products* is then outlined in Appendix.

Finally, we would like to indicate that the cleft extensions of the present paper can be placed in a broader context. A (weak) entwining structure (A, D, ψ) determines a coring extension D of the canonical A-coring $\mathscr{C}_\psi$ in the sense of [13]. The cleft property of an entwining structure can be formulated as a feature of A as an entwined module (i.e. a $\mathscr{C}_\psi$-comodule). Although it is not possible to find a cleft entwining structure behind a cleft extension $B \subseteq A$ of a Hopf algebroid $\mathscr{H}$ (with left L-bialgebroid $\mathscr{H}_L$ and right R-bialgebroid $\mathscr{H}_R$), there is still an associated coring extension. Namely, the constituent L-coring in $\mathscr{H}$ is a right extension of the A-coring $\mathscr{C} := A \otimes_R \mathscr{H}_R$, such that A is a $\mathscr{C}$-comodule. Inspired by this observation, a unified approach to all known notions of cleft extensions in terms of coring extensions is developed in [11].

Notation Throughout this paper we work over an associative unital commutative ring k. An algebra means an associative unital k-algebra. Unit elements are denoted by 1 and multiplications by μ (or by 1_R, μ_R if the algebra R needs to be specified). Categories of left, right, and bimodules for an algebra R are denoted by ${}_R\mathbf{M}$, $\mathbf{M}_R$ and ${}_R\mathbf{M}_R$, respectively. Their hom-sets are denoted by $\mathrm{Hom}_{R,-}(-,-)\mathrm{Hom}_{-,R}(-,-)$ and $\mathrm{Hom}_{R,R}(-,-)$, respectively.

Categories of left and right comodules for a coring $\mathscr{C}$ are denoted by ${}^{\mathscr{C}}\mathbf{M}$ and $\mathbf{M}^{\mathscr{C}}$, respectively. For their hom-sets we write $\mathrm{Hom}^{\mathscr{C},-}(-,-)$ and $\mathrm{Hom}^{-,\mathscr{C}}(-,-)$, respectively.

2 Preliminaries

2.1 Bialgebroids

A bialgebroid [24, 27] can be considered as a generalisation of the notion of a bialgebra to arbitrary (non-commutative) base algebras. A (left) bialgebroid over a base algebra L consists of an $L \otimes_k L^{op}$-ring structure (H, μ, η) and an L-coring structure (H, γ, π) on the same k-module H. Denoting the restriction of the unit map $\eta : L \otimes_k L^{op} \to H$ to$L \otimes_k 1_H$ (the so called *source map*) by s and its restriction to $1_H \otimes_k L^{op}$(the *target map*) by t, the bimodule structure of the L-coring is given by

$$lhl' = s(l)t(l')h, \quad \text{for all} \quad l, l' \in L, \; h \in H.$$

The range of the coproduct is required to be in the *Takeuchi product*

$$H \times_L H := \left\{ \sum_i h_i \underset{L}{\otimes} h'_i \in H \underset{L}{\otimes} H \;\middle|\; \sum_i h_i t(l) \underset{L}{\otimes} h'_i = \sum_i h_i \underset{L}{\otimes} h'_i s(l) \;\forall l \in L \right\},$$

which is, indeed, an algebra by factor wise multiplication. The following compatibility conditions are required between the $L \otimes_k L^{op}$-ring and the L-coring structures.

$$\gamma(1_H) = 1_H \underset{L}{\otimes} 1_H, \tag{2.1}$$

$$\gamma(hh') = \gamma(h)\gamma(h'), \tag{2.2}$$

$$\pi(1_H) = 1_L, \tag{2.3}$$

$$\pi\left(hs(\pi(h'))\right) = \pi(hh'), \tag{2.4}$$

$$\pi\left(ht(\pi(h'))\right) = \pi(hh'), \tag{2.5}$$

for all $h, h' \in H$.

The L-L bimodule structure of the coring underlying a left bialgebroid is defined in terms of the multiplication by $s(l)$ and $t(l)$ on the left. Right bialgebroids are defined analogously in terms of multiplications on the right. For more details we refer to [23].

Thus a bialgebroid is given by the following data: k-algebras H and L, and maps s (the source), t (the target), γ (the coproduct) and π (the counit). We write $\mathscr{L} = (H, L, s, t, \gamma, \pi)$.

Note that if $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ is a left bialgebroid, then so is the co-opposite $\mathscr{L}_{cop} = (H, L^{op}, t, s, \gamma^{op}, \pi)$, where L^{op} denotes the algebra that is isomorphic to L as a k-module, with multiplication opposite to the one in L, and $\gamma^{op} : H \to H \otimes_{L^{op}} H$, $h \mapsto h_{(2)} \otimes_{L^{op}} h_{(1)}$ is the coproduct, opposite to $\gamma : H \to H \otimes_L H$, $h \mapsto h_{(1)} \otimes_L h_{(2)}$. The opposite, $\mathscr{L}^{op} = (H^{op}, L, t, s, \gamma, \pi)$ is a right bialgebroid.

2.2 Hopf Algebroids

Hopf algebroids with bijective antipodes have been introduced in [10]. In [7] the definition was extended by relaxing the bijectivity of the antipode.

A Hopf algebroid consists of two (a left and a right) bialgebroid structures on the same total algebra. The source and target maps of the left bialgebroid $\mathscr{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$ and of the right bialgebroid $\mathscr{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$ are related by the following axioms.

$$\begin{aligned} &s_L \circ \pi_L \circ t_R = t_R, \qquad t_L \circ \pi_L \circ s_R = s_R \quad \text{and} \\ &s_R \circ \pi_R \circ t_L = t_L, \qquad t_R \circ \pi_R \circ s_L = s_L. \end{aligned} \tag{2.6}$$

These conditions imply that the left coproduct γ_L is R-R bilinear and the right coproduct γ_R is L-L bilinear. Each coproduct is required to be left and right colinear with respect to the other bialgebroid structure, i.e. the following axioms are imposed:

$$\left(\gamma_L \underset{R}{\otimes} H\right) \circ \gamma_R = \left(H \underset{L}{\otimes} \gamma_R\right) \circ \gamma_L \qquad \text{and} \qquad \left(\gamma_R \underset{L}{\otimes} H\right) \circ \gamma_L = \left(H \underset{R}{\otimes} \gamma_L\right) \circ \gamma_R. \tag{2.7}$$

An $R \otimes_k L$-$R \otimes_k L$ bilinear map $S : H \to H$, i.e. a k-linear map, such that

$$S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l) \quad \text{and} \quad S(t_R(r)ht_L(l)) = s_L(l)S(h)s_R(r), \tag{2.8}$$

for $r \in R, l \in L$ and $h \in H$, is called an *antipode* if

$$\mu_H \circ \left(S \underset{L}{\otimes} H\right) \circ \gamma_L = s_R \circ \pi_R \qquad \text{and} \qquad \mu_H \circ \left(H \underset{R}{\otimes} S\right) \circ \gamma_R = s_L \circ \pi_L. \tag{2.9}$$

For a Hopf algebroid we use the notation $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$.

Since in a Hopf algebroid there are two coring structures present, we use two versions of Sweedler's index notation for coproducts. For any $h \in H$, we write $\gamma_R(h) = h^{(1)} \otimes_R h^{(2)}$ (with upper indices) for the right coproduct and $\gamma_L(h) = h_{(1)} \otimes_L h_{(2)}$ (with lower indices) for the left coproduct.

Remark 2.1 In the formulation of Hopf algebroid axioms, given in Eqs. 2.6, 2.7, 2.8 and 2.9, the left bialgebroid $\mathscr{H}_L$ and the right bialgebroid $\mathscr{H}_R$ play symmetric roles. It turns out, however, that this set of axioms can slightly be reduced. Namely, the second equality in Eq. 2.8 can be derived from the other axioms. This can be seen by the following computation (and its symmetric version, in which the order of multiplication and roles of $\mathscr{H}_L$ and $\mathscr{H}_R$ are interchanged). For $l \in L$ and $h \in H$,

$$\begin{aligned} S(ht_L(l)) &= S\left(t_L\left(\pi_L\left(h_{(2)}\right)\right) h_{(1)} t_L(l)\right) = S\left(h_{(1)} t_L(l)\right) s_L\left(\pi_L\left(h_{(2)}\right)\right) \\ &= S\big(h_{(1)}\big)\, s_L\left(\pi_L\left(h_{(2)} s_L(l)\right)\right) = S\big(h_{(1)}\big)\, {h_{(2)}}^{(1)} s_L(l) S\Big({h_{(2)}}^{(2)}\Big) \\ &= S\big({h^{(1)}}_{(1)}\big)\, {h^{(1)}}_{(2)} s_L(l) S\big(h^{(2)}\big) = s_R\left(\pi_R\left(h^{(1)}\right)\right) s_L(l) S\big(h^{(2)}\big) \\ &= s_L(l) s_R\left(\pi_R\left(h^{(1)}\right)\right) S\big(h^{(2)}\big) = s_L(l) S\left(h^{(2)} t_R\left(\pi_R\left(h^{(1)}\right)\right)\right) \\ &= s_L(l) S(h). \end{aligned}$$

The first equality follows by the fact that π_L is counit of γ_L and the last one follows since π_R is counit of γ_R. The second and the penultimate equalities follow by the first equality in Eq. 2.8. The third equality follows by the bialgebroid axiom, requiring that the range of γ_L is in the Takeuchi product $H \times_L H$. The fourth equality follows by Eqs. 2.9 and 2.6, as the latter implies – together with the left R-linearity of γ_R – that $\gamma_R(hs_L(l)) = h^{(1)} s_L(l) \otimes_R h^{(2)}$. The fifth equality is a consequence of the right $\mathscr{H}_R$-colinearity of γ_L, i.e. Eq. 2.7. The sixth equality follows by Eq. 2.9, and the seventh one follows by Eq. 2.6, implying $s_L(l) s_R(r) = s_R(r) s_L(l)$, for $r \in R$ and $l \in L$.

It is proven in [7, Proposition 2.3] that the antipode of a Hopf algebroid is both an anti-multiplicative map, i.e. $S(hh') = S(h')S(h)$, for $h, h' \in H$, and an anti-comultiplicative map, i.e. $S(h)_{(1)} \otimes_L S(h)_{(2)} = S(h^{(2)}) \otimes_L S(h^{(1)})$ and $S(h)^{(1)} \otimes_R S(h)^{(2)} = S(h_{(2)}) \otimes_R S(h_{(1)})$, for $h \in H$, (note the appearance of left and right coproducts in both formulae). The maps

$$\pi_R \circ t_L : L^{op} \to R \qquad \text{and} \qquad \pi_L \circ s_R : R \to L^{op} \tag{2.10}$$

are inverse algebra isomorphisms.

Note that for a Hopf algebroid $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ also $\mathscr{H}^{op}_{cop} = ((\mathscr{H}_R)^{op}_{cop}, (\mathscr{H}_L)^{op}_{cop}, S)$ is a Hopf algebroid. If the antipode S is bijective, then also $\mathscr{H}_{cop} = ((\mathscr{H}_L)_{cop}, (\mathscr{H}_R)_{cop}, S^{-1})$ and $\mathscr{H}^{op} = ((\mathscr{H}_R)^{op}, (\mathscr{H}_L)^{op}, S^{-1})$ are Hopf algebroids.

Convention Throughout, whenever it is said 'Hopf algebroid $\mathscr{H}$,' it is meant a Hopf algebroid with all the structure modules and maps as in this section.

2.3 Comodule Algebras for Bialgebroids

Let $\mathscr{R} = (H, R, s, t, \gamma, \pi)$ be a right bialgebroid and M a *right $\mathscr{R}$-comodule*, that is, a right comodule for the R-coring (H, γ, π). This means [17, 18.1] that M is a right R-module and there exists a right R-linear coassociative and counital coaction, $\rho : M \to M \otimes_R H$, $m \mapsto m^{[0]} \otimes_R m^{[1]}$ (note the upper Sweedler indices indicating the involvement of a right bialgebroid). By the power of the bialgebroid structure, M can be equipped with a unique left R-action such that the range of ρ is in the Takeuchi product

$$M \times_R H := \left\{ \sum_i m_i \underset{R}{\otimes} h_i \in M \underset{R}{\otimes} H \;\middle|\; \sum_i r m_i \underset{R}{\otimes} h_i = \sum_i m_i \underset{R}{\otimes} t(r) h_i \; \forall\, r \in R \right\}. \quad (2.11)$$

The left R-multiplication in M takes the form since it commutes with the right R-action, M becomes an R-R bimodule in this way

$$rm = m^{[0]} \pi\left(t(r) m^{[1]}\right) \equiv m^{[0]} \pi\left(s(r) m^{[1]}\right), \qquad \text{for all } r \in R,\ m \in M. \quad (2.12)$$

One checks that any $\mathscr{R}$-colinear map is R-R bilinear. In particular the coaction satisfies

$$\rho\left(r m r'\right) = m^{[0]} \underset{R}{\otimes} s(r) m^{[1]} s(r'), \qquad \text{for all } r, r' \in R,\ m \in M. \quad (2.13)$$

The category of right $\mathscr{R}$-comodules is a monoidal category with a strict monoidal functor to the category ${}_R\mathbf{M}_R$ of R-R bimodules [25, Proposition 5.6]. The R-action and $\mathscr{R}$-coaction on the tensor product of two comodules M and N are given by

$$\left(m \underset{R}{\otimes} n\right) \cdot r = m \underset{R}{\otimes} nr, \quad \left(m \underset{R}{\otimes} n\right)^{[0]} \underset{R}{\otimes} \left(m \underset{R}{\otimes} n\right)^{[1]} = \left(m^{[0]} \underset{R}{\otimes} n^{[0]}\right) \underset{R}{\otimes} m^{[1]} n^{[1]}, \quad (2.14)$$

for all $r \in R$, $m \otimes_R n \in M \otimes_R N$.

A right *$\mathscr{R}$-comodule algebra* is a monoid in the monoidal category of right $\mathscr{R}$-comodules; hence, in particular, it is an R-ring.

The R-coring (H, γ, π), underlying a right R-bialgebroid $\mathscr{R}$, possesses a grouplike element 1_H. The *coinvariants* of a right $\mathscr{R}$-comodule M with respect to the grouplike element 1_H are the elements of

$$M^{co\mathscr{R}} = \left\{ m \in M \;\middle|\; m^{[0]} \underset{R}{\otimes} m^{[1]} = m \underset{R}{\otimes} 1_H \right\}.$$

It is straightforward to check that if A is a right $\mathscr{R}$-comodule algebra, then its coinvariants form a subalgebra $B := A^{co\mathscr{R}}$. In this case the algebra extension $B \subseteq A$ is termed a right *$\mathscr{R}$-extension*.

A right $\mathscr{R}$-extension $B \subseteq A$ is a right *$\mathscr{R}$-Galois* extension if the canonical map

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H, \qquad a \underset{B}{\otimes} a' \mapsto a a'^{[0]} \underset{R}{\otimes} a'^{[1]}, \quad (2.15)$$

is bijective, i.e. the A-coring $A \otimes_R H$ with the coproduct $A \otimes_R \gamma$, the counit $A \otimes_R \pi$ and with the A-A bimodule structure $a_1 (a \otimes_R h) a_2 = a_1 a a_2{}^{[0]} \otimes_R h a_2{}^{[1]}$, is a Galois coring [21].

Analogously, a right comodule N, with coaction $n \mapsto n_{[0]} \otimes_L n_{[1]}$ (note the lower Sweedler indices indicating the involvement of a left bialgebroid), for the L-coring

(H, γ, π), underlying a left bialgebroid $\mathscr{L} = (H, L, s, t, \gamma, \pi)$, can be equipped with a left L-action

$$ln = n_{[0]}\pi\left(n_{[1]}s(l)\right) \equiv n_{[0]}\pi\left(n_{[1]}t(l)\right), \qquad \text{for all } l \in L, n \in N. \tag{2.16}$$

The category of right $\mathscr{L}$-comodules is a monoidal category with monoidal product, the module tensor product over L^{op}. The right L-action and $\mathscr{L}$-coaction on the tensor product of two $\mathscr{L}$-comodules M and N are

$$\left(m \underset{L^{op}}{\otimes} n\right) \cdot l = ml \underset{L^{op}}{\otimes} n, \qquad \left(m \underset{L^{op}}{\otimes} n\right)_{[0]} \underset{L}{\otimes} \left(m \underset{L^{op}}{\otimes} n\right)_{[1]} = \left(m_{[0]} \underset{L^{op}}{\otimes} n_{[0]}\right) \underset{L}{\otimes} m_{[1]} n_{[1]},$$

for all $l \in L, m \otimes_L n \in M \otimes_L N$. Right *$\mathscr{L}$-comodule algebras* are defined as monoids in the monoidal category of right $\mathscr{L}$-comodules – hence they are, in particular, L^{op}-rings. *Coinvariants* are defined with respect to the grouplike element 1_H. An algebra extension $B \subseteq A$ is called a right *$\mathscr{L}$-extension* if A is a right $\mathscr{L}$-comodule algebra and $B = A^{co\mathscr{L}}$. A right $\mathscr{L}$-extension $B \subseteq A$ is said to be right *$\mathscr{L}$-Galois* if the canonical map

$$\mathrm{can}_L : A \underset{B}{\otimes} A \to A \underset{L}{\otimes} H, \qquad a \underset{B}{\otimes} a' \mapsto a_{[0]} a' \underset{L}{\otimes} a_{[1]}, \tag{2.17}$$

is bijective.

Right comodules for a left bialgebroid $\mathscr{L}$ are canonically identified with left comodules for the co-opposite bialgebroid $\mathscr{L}_{cop}$, thus resulting in a monoidal equivalence ${}^{\mathscr{L}_{cop}}\mathbf{M} \simeq \mathbf{M}^{\mathscr{L}}$. This identification leads to analogous notions of left comodule algebras, left $\mathscr{L}$-extensions and left $\mathscr{L}$-Galois extensions.

2.4 Comodule Algebras for Hopf Algebroids

By the Hopf algebroid axioms (2.7), the total algebra H of a Hopf algebroid $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ is both an $\mathscr{H}_L$-$\mathscr{H}_R$ bicomodule with coactions provided by the coproducts γ_L and γ_R, and an $\mathscr{H}_R$-$\mathscr{H}_L$ bicomodule with coactions γ_R and γ_L. That is, in the terminology of [13], the coring underlying $\mathscr{H}_R$ is a *right extension* of the coring underlying $\mathscr{H}_L$, and vice versa. By [13, Theorem 2.6], this implies that there exist unique k-additive functors $U : \mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}^{\mathscr{H}_L}$ and $V : \mathbf{M}^{\mathscr{H}_L} \to \mathbf{M}^{\mathscr{H}_R}$, such that the forgetful functors $F_R : \mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}_k$ and $F_L : \mathbf{M}^{\mathscr{H}_L} \to \mathbf{M}_k$ satisfy

$$F_R = F_L \circ U \qquad \text{and} \qquad F_L = F_R \circ V,$$

(meaning that $U(M)$ or $V(M)$ are equal to M as k-modules). Explicit forms of functors between comodule categories corresponding to coring extensions can be found in [13]. The application to the present situation gives that the functor U maps a right $\mathscr{H}_R$-comodule M, with coaction $m \mapsto m^{[0]} \otimes_R m^{[1]}$, to the right L-module

$$ml = m^{[0]}\pi_R\left(t_L(l) m^{[1]}\right) \equiv \pi_R(t_L(l))m, \qquad \text{for all } l \in L,\ m \in M, \tag{2.18}$$

with right $\mathscr{H}_L$-coaction

$$m \mapsto m^{[0]}\pi_R\left(m^{[1]}{}_{(1)}\right) \underset{L}{\otimes} m^{[1]}{}_{(2)}, \qquad \text{for all } m \in M. \tag{2.19}$$

On sets of comodule maps U acts as the identity function.

Analogously, the functor V maps a right $\mathscr{H}_L$-comodule N, with coaction $n \mapsto n_{[0]} \otimes_L n_{[1]}$, to the right R-module

$$nr = n_{[0]}\pi_L\left(n_{[1]}s_R(r)\right) \equiv \pi_L(s_R(r))n, \qquad \text{for all } r \in R,\ n \in N, \tag{2.20}$$

with right $\mathscr{H}_R$-coaction

$$n \mapsto n_{[0]}\pi_L({n_{[1]}}^{(1)}) \underset{R}{\otimes} {n_{[1]}}^{(2)}, \qquad \text{for all } n \in N. \tag{2.21}$$

On sets of comodule maps V acts as the identity function.

Theorem 2.2 *Let $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ be a Hopf algebroid. The functors $U : \mathbf{M}^{\mathscr{H}_R} \to \mathbf{M}^{\mathscr{H}_L}$ in Eq. 2.18 and (2.19), and $V : \mathbf{M}^{\mathscr{H}_L} \to \mathbf{M}^{\mathscr{H}_R}$ in Eq. 2.20 and (2.21) are strict monoidal inverse isomorphisms.*

Proof In order to see that U is strict monoidal we compute the $\mathscr{H}_L$-comodule structure on $U(M \otimes_R N)$, for two $\mathscr{H}_R$-comodules M and N, with respective coactions $m \mapsto m^{[0]} \otimes_R m^{[1]}$ and $n \mapsto n^{[0]} \otimes_R n^{[1]}$.

The right L-module $U(M \otimes_R N)$ can be identified with $U(M) \otimes_{L^{op}} U(N)$ via the algebra isomorphism (2.10). In view of Eq. 2.18, the right L-action on this module is given by

$$\left(m \underset{R}{\otimes} n\right) l = \pi_R(t_L(l)) \left(m \underset{R}{\otimes} n\right) = \pi_R(t_L(l)) m \underset{R}{\otimes} n = ml \underset{R}{\otimes} n,$$

for all $l \in L$ and $m \otimes_R n \in M \otimes_R N$. By (2.19), the right $\mathscr{H}_L$-coaction on $U(M \otimes_R N)$ maps an element $m \otimes_R n$ to

$$\left(m \underset{R}{\otimes} n\right)^{[0]} \pi_R\left(\left(m \underset{R}{\otimes} n\right)^{[1]}{}_{(1)}\right) \underset{L}{\otimes} \left(m \underset{R}{\otimes} n\right)^{[1]}{}_{(2)}. \tag{2.22}$$

In light of the form (2.14) of the $\mathscr{H}_R$-coaction on $M \otimes_R N$ and the bialgebroid axiom (2.2), the expression (2.22) is equal to

$$\left(m^{[0]} \underset{R}{\otimes} n^{[0]} \pi_R\left(m^{[1]}{}_{(1)} n^{[1]}{}_{(1)}\right)\right) \underset{L}{\otimes} m^{[1]}{}_{(2)} n^{[1]}{}_{(2)}.$$

By the right bialgebroid analogue of axiom (2.5) and the fact that by construction the range of the $\mathscr{H}_R$-coaction on N is in the Takeuchi product (cf. Eq. 2.11), the expression (2.22) is equal to

$$\begin{aligned}&\left(m^{[0]} \underset{R}{\otimes} \pi_R\left(m^{[1]}{}_{(1)}\right) n^{[0]} \pi_R\left(n^{[1]}{}_{(1)}\right)\right) \underset{L}{\otimes} m^{[1]}{}_{(2)} n^{[1]}{}_{(2)} \\ &\quad = \left(m^{[0]} \pi_R\left(m^{[1]}{}_{(1)}\right) \underset{R}{\otimes} n^{[0]} \pi_R\left(n^{[1]}{}_{(1)}\right)\right) \underset{L}{\otimes} m^{[1]}{}_{(2)} n^{[1]}{}_{(2)}.\end{aligned} \tag{2.23}$$

Using the canonical isomorphism of right L-modules, $U(M \otimes_R N) \simeq U(M) \otimes_{L^{op}} U(N)$, and comparing Eq. 2.23 with (2.19), we conclude that the $\mathscr{H}_L$-coactions on $U(M \otimes_R N)$ and on $U(M) \otimes_{L^{op}} U(N)$ are equal to each other, hence U is strict monoidal. Let M be a right $\mathscr{H}_R$-comodule with coaction $m \mapsto m^{[0]} \otimes_R m^{[1]}$. Using the forms (2.16) of the left L-action on the right $\mathscr{H}_L$-comodule $U(M)$ and Eq. 2.18

of the right L-action on the right $\mathscr{H}_R$-comodule M, the left L-action on $U(M)$ is computed to be

$$L\underset{k}{\otimes}M \to M, \qquad l\underset{k}{\otimes}m \mapsto lm = m\pi_R(t_L(l)). \tag{2.24}$$

Hence applying Eq. 2.20, we can relate the right R-action on $V\big(U(M)\big)$ to the left L-action on $U(M)$, and applying (2.24), also to the right R-action on M. After these steps we arrive at the right R-action on $V\big(U(M)\big)$,

$$M\underset{k}{\otimes}R \to M, \qquad m\underset{k}{\otimes}r \mapsto \pi_L(s_R(r))m = m\pi_R\big(t_L\big(\pi_L(s_R(r))\big)\big) = mr,$$

which is equal to the right R-action on M. The last equality of the computation follows by the Hopf algebroid axiom (2.6).

The right $\mathscr{H}_R$-coaction on $V\big(U(M)\big)$ is computed using (2.21), Eq. 2.18 and (2.19). It maps $m \in M$ to

$$\pi_R\left(t_L\left(\pi_L\left({m^{[1]}}_{(2)}{}^{(1)}\right)\right)\right) m^{[0]}\pi_R\left({m^{[1]}}_{(1)}\right) \underset{R}{\otimes} {m^{[1]}}_{(2)}{}^{(2)}. \tag{2.25}$$

Using the Hopf algebroid axiom (2.7) and the form (2.12) of the left R-action on M, the expression (2.25) can be simplified to

$$m^{[0]}\pi_R\left(t_L\left(\pi_L\left({m^{[1]}}_{(2)}\right)\right){m^{[1]}}_{(1)}{}^{(1)}s_R\left(\pi_R\left({m^{[1]}}_{(1)}{}^{(2)}\right)\right)\right)\underset{R}{\otimes}m^{[2]} = m^{[0]}\underset{R}{\otimes}m^{[1]},$$

where in the last step the counit property of π_R, the counit property of π_L, and the counitality of the $\mathscr{H}_R$-coaction on M have been used. This proves that $V \circ U$ is the identity functor on $\mathbf{M}^{\mathscr{H}_R}$. The identity $U \circ V = \mathbf{M}^{\mathscr{H}_L}$ is proven in a symmetric way. □

The content of Theorem 2.2 and [13, Theorem 2.6] can be summarised in the following commutative diagram of functors.

$$\begin{tikzcd}
\mathbf{M}^{\mathscr{H}_L} \arrow[rrrr, shift left, "V"] \arrow[dr] & & & & \mathbf{M}^{\mathscr{H}_R} \arrow[llll, shift left, "U"] \arrow[dl] \\
& {}_{L^{op}}\mathbf{M}_{L^{op}} \arrow[rr, shift left, "v"] \arrow[dr] & & {}_R\mathbf{M}_R \arrow[ll, shift left, "u"] \arrow[dl] & \\
& & \mathbf{M}_k & &
\end{tikzcd}$$

The unlabeled functors are forgetful functors and u and v are restrictions of scalars functors along the algebra isomorphisms (2.10). The functors U, V, u, v and the forgetful functors $\mathbf{M}^{\mathscr{H}_R} \to {}_R\mathbf{M}_R$ and $\mathbf{M}^{\mathscr{H}_L} \to {}_{L^{op}}\mathbf{M}_{L^{op}}$ are strict monoidal.

In view of Theorem 2.2, we do not distinguish between $\mathscr{H}_R$- and $\mathscr{H}_L$-comodules in the case of a Hopf algebroid $\mathscr{H}$: we call them simply $\mathscr{H}$-comodules. For the $\mathscr{H}_R$-coaction on a right $\mathscr{H}$-comodule M we use Sweedler type index notation $\rho : m \mapsto m^{[0]} \otimes_R m^{[1]}$ (with upper indices) and for the corresponding $\mathscr{H}_L$-coaction we write $\lambda : m \mapsto m_{[0]} \otimes_L m_{[1]}$ (with lower indices), for $m \in M$. Also, by an $\mathscr{H}$-colinear map we mean an $\mathscr{H}_R$- or, equivalently, an $\mathscr{H}_L$-colinear map. Note that in particular a right

$\mathcal{H}_R$-coaction, being $\mathcal{H}_R$-colinear (i.e. coassociative), is also $\mathcal{H}_L$-colinear, and a right $\mathcal{H}_L$-coaction is $\mathcal{H}_R$-colinear.

By the strict monoidality of the functors U and V, a right $\mathcal{H}_R$-comodule algebra A is also a right $\mathcal{H}_L$-comodule algebra, and vice versa. Hence we can call A an $\mathcal{H}$-comodule algebra. It is, in particular, an R-ring and an L^{op}-ring. By Eq. 2.18, the unit of the L^{op}-ring A can be expressed in terms of the unit $\eta_R : R \to A$ of the R-ring as

$$L^{op} \to A, \qquad l \mapsto \eta_R \circ \pi_R \circ t_L.$$

Using explicit forms (2.19) and (2.21) of functors U and V, it is easy to see that the $\mathcal{H}_R$-coinvariants and the $\mathcal{H}_L$-coinvariants of an $\mathcal{H}$-comodule are the same. In particular, an algebra extension $B \subseteq A$ is a right $\mathcal{H}_R$-extension if and only if it is a right $\mathcal{H}_L$-extension. The notions of right $\mathcal{H}_R$-Galois extensions and of right $\mathcal{H}_L$-Galois extensions are known to coincide, however, only if the antipode of the Hopf algebroid $\mathcal{H}$ is bijective [8, Lemma 3.3].

By applying the same arguments to the Hopf algebroid $\mathcal{H}^{op}_{cop}$ and using the identification of right comodules for a bialgebroid and left comodules for its co-opposite, one derives analogous results for left comodules.

With the help of the antipode, to any left $\mathcal{H}$-comodule M with $\mathcal{H}_R$-coaction $m \mapsto m^{[-1]} \otimes_R m^{[0]}$, one associates a right $\mathcal{H}$-comodule M with $\mathcal{H}_L$-coaction

$$M \to M \underset{L}{\otimes} H, \qquad m \mapsto m^{[0]} \underset{L}{\otimes} S\left(m^{[-1]}\right), \tag{2.26}$$

where the L-module structures are defined via the algebra isomorphisms (2.10). If the antipode is bijective, then this results in an anti-monoidal isomorphism ${}^{\mathcal{H}_R}\mathbf{M} \simeq \mathbf{M}^{(\mathcal{H}_R)_{cop}} \to \mathbf{M}^{\mathcal{H}_L}$.

3 $\mathcal{H}$-Cleft Extensions

Recall that to an L-ring A (with multiplication $\mu : A \otimes_L A \to A$ and unit map $\eta : L \to A$) and an L-coring H (with comultiplication $\gamma : H \to H \otimes_L H$ and counit $\pi : H \to L$), one associates a *convolution algebra* $\mathrm{Hom}_{L,L}(H, A)$, with multiplication $j \diamond j' : = \mu \circ (j \otimes_L j') \circ \gamma$ and unit $\eta \circ \pi$. The first aim of this section is to develop a generalisation of the notion of a convolution algebra and, in particular, of a convolution inverse suitable for Hopf algebroids.This will make it possible to interpret in particular the antipode of a Hopf algebroid as the convolution inverse of the identity map.

As explained in Section 2.2, a Hopf algebroid is built on a k-module with two coring structures. Although we are primarily interested in Hopf algebroids, in general there is no need to put any special restrictions on these coring structures. Dually, one can consider a k-module with ring structures over two different rings. In this more general situation the convolution algebra (which is simply a k-linear category with a single object) can be generalised to a Morita context (i.e. a k-linear category with two objects). The notion of a *convolution inverse* is introduced within this *convolution category*.

Let L and R be k-algebras and let H and A be k-modules. Assume that A is an L-ring (with multiplication $\mu_L : A \otimes_L A \to A$ and unit $\eta_L : L \to A$) and an R-ring (with multiplication $\mu_R : A \otimes_R A \to A$ and unit $\eta_R : R \to A$). Assume that A is an

L-R and R-L bimodule with respect to the corresponding module structures, μ_L is R-R bilinear and μ_R is L-L bilinear, and that

$$\mu_L \circ \left(A \underset{L}{\otimes} \mu_R\right) = \mu_R \circ \left(\mu_L \underset{R}{\otimes} A\right), \qquad \mu_R \circ \left(A \underset{R}{\otimes} \mu_L\right) = \mu_L \circ \left(\mu_R \underset{L}{\otimes} A\right). \tag{3.1}$$

Dually, assume that H is an L-coring (with comultiplication $\gamma_L : H \to H \otimes_L H$ and counit $\pi_L : H \to L$) and an R-coring (with comultiplication $\gamma_R : H \to H \otimes_R H$ and counit $\pi_R : H \to R$). Assume further that H is an L-R and R-L bimodule with respect to the corresponding module structures, such that γ_L is R-R bilinear, γ_R is L-L bilinear and

$$\left(H \underset{L}{\otimes} \gamma_R\right) \circ \gamma_L = \left(\gamma_L \underset{R}{\otimes} H\right) \circ \gamma_R, \qquad \left(H \underset{R}{\otimes} \gamma_L\right) \circ \gamma_R = \left(\gamma_R \underset{L}{\otimes} H\right) \circ \gamma_L. \tag{3.2}$$

To the above data one associates a k-linear *convolution category* $\mathsf{Conv}(H, A)$ as follows. $\mathsf{Conv}(H, A)$ has two objects, R and L, and morphisms

$$\mathsf{Conv}(H, A)(P, Q) = \mathrm{Hom}_{Q,P}(H, A), \qquad P, Q \in \{L, R\},$$

with composition $\diamond$, defined for all $\phi \in \mathrm{Hom}_{P,Q}(H, A)$ and $\psi \in \mathrm{Hom}_{Q,S}(H, A)$, $P, Q, S \in \{L, R\}$,

$$\phi \diamond \psi = \mu_Q \circ \left(\phi \underset{Q}{\otimes} \psi\right) \circ \gamma_Q \in \mathrm{Hom}_{P,S}(H, A).$$

Note that the identity morphism in $\mathsf{Conv}(H, A)(P, P)$ is $\eta_P \circ \pi_P$. The conditions (3.1) and (3.2) together with coassociativity of the coproducts in H and associativity of products in A ensure that the composition $\diamond$ is an associative operation.

Definition 3.1 Let $\mathsf{Conv}(H, A)$ be a convolution category and let j be a morphism in $\mathsf{Conv}(H, A)$. A retraction of j in $\mathsf{Conv}(H, A)$ is called a *left convolution inverse of j* and a section of j in $\mathsf{Conv}(H, A)$ is called a *right convolution inverse of j*. If j is an isomorphism in $\mathsf{Conv}(H, A)$, then it is said to be *convolution invertible*; its inverse is called the *convolution inverse of j* and is denoted by j^{c}.

Remark 3.2

(1) A k-linear category with a single object a can be identified with the k-algebra $\mathrm{End}(a)$ of the morphisms in the category. In a similar manner, a k-linear category with two objects a and b can be identified with a Morita context as follows. The composition of morphisms makes k-modules $\mathrm{Hom}(a, b)$ and $\mathrm{Hom}(b, a)$ bimodules for k-algebras $\mathrm{End}(a)$ and $\mathrm{End}(b)$. Furthermore, the restriction of the composition to the map $\mathrm{Hom}(a, b) \underset{k}{\otimes} \mathrm{Hom}(b, a) \to \mathrm{End}(b)$ is an $\mathrm{End}(a)$-balanced $\mathrm{End}(b)$-bimodule map. That is, it is a composite of the canonical epimorphism $\mathrm{Hom}(a, b) \otimes_k \mathrm{Hom}(b, a) \to \mathrm{Hom}(a, b) \otimes_{\mathrm{End}(a)} \mathrm{Hom}$

(b, a), and an $\mathrm{End}(b)$-bimodule map, $F_a : \mathrm{Hom}(a, b) \otimes_{\mathrm{End}(a)} \mathrm{Hom}(b, a) \to \mathrm{End}(b)$. Similarly, the map $\mathrm{Hom}(b, a) \underset{k}{\otimes} \mathrm{Hom}(a, b) \to \mathrm{End}(a)$, obtained by restricting the composition, factors through the canonical epimorphism and the $\mathrm{End}(a)$-bimodule map, $F_b : \mathrm{Hom}(b, a) \otimes_{\mathrm{End}(b)} \mathrm{Hom}(a, b) \to \mathrm{End}(a)$. Using the associativity of the composition of the morphisms in a category, one easily checks that the 6-tuple $(\mathrm{End}(a), \mathrm{End}(b), \mathrm{Hom}(b, a), \mathrm{Hom}(a, b), F_a, F_b)$ is a Morita context. Clearly, there is a category of this kind behind any Morita context.

In particular, the convolution category $\mathsf{Conv}(H, A)$ can be identified with a Morita context connecting convolution algebras $\mathrm{Hom}_{L,L}(H, A)$ and $\mathrm{Hom}_{R,R}(H, A)$.

(2) In the case $L = R, \gamma_L = \gamma_R, \pi_L = \pi_R, \mu_R = \mu_L, \eta_R = \eta_L$, i.e. when there is one, say, L-coring H and one, say, L-ring A, the convolution category $\mathsf{Conv}(H, A)$ consists of a single object. The algebras in the corresponding Morita context are both equal to the convolution algebra $\mathrm{Hom}_{L,L}(H, A)$, the bimodules are the regular bimodules and the connecting homomorphisms are both equal to the identity map of $\mathrm{Hom}_{L,L}(H, A)$. In a word: the Morita context reduces to the convolution algebra. Thus an L-L bimodule map j is convolution invertible in the sense of Definition 3.1 if and only if it is an invertible element of the convolution algebra $\mathrm{Hom}_{L,L}(H, A)$.

(3) Conditions 3.1, imposed on the R-ring and L-ring structures of A, imply that the underlying k-algebras are isomorphic via the map $A \ni a \mapsto \mu_R(a \otimes_R \eta_L(1_L))$, with the inverse $a \mapsto \mu_L(a \otimes_L \eta_R(1_R))$.

(4) Conditions 3.2, imposed on the two coring structures of H, imply that the L-coring H is a left (and right) extension of the R-coring H, while the R-coring H is a right (and left) extension of the L-coring H, with the coactions given by the coproducts, in the sense of [13].

We can now exemplify the contents of Definition 3.1 with the main case of interest, whereby the coring structures on H constitute bialgebroids. Consider a right bialgebroid $\mathscr{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$ and a left bialgebroid $\mathscr{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$ on the same total algebra H, which satisfy conditions (2.6) and (2.7). In this situation, compatibility conditions for coring structures on H in the definition of a convolution category, including Eq. 3.2, are satisfied. For a target of convolution invertible maps take an $R \otimes_k L$-ring A. In this case the unit maps η_R and η_L are obtained as the restrictions of the unit map $R \otimes_k L \to A$ to $R \otimes_k 1_L$ and to $1_R \otimes_k L$, respectively. There is no need to distinguish between the products of A as an R-ring and as an L-ring, so we write simply μ_A for the product in A, and it becomes clear from the context, how this should be understood. Since we are dealing with a single product, it makes sense to denote the action of μ_A on elements by juxtaposition. One immediately checks that all the compatibility conditions between the L-, R-ring structures on A in the definition of a convolution category are satisfied, in particular (3.1) follow by the associativity of μ_A. All this means that, for two bialgebroids $\mathscr{H}_L$ and $\mathscr{H}_R$ on the same total algebra H, such that Eqs. 2.6 and 2.7 hold, and an $R \otimes_k L$-ring A, there is a convolution category $\mathsf{Conv}(H, A)$. We can now make explicit the contents of Definition 3.1 in this case. This essentially means describing explicitly all the L-, and R-actions involved.

For left and right bialgebroids $\mathscr{H}_L$ and $\mathscr{H}_R$ on the same total algebra H, such that Eqs. 2.6 and 2.7 hold, and an $R\otimes_k L$-ring A, a map $j : H \to A$ is an L-R bimodule map provided

$$j(s_L(l)\, h\, s_R(r)) = \eta_L(l)\; j(h)\; \eta_R(r), \quad \text{for all} \quad l \in L, r \in R,\; h \in H. \tag{3.3}$$

Similarly, $\tilde{j} : H \to A$ is an R-L bimodule map if

$$\tilde{j}(t_L(l)\, h\, t_R(r)) = \eta_R(r)\, \tilde{j}(h) \eta_L(l), \quad \text{for all} \quad l \in L, r \in R,\; h \in H. \tag{3.4}$$

A right convolution inverse of $j \in \mathrm{Hom}_{L,R}(H, A)$ is a map $\tilde{j} \in \mathrm{Hom}_{R,L}(H, A)$ such that

$$\mu_A \circ \left(j \underset{R}{\otimes} \tilde{j} \right) \circ \gamma_R = \eta_L \circ \pi_L. \tag{3.5}$$

A left convolution inverse of j is a map $\hat{j} \in \mathrm{Hom}_{R,L}(H, A)$ such that

$$\mu_A \circ \left(\hat{j} \underset{L}{\otimes} j \right) \circ \gamma_L = \eta_R \circ \pi_R. \tag{3.6}$$

Obviously, by the associativity of the composition in $\mathsf{Conv}(H, A)$, if a map $j : H \to A$ satisfying Eq. 3.3 has both left and right convolution inverses, then they coincide and hence the convolution inverse of an L-R bimodule map j is unique.

Example 3.3 Let $\mathscr{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$ be a left bialgebroid and $\mathscr{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$ be a right bialgebroid, on the same total algebra H. Assume that the compatibility conditions (2.6) and (2.7) hold. Consider the $R \otimes_k L$-ring structure on H, defined by the unit map $R \otimes_k L \to H$, $r \otimes_k l \mapsto s_R(r) s_L(l) \equiv s_L(l) s_R(r)$. It gives rise to a convolution category $\mathsf{Conv}(H, H)$. In light of Eq. 3.3, the identity map of H is an element of $\mathsf{Conv}(H, H)(R, L)$. By Eqs. 3.4, 3.5 and 3.6, the identity map possesses a convolution inverse S if and only if the first equality in Eqs. 2.8 and 2.9 hold true. Hence it follows by Remark 2.1 that $(\mathscr{H}_L, \mathscr{H}_R, S)$ is a Hopf algebroid if and only if S is convolution inverse of the identity map in $\mathsf{Conv}(H, H)$ (in the same way as the antipode of a Hopf algebra H over a ring k is the inverse of the identity map in the convolution algebra $\mathrm{End}_k(H)$).

Example 3.4 Example 3.3 can be extended as follows. Take a Hopf algebroid $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ and a left $\mathscr{H}_L$-module algebra B. (The role of B is played by the base algebra L in Example 3.3.) The smash product algebra $A := B \rtimes H$ is defined as the k-module $B \otimes_L H$ with product

$$(b \rtimes h) \left(b' \rtimes h'\right) := b \left(h_{(1)} \cdot b'\right) \rtimes h_{(2)} h',$$

(cf. [23, Section 2.3]). Here, the left L-module structure on H is given by the multiplication by $s_L(l)$ on the left. A is an R-ring with $\eta_R(r) = 1_B \rtimes s_R(r)$ (and hence an L^{op}-ring with unit $l \mapsto 1_B \rtimes t_L(l)$) and an L-ring with unit $\eta_L(l) = 1_B \rtimes s_L(l)$. Since the elements $\eta_R(r)$ and $\eta_L(l)$ commute in A, for any $r \in R$ and $l \in L$, A is an $R \otimes_k L$-ring.

The L-R bimodule map $j : H \to A, h \mapsto 1_B \rtimes h$ is convolution invertible with the inverse $j^{\mathsf{c}} : H \to A, h \mapsto 1_B \rtimes S(h)$ (cf. Eq. 2.9).

The notion of a convolution inverse, once established, plays the fundamental role in the definition of a cleft extension of a Hopf algebroid, which we describe presently.

Let $\mathscr{H}$ be a Hopf algebroid and A a right $\mathscr{H}$-comodule algebra. Then A is, in particular, an R-ring. The unit of this R-ring, the algebra homomorphism $R \to A$, is denoted by η_R. The coinciding k-subalgebra of $\mathscr{H}_R$- and of $\mathscr{H}_L$-coinvariants in A is denoted by B.

Assume that A is also an L-ring, with unit $\eta_L : L \to A$, and B is an L-subring of A. The latter implies that both the $\mathscr{H}_R$-coaction ρ^A, and the $\mathscr{H}_L$-coaction λ^A are left L-linear. Since ρ^A is R-R bilinear (cf. Eq. 2.13),

$$\rho^A\left(b\,\eta_R(r)\right) = b \underset{R}{\otimes} s_R(r) = \rho^A\left(\eta_R(r)b\right), \qquad \text{for all } r \in R,\ b \in B.$$

Thus it follows that B is in the commutant of the image of η_R.

Recall from Section 2.3 that any right $\mathscr{H}$-colinear map $j : H \to A$ is right R-linear in the sense of Eq. 3.3 and left R-linear in the sense that

$$j(s_R(r)h) = \eta_R(r)\, j(h), \tag{3.7}$$

for all $r \in R$ and $h \in H$ (cf. Eq. 2.12).

Definition 3.5 Let $\mathscr{H}$ be a Hopf algebroid and A a right $\mathscr{H}$-comodule algebra. Denote by $\eta_R(r) = r \cdot 1_A = 1_A \cdot r$ the unit map of the corresponding R-ring structure of A. Let B be the subalgebra of $\mathscr{H}$-coinvariants in A. The extension $B \subseteq A$ is called *$\mathscr{H}$-cleft* if

(a) A is an L-ring (with unit $\eta_L : L \to A$) and B is an L-subring of A;
(b) There exists a convolution invertible left L-linear right $\mathscr{H}$-colinear morphism $j : H \to A$.

A map j, satisfying condition (b), is called a *cleaving map*.

Condition (b) in Definition 3.5 means, in particular, that a cleaving map is L-R bilinear in the sense of Eq. 3.3.

Example 3.6 Consider a smash product algebra $A = B \rtimes H$ of Example 3.4. Similarly to [8, Example 3.7], A is a right $\mathscr{H}$-comodule algebra with $\mathscr{H}_R$-coaction $B \otimes_L \gamma_R$. The subalgebra of $\mathscr{H}$-coinvariants in A is $B \rtimes 1_H$. It is an L-subring of A. Since the convolution invertible map $j : H \to A$, $h \mapsto 1_B \rtimes h$ in Example 3.4 is right $\mathscr{H}$-colinear, $B \subseteq A$ is an $\mathscr{H}$-cleft extension.

In particular, let $N \subseteq M$ be a depth 2 (or D2, for short) extension of algebras [23, Definition 3.1]. It has been proven in [23, Theorem 4.1] that the algebra $\mathrm{End}_{N,N}(M)$ of N-N bilinear endomorphisms of M is a left bialgebroid and M is its left module algebra. By [23, Corollary 4.5], the algebra $\mathrm{End}_{-N}(M)$ of right N-linear endomorphisms of M, with multiplication given by composition, is isomorphic to the smash product algebra $M \rtimes \mathrm{End}_{N,N}(M)$.

If the D2 extension $N \subseteq M$ is also a Frobenius extension, then $\mathrm{End}_{N,N}(M)$ is a Hopf algebroid. Hence we can conclude that for any D2 Frobenius extension $N \subseteq M$, the extension $M \subseteq \mathrm{End}_{-N}(M)$ (where the inclusion is given by the left multiplication) is a cleft extension of the Hopf algebroid $\mathrm{End}_{N,N}(M)$.

Lemma 3.7 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension, with a cleaving map j. Then*

$$j^{\mathsf{c}}(t_R(r)\,h) = j^{\mathsf{c}}(h)\,\eta_R(r), \qquad \textit{for all } \; r \in R, \; h \in H. \tag{3.8}$$

Proof Use the counit property of π_L (in the first equality), right L-linearity of j^{c}, i.e. Eq. 3.4 (in the second one), the fact that, since B is an L-subring of A, the images of η_L and of η_R commute in A (in the third one), the assumption that j^{c} is right convolution inverse of j, i.e. Eq. 3.5 (in the fourth one), the left R linearity of j, i.e. Eq. 3.7 (in the fifth one), axiom (2.7) (in the sixth one), the identity $\gamma_L(s_R(r)h) = h_{(1)} \otimes_L s_R(r)h_{(2)}$, for $h \in H$ and $r \in R$, and the assumption that j^{c} is a left convolution inverse of j, i.e. Eq. 3.6 (in the seventh one), the left R-linearity of j^{c}, i.e. Eq. 3.4 (in the penultimate one) and the counit property of π_R (in the last one) to compute

$$\begin{aligned}
j^{\mathsf{c}}(h)\eta_R(r) &= j^{\mathsf{c}}\left(t_L\left(\pi_L\left(h_{(2)}\right)\right)h_{(1)}\right)\eta_R(r) = j^{\mathsf{c}}\left(h_{(1)}\right)\eta_L\left(\pi_L\left(h_{(2)}\right)\right)\eta_R(r) \\
&= j^{\mathsf{c}}\left(h_{(1)}\right)\eta_R(r)\eta_L\left(\pi_L\left(h_{(2)}\right)\right) = j^{\mathsf{c}}\left(h_{(1)}\right)\eta_R(r)\,j\left({h_{(2)}}^{(1)}\right)j^{\mathsf{c}}\left({h_{(2)}}^{(2)}\right) \\
&= j^{\mathsf{c}}\left(h_{(1)}\right)j\left(s_R(r){h_{(2)}}^{(1)}\right)j^{\mathsf{c}}\left({h_{(2)}}^{(2)}\right) = j^{\mathsf{c}}\left({h^{(1)}}_{(1)}\right)j\left(s_R(r){h^{(1)}}_{(2)}\right)j^{\mathsf{c}}\left(h^{(2)}\right) \\
&= \eta_R\left(\pi_R(s_R(r)h^{(1)})\right)j^{\mathsf{c}}\left(h^{(2)}\right) = j^{\mathsf{c}}\left(h^{(2)}t_R\left(\pi_R\left(s_R(r)h^{(1)}\right)\right)\right) = j^{\mathsf{c}}\left(t_R(r)h\right),
\end{aligned}$$

for $h \in H$ and $r \in R$. □

In the case of a Hopf algebra cleft extension, the convolution inverse of a cleaving map is a right colinear map, where the right coaction in the Hopf algebra is given by the coproduct followed by the antipode and a flip. In the case of a Hopf algebroid there are two coactions, one for each constituent bialgebroid, related by the isomorphism functors in Theorem 2.2. The following lemma shows the behaviour of the convolution inverse of a cleaving map with respect to these right coactions.

Lemma 3.8 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension with a cleaving map j. Then, for all $h \in H$,*

$$\rho^A\left(j^{\mathsf{c}}(h)\right) = j^{\mathsf{c}}\left(h_{(2)}\right)\underset{R}{\otimes}S\left(h_{(1)}\right), \tag{3.9}$$

and, equivalently,

$$\lambda^A\left(j^{\mathsf{c}}(h)\right) = j^{\mathsf{c}}\left(h^{(2)}\right)\underset{L}{\otimes}S\left(h^{(1)}\right). \tag{3.10}$$

Proof Combining the module map property of the antipode, $S(t_L(l)h) = S(h)s_L(l)$, for all $l \in L$, $h \in H$, with the Hopf algebroid axiom $s_L = t_R \circ \pi_R \circ s_L$ and using Eq. 3.8, one shows that the expression on the right hand side of Eq. 3.9 belongs to the appropriate R-module tensor product. Next using Eq. 3.4 one finds that it is an element of the Takeuchi product $A \times_R H$, defined in Eq. 2.11, i.e.

$$\eta_R(r)\,j^{\mathsf{c}}\left(h_{(2)}\right)\underset{R}{\otimes}S\left(h_{(1)}\right) = j^{\mathsf{c}}\left(h_{(2)}\right)\underset{R}{\otimes}t_R(r)S\left(h_{(1)}\right),$$

for all $r \in R$, $h \in H$. $A \times_R H$ is an $R \otimes_k L$-ring with factorwise multiplication and unit maps

$$\eta_R^{\times} : R \to A \times_R H, \quad r \mapsto 1_A \underset{R}{\otimes} s_R(r) \quad \text{and} \quad \eta_L^{\times} : L \to A \times_R H \quad l \mapsto \eta_L(l)\underset{R}{\otimes}1_H,$$

such that $\rho^A : A \to A \times_R H$ is a homomorphism of $R \otimes_k L$-rings. Furthermore, $\rho^A \circ j^c : H \to A \times_R H$ is the convolution inverse of $\rho^A \circ j$. We claim that the map

$$\tilde{\Theta} : H \to A \times_R H, \qquad h \mapsto j^c\left(h_{(2)}\right) \underset{R}{\otimes} S\left(h_{(1)}\right),$$

is a right convolution inverse of $\rho^A \circ j$.

Take any $h \in H, l \in L$ and $r \in R$. By the Hopf algebroid identity $\gamma_L(t_L(l)ht_R(r)) = h_{(1)}t_R(r) \otimes_L t_L(l)h_{(2)}$, the module map property of the antipode, $S(ht_R(r)) = s_R(r)S(h)$, and the right L-linearity of j^c it follows that

$$\begin{aligned}\tilde{\Theta}\left(t_L(l)ht_R(r)\right) &= j^c\left(t_L(l)h_{(2)}\right) \underset{R}{\otimes} S\left(h_{(1)}t_R(r)\right)\\ &= j^c\left(h_{(2)}\right)\eta_L(l) \underset{R}{\otimes} s_R(r)S\left(h_{(1)}\right) = \eta_R^{\times}(r)\tilde{\Theta}(h)\eta_L^{\times}(l),\end{aligned}$$

that is, $\tilde{\Theta}$ satisfies Eq. 3.4. Using the right $\mathscr{H}_R$-colinearity of j and the coassociativity of γ_R, one computes,

$$\begin{aligned}\left[\mu_{A\times_R H} \circ \left(\rho^A \circ j \underset{R}{\otimes} \tilde{\Theta}\right) \circ \gamma_R\right](h) &= j\left(h^{(1)}\right) j^c\left({h^{(2)}}^{(2)}{}_{(2)}\right) \underset{R}{\otimes} {h^{(2)}}^{(1)} S\left({h^{(2)}}^{(2)}{}_{(1)}\right)\\
&= j\left(h^{(1)}\right) j^c\left({h^{(2)}}_{(2)}\right) \underset{R}{\otimes} {{h^{(2)}}_{(1)}}^{(1)} S\left({{h^{(2)}}_{(1)}}^{(2)}\right)\\
&= j\left(h^{(1)}\right) j^c\left({h^{(2)}}_{(2)}\right) \underset{R}{\otimes} s_L\left(\pi_L\left({h^{(2)}}_{(1)}\right)\right)\\
&= j\left(h^{(1)}\right) j^c\left({h^{(2)}}_{(2)}\right) \eta_R\left(\pi_R\left(s_L\left(\pi_L\left({h^{(2)}}_{(1)}\right)\right)\right)\right) \underset{R}{\otimes} 1_H\\
&= j\left(h^{(1)}\right) j^c\left(s_L\left(\pi_L\left({h^{(2)}}_{(1)}\right)\right){h^{(2)}}_{(2)}\right) \underset{R}{\otimes} 1_H\\
&= j\left(h^{(1)}\right) j^c\left(h^{(2)}\right) \underset{R}{\otimes} 1_H = \eta_L\left(\pi_L(h)\right) \underset{R}{\otimes} 1_H\\
&= \left[\eta_L^{\times} \circ \pi_L\right](h),\end{aligned}$$

where the second equality follows by the Hopf algebroid axiom (2.7), the third one by the antipode axiom (2.9), the fourth one by the axiom $s_L = t_R \circ \pi_R \circ s_L$ in Eq. 2.6, the fifth one by Eq. 3.8 and the penultimate one by Eq. 3.5. This proves that $\tilde{\Theta}$ satisfies Eq. 3.5, hence $\tilde{\Theta}$ is a right convolution inverse of $\rho^A \circ j$. In view of the uniqueness of a convolution inverse this implies Eq. 3.9.

By Theorem 2.2, the right $\mathscr{H}_R$-colinearity of j^c (i.e. property (3.9)) is equivalent to its $\mathscr{H}_L$-colinearity (i.e. property (3.10)). □

Remark 3.9 Recall from Section 2.4 that if the antipode of a Hopf algebroid $\mathscr{H}$ is bijective, then there exists an anti-monoidal isomorphism between the categories of right $\mathscr{H}$-comodules and right $\mathscr{H}_{cop}$-comodules. Hence in this case, in light of the explicit form (2.26) of the relation between the $\mathscr{H}_L$ and $(\mathscr{H}_R)_{cop}$-coactions, an algebra extension $B \subseteq A$ is a right $\mathscr{H}$-extension if and only if $B^{op} \subseteq A^{op}$ is a right $\mathscr{H}_{cop}$-extension. Furthermore, $B \subseteq A$ is an $\mathscr{H}$-cleft extension if and only if $B^{op} \subseteq A^{op}$ is an $\mathscr{H}_{cop}$-cleft extension. Indeed, by Lemma 3.8, if $j : H \to A$ is a cleaving map for the $\mathscr{H}$-cleft extension $B \subseteq A$, then its convolution inverse j^c is a cleaving map for the $\mathscr{H}_{cop}$-cleft extension $B^{op} \subseteq A^{op}$.

Our next aim is to prove that, in parallel to the Hopf algebra case, an $\mathscr{H}$-cleft extension can be equivalently characterised as a Galois extension with the normal basis property. This is the main result of this section. The main difference with the Hopf algebra case is that a cleft $\mathscr{H}$-extension is a Galois extension with respect to the *right* bialgebroid $\mathscr{H}_R$ but it has a normal basis property with respect to the *left* bialgebroid $\mathscr{H}_L$. In preparation for this we state the following two lemmas.

Lemma 3.10 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension with a cleaving map j. Then, for all $a \in A$, $a^{[0]} j^{\mathsf{c}}(a^{[1]}) \in B$.*

Proof This is checked by applying ρ^A to $a^{[0]} j^{\mathsf{c}}(a^{[1]})$, noting that ρ^A is an algebra map and j^{c} satisfies Eq. 3.9, and then repeating the same steps as in the verification that $\tilde{\Theta}$ satisfies Eq. 3.5 in the proof of Lemma 3.8. □

Lemma 3.11 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension. Then the inclusion $B \subseteq A$ splits in the category of left B-modules. If, in addition, the antipode of $\mathscr{H}$ is bijective, then the inclusion $B \subseteq A$ splits also in the category of right B-modules.*

Proof A left B-linear splitting of the inclusion $B \to A$ is given by the map

$$A \to B, \qquad a \mapsto a^{[0]} j^{\mathsf{c}}\left(a^{[1]}\right) j(1_H), \tag{3.11}$$

where j is a cleaving map. The element $a^{[0]} j^{\mathsf{c}}(a^{[1]})$ belongs to B for any $a \in A$ by Lemma 3.10 and $j(1_H)$ is an element of B by the colinearity of j and the unitality of ρ^A. The left B-linearity of the map (3.11) follows by the left B-linearity of ρ^A. Finally, for all $b \in B$,

$$b^{[0]} j^{\mathsf{c}}(b^{[1]}) j(1_H) = b \;\; j^{\mathsf{c}}(1_H) j(1_H) = b \;\; \eta_R\big(\pi_R(1_H)\big) = b,$$

where the penultimate equality follows by the fact that j^{c} is the convolution inverse of j and the unitality of γ_L. If the antipode is bijective, then, by Remark 3.9, the map

$$A \to B, \qquad a \mapsto j^{\mathsf{c}}(1_H)\, j\big(S^{-1}\left(a_{[1]}\right)\big)\, a_{[0]},$$

is a right B-linear section of the inclusion $B \subseteq A$. □

Notice that $B \otimes_L H$ is a left B-module via the regular B-module structure of the first tensor factor, and – since the coproducts γ_R and γ_L are left L-linear – also a right $\mathscr{H}$-comodule via the regular $\mathscr{H}$- comodule structure of the second tensor factor.

Theorem 3.12 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ a right $\mathscr{H}$-extension. Then the following statements are equivalent:*

(1) *$B \subseteq A$ is an $\mathscr{H}$-cleft extension.*
(2) (a) *The extension $B \subseteq A$ is $\mathscr{H}_R$-Galois;*
    (b) *$A \simeq B \otimes_L H$ as left B-modules and right $\mathscr{H}$-comodules.*

Proof

(1) ⇒ (2)(a) Suppose that $B \subseteq A$ is a cleft $\mathscr{H}$-extension with a cleaving map j and consider the map

$$\Phi : A\underset{R}{\otimes}H \to A\underset{B}{\otimes}A, \qquad a\underset{R}{\otimes}h \mapsto aj^{\mathsf{c}}\left(h_{(1)}\right)\underset{B}{\otimes}j\left(h_{(2)}\right).$$

By Eqs. 3.4 and 3.3, $j^{\mathsf{c}}(h_{(1)})\otimes_L j(h_{(2)})$ is a well defined element of $A\otimes_L A$. Since B is an L-ring, Φ is a well defined map. We claim that Φ is the inverse of the $\mathscr{H}_R$-canonical map (2.15). Take any $a\otimes_R h \in A\otimes_R H$ and compute

$$\begin{aligned}
\mathrm{can}_R\left(\Phi\left(a\underset{R}{\otimes}h\right)\right) &= aj^{\mathsf{c}}\left(h_{(1)}\right)j\left(h_{(2)}\right)^{[0]}\underset{R}{\otimes}j\left(h_{(2)}\right)^{[1]}\\
&= aj^{\mathsf{c}}\left(h_{(1)}\right)j\left({h_{(2)}}^{(1)}\right)\underset{R}{\otimes}{h_{(2)}}^{(2)}\\
&= aj^{\mathsf{c}}\left({h^{(1)}}_{(1)}\right)j\left({h^{(1)}}_{(2)}\right)\underset{R}{\otimes}h^{(2)}\\
&= a\eta_R\left(\pi_R\left(h^{(1)}\right)\right)\underset{R}{\otimes}h^{(2)}\\
&= a\underset{R}{\otimes}h^{(2)}t_R\left(\pi_R\left(h^{(1)}\right)\right) = a\underset{R}{\otimes}h,
\end{aligned}$$

where the second equality follows by the right $\mathscr{H}_R$-colinearity of j, the third one by Eq. 2.7, and the fourth one by Eq. 3.6. On the other hand, for all $a\otimes_B a' \in A\otimes_B A$,

$$\begin{aligned}
\Phi\left(\mathrm{can}_R\left(a\underset{B}{\otimes}a'\right)\right) &= aa'^{[0]}j^{\mathsf{c}}\left({a'^{[1]}}_{(1)}\right)\underset{B}{\otimes}j\left({a'^{[1]}}_{(2)}\right)\\
&= a{a'_{[0]}}^{[0]}j^{\mathsf{c}}\left({a'_{[0]}}^{[1]}\right)\underset{B}{\otimes}j\left(a'_{[1]}\right)\\
&= a\underset{B}{\otimes}{a'_{[0]}}^{[0]}j^{\mathsf{c}}\left({a'_{[0]}}^{[1]}\right)j\left(a'_{[1]}\right)\\
&= a\underset{B}{\otimes}a'^{[0]}j^{\mathsf{c}}\left({a'^{[1]}}_{(1)}\right)j({a'^{[1]}}_{(2)})\\
&= a\underset{B}{\otimes}a'^{[0]}\eta_R\left(\pi_R(a'^{[1]})\right) = a\underset{B}{\otimes}a',
\end{aligned}$$

where the second and the fourth equalities follow by the right $\mathscr{H}_L$-colinearity of ρ^A, the third one by Lemma 3.10, the fifth one by Eq. 3.6 and the last one by the counitality of ρ^A. Thus Φ is the inverse of the canonical map, as claimed.

(1) ⇒ (2)(b) Given a cleaving map j, consider the left B-linear map

$$\kappa : A \to B\underset{L}{\otimes}H, \quad a \mapsto {a_{[0]}}^{[0]}j^{\mathsf{c}}({a_{[0]}}^{[1]})\underset{L}{\otimes}a_{[1]} = a^{[0]}j^{\mathsf{c}}\left({a^{[1]}}_{(1)}\right)\underset{L}{\otimes}{a^{[1]}}_{(2)}. \tag{3.12}$$

The equality of two forms of κ follows by the right $\mathscr{H}_L$-colinearity of ρ^A. Furthermore, Lemma 3.10 implies that the image of κ is in $B\otimes_L H$. In the opposite direction, define the left B-linear map $\nu : B\otimes_L H \to A$, $b\otimes_L h \mapsto b\,j(h)$, which is right $\mathscr{H}$-colinear by the right colinearity of a cleaving map and the left B-linearity of the coaction. The map ν is well defined in view of Eq. 3.3. For all $b \otimes_L h \in B \otimes_L H$,

$$\kappa\left(\nu\left(b \underset{L}{\otimes} h\right)\right) = b\, j(h)_{[0]}{}^{[0]} j^{\mathsf{c}}\left(j(h)_{[0]}{}^{[1]}\right) \underset{L}{\otimes} j(h)_{[1]}$$
$$= b\, j\left(h_{(1)}{}^{(1)}\right) j^{\mathsf{c}}\left(h_{(1)}{}^{(2)}\right) \underset{L}{\otimes} h_{(2)}$$
$$= b\,\eta_L\left(\pi_L\left(h_{(1)}\right)\right) \underset{L}{\otimes} h_{(2)} = b \underset{L}{\otimes} s_L\left(\pi_L\left(h_{(1)}\right)\right) h_{(2)} = b \underset{L}{\otimes} h,$$

where the second equality follows by the $\mathscr{H}$-colinearity of j, and the third one by Eq. 3.5. On the other hand, Eq. 3.6 and the counitality of ρ^A imply, for all $a \in A$,

$$\nu\left(\kappa(a)\right) = a^{[0]} j^{\mathsf{c}}\left(a^{[1]}{}_{(1)}\right) j\left(a^{[1]}{}_{(2)}\right) = a^{[0]}\eta_R\left(\pi_R\left(a^{[1]}\right)\right) = a.$$

This means that ν is the left B-linear right $\mathscr{H}$-colinear inverse of κ, hence κ is the required isomorphism.

(2) $\Rightarrow$ (1) Suppose that the canonical map (2.15) is bijective and write $\tau = \mathrm{can}_R^{-1}(1_A \underset{R}{\otimes} -) : H \to A \otimes_B A$ for the translation map. In explicit calculations we use a Sweedler type notation, for all $h \in H$, $\tau(h) = h^{\{1\}} \otimes_B h^{\{2\}}$ (summation understood). Let $\kappa : A \to B \otimes_L H$ be an isomorphism of left B-modules and of right $\mathscr{H}$-comodules and define maps $H \to A$

$$j := \kappa^{-1}\left(1_B \underset{L}{\otimes} -\right), \qquad \tilde{j} := \left[A \underset{B}{\otimes}\left(B \underset{L}{\otimes} \pi_L\right) \circ \kappa\right] \circ \tau.$$

We claim that j is a cleaving map and $\tilde{j}$ is its convolution inverse. First note that since κ^{-1} is left B-linear, it is in particular left L-linear, hence also j is left L-linear. Since κ^{-1} is also right $\mathscr{H}$-colinear, so is j. Furthermore, the canonical map is left A-linear, hence also left R-linear. Therefore, its inverse is left R-linear, implying that, for all $h \in H$ and $r \in R$, $\tau(h\, t_R(r)) = \eta_R(r) h^{\{1\}} \otimes_B h^{\{2\}}$. With this property of the translation map at hand, one immediately finds that, for all $h \in H$ and $r \in R$, $\tilde{j}(h t_R(r)) = \eta_R(r)\tilde{j}(h)$. On the other hand, by Eq. 2.13, for all $a, a' \in A$ and $r \in R$, $\mathrm{can}_R(a \otimes_B \eta_R(r) a') = a a'^{[0]} \otimes_R s_R(r) a'^{[1]}$. This implies that $\tau(s_R(r)\, h) = h^{\{1\}} \otimes_B \eta_R(r) h^{\{2\}}$. Thus, in view of the Hopf algebroid axiom $t_L = s_R \circ \pi_R \circ t_L$, one finds, for all $h \in H$ and $l \in L$,

$$\tilde{j}(t_L(l)\, h) = \left[A \underset{B}{\otimes}\left(B \underset{L}{\otimes} \pi_L\right) \circ \kappa\right] \left(\tau\left(t_L(l) h\right)\right)$$
$$= \left[A \underset{B}{\otimes}\left(B \underset{L}{\otimes} \pi_L\right) \circ \kappa\right] \left(h^{\{1\}} \underset{B}{\otimes} \eta_R\left(\pi_R\left(t_L(l)\right)\right) h^{\{2\}}\right).$$

Since κ is right $\mathscr{H}_R$-colinear, it is in particular left R-linear, where the left R-module structure of $B \otimes_L H$ is given by $r \cdot (b \otimes_L h) = b \otimes_L s_R(r) h$, (cf. Eq. 2.12). By the right L-linearity of π_L and the axiom $t_L = s_R \circ \pi_R \circ t_L$, one therefore concludes that $\tilde{j}(t_L(l)\, h) = \tilde{j}(h)\eta_L(l)$, as required. This proves that $\tilde{j}$ satisfies Eq. 3.4. It remains to check Eqs. 3.5 and 3.6:

$$\begin{aligned}\mu_A \circ \left(j \underset{R}{\otimes} \tilde{j} \right) \circ \gamma_R &= \mu_A \circ \left\{ j \underset{R}{\otimes} \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \right] \circ \tau \right\} \circ \gamma_R \\ &= \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \right] \circ \mathrm{can}_R^{-1} \circ \left(j \underset{R}{\otimes} H \right) \circ \gamma_R \\ &= \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \right] \circ \mathrm{can}_R^{-1} \circ \rho^A \circ j \\ &= \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \right] \circ \left(1_A \underset{B}{\otimes} j(-) \right) = \eta_L \circ \pi_L,\end{aligned}$$

where the second equality follows by the left A-linearity of the canonical map can_R, hence of can_R^{-1}, the third one by the right $\mathscr{H}_R$-colinearity of j and the fourth one by the explicit form (2.15) of can_R. Furthermore,

$$\begin{aligned}\mu_A \circ \left(\tilde{j} \underset{L}{\otimes} j \right) \circ \gamma_L &= \mu_A \circ \left\{ \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \right] \circ \tau \underset{L}{\otimes} j \right\} \circ \gamma_L \\ &= \mu_A \circ \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} \pi_L \right) \circ \kappa \underset{L}{\otimes} j \right] \circ \left(A \underset{B}{\otimes} \lambda^A \right) \circ \tau \\ &= \mu_A \circ \left(A \underset{B}{\otimes} B \underset{L}{\otimes} \pi_L \underset{L}{\otimes} j \right) \circ \left(A \underset{B}{\otimes} B \underset{L}{\otimes} \gamma_L \right) \circ \left(A \underset{B}{\otimes} \kappa \right) \circ \tau \\ &= \mu_A \circ \left[A \underset{B}{\otimes} \left(B \underset{L}{\otimes} j \right) \circ \kappa \right] \circ \tau = \mu_A \circ \tau = \eta_R \circ \pi_R,\end{aligned}$$

where the second equality follows by the $\mathscr{H}_L$-colinearity of τ, the third one by the $\mathscr{H}_L$-colinearity of κ, the penultimate one by the left B-linearity of κ and the last one by $(A \otimes_R \pi_R) \circ \mathrm{can}_R = \mu_A$ and the definition of the translation map τ. □

By Remark 3.9, the following 'left handed version' of Theorem 3.12 (1) ⇒ (2)(b) can be formulated.

Corollary 3.13 *Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode and $B \subseteq A$ an $\mathscr{H}$-cleft extension with a cleaving map j. Then the right B-linear left $\mathscr{H}$-colinear map*

$$A \to H \underset{L}{\otimes} B,\ a \mapsto S^{-1}\left(a_{[1]}\right)_{(1)} \underset{L}{\otimes} j\left(S^{-1}\left(a_{[1]}\right)_{(2)} \right) a_{[0]} \equiv S^{-1}\left(a^{[1]}\right) \underset{L}{\otimes} j\left(S^{-1}\left(a^{[0]}{}_{[1]} \right) \right) a^{[0]}{}_{[0]}, \tag{3.13}$$

is an isomorphism.

The following is an immediate consequence of Theorem 3.12.

Corollary 3.14 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension. If H is a projective left L-module, then A is a faithfully flat left B-module.*

Proof By Theorem 3.12 (1) ⇒ (2)(b), $A \simeq B \otimes_L H$ as left B-modules. Since H is projective as a left L-module, A is projective as a left B-module. Together with Lemma 3.11 this implies the claim. □

If the antipode of a Hopf algebroid $\mathscr{H}$ is bijective, then, by [8, Lemma 3.3], an extension $B \subseteq A$ is a right $\mathscr{H}_R$-Galois extension if and only if it is a right $\mathscr{H}_L$-Galois extension. By [8, Lemma 4.1], this is further equivalent to the left $\mathscr{H}_R$-Galois and also to the left $\mathscr{H}_L$-Galois property of the extension. Hence repeating the steps in the proof of [9, Proposition 4.1], we conclude that Lemma 3.11, Theorem 3.12 and Remark 3.9 imply the following

Corollary 3.15 *Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode and $B \subseteq A$ an $\mathscr{H}$-cleft extension. Then A is an R-relative injective right and left $\mathscr{H}_R$-comodule, and an L-relative injective left and right $\mathscr{H}_L$-comodule.*

4 Crossed Products with Hopf Algebroids

One of the main results in the theory of cleft extensions of Hopf algebras is the equivalent characterisation of such extensions as crossed product algebras with an invertible cocycle (cf. [20, Theorem 11] [6, Theorem 1.18]). The aim of this section is to derive such a characterisation for Hopf algebroid cleft extensions. First we need to develop a suitable theory of crossed products, generalising that of [20] and [5]. We start by extending the notion of a *measuring* [26, p. 139].

Definition 4.1 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring with unit map $\iota : L \to B$. $\mathscr{L}$ *measures* B if there exits a k-linear map, called a *measuring*, $H \otimes_k B \to B$, $h \otimes_k b \mapsto h \cdot b$ such that, for all $h \in H, l \in L, b, b' \in B$,

(a) $h \cdot 1_B = \iota\big(\pi(h)\big)$;
(b) $\big(t(l)h\big) \cdot b = (h \cdot b)\iota(l)$ and $\big(s(l)h\big) \cdot b = \iota(l)(h \cdot b)$;
(c) $h \cdot (b\, b') = (h_{(1)} \cdot b)(h_{(2)} \cdot b')$.

Note that condition (b) means simply that a measuring is an L-L bimodule map, where H is viewed as an L-L bimodule via the left multiplication by s and t. A left $\mathscr{L}$-module algebra B is measured by $\mathscr{L}$ with a measuring provided by the left H-multiplication in B.

Definition 4.2 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and $\iota : L \to B$ an L-ring, measured by $\mathscr{L}$. A B-valued *2-cocycle* σ on $\mathscr{L}$ is a k-linear map $H \otimes_{L^{op}} H \to B$ (where the right and left L^{op}-module structures on H are given by right and left multiplication by $t(l)$, respectively) satisfying

(a) $\sigma(s(l)h, k) = \iota(l)\sigma(h, k)$ and $\sigma(t(l)h, k) = \sigma(h, k)\iota(l)$;
(b) $(h_{(1)} \cdot \iota(l))\, \sigma(h_{(2)}, k) = \sigma(h, s(l)k)$;
(c) $\sigma(1, h) = \iota\big(\pi(h)\big) = \sigma(h, 1)$;
(d) $[h_{(1)} \cdot \sigma(k_{(1)}, m_{(1)})]\, \sigma(h_{(2)}, k_{(2)}m_{(2)}) = \sigma(h_{(1)}, k_{(1)})\, \sigma(h_{(2)}k_{(2)}, m)$,

for all $h, k, m \in H, l \in L$.

An $\mathscr{L}$-measured L-ring B is called a *σ-twisted left $\mathscr{L}$-module* if a 2-cocycle σ satisfies

(e) $1_H \cdot b = b$,
(f) $[h_{(1)} \cdot (k_{(1)} \cdot b)]\, \sigma(h_{(2)}, k_{(2)}) = \sigma(h_{(1)}, k_{(1)})\, (h_{(2)}k_{(2)} \cdot b)$,

for all $h, k \in H$, $b \in B$.

Conditions (c) in Definition 4.2 determine the normalisation of σ and (d) is a cocycle condition. These have the same form as corresponding conditions in the bialgebra case. Conditions (a) determine the module map properties of σ while (b) ensures that σ is properly L-balanced; both are needed for (d) (and (f)) to make sense. Condition (e) sates that a measuring is a *weak action* (cf. [5, Definition 1.1]).

Similarly to the bialgebra case, the map $\sigma(h, h') := \iota\big(\pi(h h')\big)$ is a (trivial) cocycle for an $\mathscr{L}$-measured L-ring B with unit ι, provided that the measuring restricts to the action on L, $h \cdot \iota(l) = \iota\big(\pi(hs(l))\big)$, for $h \in H$ and $l \in L$. A twisted left $\mathscr{L}$-module corresponding to this trivial cocycle σ is simply a left $\mathscr{L}$-module algebra.

Proposition 4.3 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and $\iota : L \to B$ an L-ring, measured by $\mathscr{L}$. Let $\sigma : H \otimes_{L^{op}} H \to B$ be a map, satisfying properties (a) and (b) in Definition 4.2. Consider the k-module $B \otimes_L H$, where the left L-module structure on H is given by multiplication by $s(l)$ on the left. $B \otimes_L H$ is an associative algebra with unit $1_B \otimes_L 1_H$ and product*

$$\begin{aligned}
&\Big(B \underset{L}{\otimes} H\Big) \underset{k}{\otimes} \Big(B \underset{L}{\otimes} H\Big) \to \Big(B \underset{L}{\otimes} H\Big), \\
&\Big(b \underset{L}{\otimes} h\Big) \underset{k}{\otimes} \Big(b' \underset{L}{\otimes} h'\Big) \mapsto b\, \big(h_{(1)} \cdot b'\big)\, \sigma\, \big(h_{(2)}, h'_{(1)}\big) \underset{L}{\otimes} h_{(3)} h'_{(2)},
\end{aligned} \tag{4.1}$$

if and only if σ is a cocycle and B is a σ-twisted $\mathscr{L}$-module. The resulting associative algebra is called a crossed product *of B with $\mathscr{L}$ and is denoted by $B\#_\sigma \mathscr{L}$.*

Note that the smash product algebra in Example 3.4 is a crossed product with a trivial cocycle.

Proof The element $1_B\#1_H$ is a left unit if and only if

$$b\#h = (1_H \cdot b)\, \sigma\, \big(1_H, h_{(1)}\big)\, \# h_{(2)}, \qquad \text{for all } b\#h \in B \underset{L}{\otimes} H. \tag{4.2}$$

If $\sigma(1_H, h) = \iota\big(\pi(h)\big)$ and $1_H \cdot b = b$, then Eq. 4.2 obviously holds. On the other hand, applying $B \otimes_L \pi$ to Eq. 4.2 we arrive at the identity

$$b\,\iota\,(\pi(h)) = (1_H \cdot b)\ \sigma\ (1_H, h)\,, \qquad \text{for all } b \in B,\ h \in H. \tag{4.3}$$

Setting $b = 1_B$ in Eq. 4.3 we obtain $\sigma(1_H, h) = \iota\big(\pi(h)\big)$, and setting $h = 1_H$ we get $1_H \cdot b = b$. Analogously, the condition that $1_B\#1_H$ is a right unit is equivalent to the condition $\sigma(h, 1_H) = \iota\big(\pi(h)\big)$, for all $h \in H$. The associative law for product (4.1) reads, for all $h, k, m \in H$, $a, b, c \in B$,

$$\begin{aligned} & a\left(h_{(1)} \cdot b\right)\sigma\left(h_{(2)}, k_{(1)}\right)\left(h_{(3)}k_{(2)} \cdot c\right)\sigma\left(h_{(4)}k_{(3)}, m_{(1)}\right)\#h_{(5)}k_{(4)}m_{(2)} \\ &\quad = a\left(h_{(1)} \cdot b\right)\left[h_{(2)} \cdot \left(k_{(1)} \cdot c\right)\right]\left[h_{(3)} \cdot \sigma\left(k_{(2)}, m_{(1)}\right)\right]\sigma(h_{(4)}, k_{(3)}m_{(2)})\#h_{(5)}k_{(4)}m_{(3)}. \end{aligned} \tag{4.4}$$

If σ is a cocycle and B is a σ-twisted module, then Eq. 4.4 obviously holds. Note that, for all $h, k \in H$,

$$\sigma\left(h_{(1)}, k_{(1)}\right)\iota\left(\pi(h_{(2)}k_{(2)})\right) = \sigma(h, k). \tag{4.5}$$

Applying $B \otimes_L \pi$ to Eq. 4.4, using Eq. 4.5 and setting $a = 1_B = b$, we arrive at

$$\sigma\left(h_{(1)}, k_{(1)}\right)\left(h_{(2)}k_{(2)} \cdot c\right)\sigma\left(h_{(3)}k_{(3)}, m\right) = \left[h_{(1)} \cdot \left(k_{(1)} \cdot c\right)\sigma\left(k_{(2)}, m_{(1)}\right)\right]\sigma(h_{(2)}, k_{(3)}m_{(2)}). \tag{4.6}$$

Setting $c = 1_B$ in Eq. 4.6 we derive the cocycle condition Definition 4.2 (d), while setting $m = 1_H$ in Eq. 4.6 we obtain Definition 4.2 (f). □

Theorem 4.4 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and $\iota : L \to B$ an L-ring. View $A = B \otimes_L H$ as a left B-module and a right $\mathscr{L}$-comodule in canonical ways (i.e. the left B-multiplication is given by product in B and the right $\mathscr{L}$-coaction is $B \otimes_L \gamma$, with the L-actions on H given by the left multiplication by s and t). Then A is a right $\mathscr{L}$-comodule algebra with unit $1_B \otimes_L 1_H$ and a left B-linear product if and only if A is a crossed product algebra as in Proposition* 4.3.

Proof The definition of the product in $B\#_\sigma\mathscr{L}$ immediately implies that $B\#_\sigma\mathscr{L}$ is a right $\mathscr{L}$-comodule algebra with a left B-linear multiplication. Conversely, suppose that A has the required $\mathscr{L}$-comodule algebra structure. Then, in particular, A is an L^{op}-ring via $L^{op} \to A, l \mapsto 1_B \otimes_L t(l)$. We use the hom-tensor relation

$$\mathrm{Hom}_{B-}^{-\mathscr{L}}\left(\left(B \underset{L}{\otimes} H\right) \underset{L^{op}}{\otimes} \left(B \underset{L}{\otimes} H\right), B \underset{L}{\otimes} H\right) \simeq \mathrm{Hom}_{L,L}\left(H \underset{L^{op}}{\otimes} \left(B \underset{L}{\otimes} H\right), B\right) \tag{4.7}$$

and the $\mathscr{L}$-colinearity of the product in A, to view the multiplication in A as an L-L bilinear map $H \otimes_{L^{op}} (B \otimes_L H) \to B$. For any $b \in B$ and $h \in H$, define

$$h \cdot b := \left(B \underset{L}{\otimes} \pi\right)\left(\left(1_B \underset{L}{\otimes} h\right)\left(b \underset{L}{\otimes} 1_H\right)\right). \tag{4.8}$$

By (4.7), the above definition implies that, conversely,

$$\left(1_B \underset{L}{\otimes} h\right)\left(b \underset{L}{\otimes} 1_H\right) = h_{(1)} \cdot b \underset{L}{\otimes} h_{(2)}. \tag{4.9}$$

Now, the assumption that $1_B \otimes_L 1_H$ is the unit in A implies condition (a) in Definition 4.1. The conditions (b) follow by the right L-linearity and the left B-linearity of the product respectively (remember that every right $\mathscr{L}$-comodule map is necessarily right L-linear). The condition (c) follows by the associativity of the product. Thus B is measured by $\mathscr{L}$ with measuring (4.8).

Next, for all $h, h' \in H$, define

$$\sigma(h, h') := \left(B \underset{L}{\otimes} \pi\right)\left(\left(1_B \underset{L}{\otimes} h\right)\left(1_B \underset{L}{\otimes} h'\right)\right). \tag{4.10}$$

Then, by (4.7),

$$\left(1_B \underset{L}{\otimes} h\right)\left(1_B \underset{L}{\otimes} h'\right) = \sigma\left(h_{(1)}, h'_{(1)}\right) \underset{L}{\otimes} h_{(2)} h'_{(2)}. \tag{4.11}$$

Since A is an L^{op}-ring, Eq. 4.10 defines a k-linear map $\sigma : H \otimes_{L^{op}} H \to B$. The conditions (a) in Definition 4.2 follow by the left B-linearity and the right L-linearity of the product respectively. To check condition (b), take any $h, k \in H$ and $l \in L$, and compute

$$\begin{aligned}\left(h_{(1)} \cdot \iota(l)\right) \sigma\left(h_{(2)}, k\right) &= \left(B \underset{L}{\otimes} \pi\right)\left[\left(1_B \underset{L}{\otimes} h\right)\left(\iota(l) \underset{L}{\otimes} 1_H\right)\left(1_B \underset{L}{\otimes} k\right)\right] \\ &= \left(B \underset{L}{\otimes} \pi\right)\left[\left(1_B \underset{L}{\otimes} h\right)\left(1_B \underset{L}{\otimes} s(l) k\right)\right] = \sigma\left(h, s(l) k\right),\end{aligned}$$

where the first and last equalities follow by the definitions of the measuring and σ and Eqs. 4.9, 4.11, and the left B-linearity of the product. Finally, for all $b, b' \in B$, $h, h' \in H$,

$$\begin{aligned}\left(b \underset{L}{\otimes} h\right)\left(b' \underset{L}{\otimes} h'\right) &= b\left[\left(1_B \underset{L}{\otimes} h\right)\left(b' \underset{L}{\otimes} 1_H\right)\left(1_B \underset{L}{\otimes} h'\right)\right] \\ &= b\left(h_{(1)} \cdot b'\right)\left[\left(1_B \underset{L}{\otimes} h_{(2)}\right)\left(1_B \underset{L}{\otimes} h'\right)\right] \\ &= b\left(h_{(1)} \cdot b'\right) \sigma\left(h_{(2)}, h'_{(1)}\right) \underset{L}{\otimes} h_{(3)} h'_{(2)},\end{aligned}$$

where we have used the left B-linearity of the product and Eqs. 4.9 and 4.11. Proposition 4.3 yields the assertion. □

Corollary 4.5 *Given a crossed product $B\#_\sigma \mathscr{L}$ and a convolution invertible map $\chi \in \mathrm{Hom}_{L,L}(H, B)$ such that $\chi(1_H) = 1_B$, define, for all $h, k \in H$ and $b \in B$,*

$$h \cdot^\chi b := \chi\left(h_{(1)}\right)\left(h_{(2)} \cdot b\right) \chi^{\mathsf{c}}\left(h_{(3)}\right), \tag{4.12}$$

$$\sigma^\chi(h, k) := \chi\left(h_{(1)}\right)\left(h_{(2)} \cdot \chi(k_{(1)})\right) \sigma\left(h_{(3)}, k_{(2)}\right) \chi^{\mathsf{c}}\left(h_{(4)} k_{(3)}\right). \tag{4.13}$$

Then B is a σ^χ-twisted $\mathscr{L}$-module with measuring (4.12). The corresponding crossed product $B\#_{\sigma^\chi}\mathscr{L}$ is called a gauge transform of $B\#_\sigma\mathscr{L}$.

Proof Any convolution invertible map $\chi \in \mathrm{Hom}_{L,L}(H, B)$ defines a left B-module right $\mathscr{L}$-comodule automorphism Φ of $B \otimes_L H$, by

$$\Phi\left(b \underset{L}{\otimes} h\right) = b\, \chi\left(h_{(1)}\right) \underset{L}{\otimes} h_{(2)}, \qquad \Phi^{-1}\left(b \underset{L}{\otimes} h\right) = b\, \chi^{\mathsf{c}}\left(h_{(1)}\right) \underset{L}{\otimes} h_{(2)}. \tag{4.14}$$

If $\chi(1_H) = 1_B$, then $\Phi(1_H \otimes_L 1_B) = 1_H \otimes_L 1_B$. We can use this isomorphism to induce a new right $\mathscr{L}$-comodule algebra structure on $B \otimes_L H$(with unit $1_H \otimes_L 1_B$)

from that of $B\#_\sigma\mathscr{L}$. In view of Theorem 4.4, this necessarily is a crossed product with the measuring and cocycle given by Eqs. 4.8 and 4.10, i.e., for all $b \in B$ and $h, k \in H$,

$$h \cdot^\chi b = \left(B \underset{L}{\otimes} \pi\right)\left(\Phi^{-1}\left(\Phi\left(1_B \underset{L}{\otimes} h\right)\Phi\left(b \underset{L}{\otimes} 1_H\right)\right)\right),$$

$$\sigma^\chi(h,k) = \left(B \underset{L}{\otimes} \pi\right)\left(\Phi^{-1}\left(\Phi(1_B \underset{L}{\otimes} h)\Phi(1_B \underset{L}{\otimes} k)\right)\right),$$

where the product is computed in $B\#_\sigma\mathscr{L}$. One easily checks that these have the form stated in Eqs. 4.12 and 4.13. □

Definition 4.6 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring. Crossed products $B\#_\sigma\mathscr{L}$ and $B\#_{\bar\sigma}\mathscr{L}$ are said to be *equivalent* if there exists a left B-module isomorphism of right $\mathscr{L}$-comodule algebras $B\#_{\bar\sigma}\mathscr{L} \to B\#_\sigma\mathscr{L}$.

Theorem 4.7 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring. Crossed products $B\#_\sigma\mathscr{L}$ and $B\#_{\bar\sigma}\mathscr{L}$ are equivalent if and only if $B\#_{\bar\sigma}\mathscr{L}$ is a gauge transform of $B\#_\sigma\mathscr{L}$.*

Proof In view of the hom-tensor relation $\mathrm{Hom}_{B-}^{-\mathscr{L}}(B \otimes_L H, B \otimes_L H) \simeq \mathrm{Hom}_{L,L}(H, B)$, there is a bijective correspondence between left B-module right $\mathscr{L}$-comodule isomorphisms $\Phi : B\#_{\bar\sigma}\mathscr{L} \to B\#_\sigma\mathscr{L}$ and convolution invertible maps $\chi \in \mathrm{Hom}_{L,L}(H, B)$. This correspondence is given by Eq. 4.14 in one direction and by $\chi(h) = (B \otimes_L \pi)(\Phi(1_B \otimes_L h))$ in the other. If Φ is also an algebra map, then $\chi(1_H) = 1_B$ and, following the same line of argument as in the proof of Corollary 4.5, one finds that the measuring corresponding to $\bar\sigma$ is given by $h \cdot^\chi b$ and that $\bar\sigma = \sigma^\chi$. Conversely, given χ and corresponding (by Eq. 4.14) isomorphism $\Phi : B\#_{\sigma^\chi}\mathscr{L} \to B\#_\sigma\mathscr{L}$, one can compute, for all $b, b' \in B$, $h, h' \in H$,

$$\begin{aligned}\Phi\left((b\#_{\sigma^\chi}h)\left(b'\#_{\sigma^\chi}h'\right)\right) &= b\left(h_{(1)} \cdot^\chi b'\right)\sigma^\chi\left(h_{(2)}, h'_{(1)}\right)\chi\left(h_{(3)}h'_{(2)}\right)\#_\sigma h_{(4)}h'_{(3)}\\ &= b\,\chi\left(h_{(1)}\right)\left(h_{(2)} \cdot b'\right)\left(h_{(3)} \cdot \chi\left(h'_{(1)}\right)\right)\sigma\left(h_{(4)}, h'_{(2)}\right)\#_\sigma h_{(5)}h'_{(3)}\\ &= b\,\chi\left(h_{(1)}\right)\left(h_{(2)} \cdot \left(b'\chi\left(h'_{(1)}\right)\right)\right)\sigma\left(h_{(3)}, h'_{(2)}\right)\#_\sigma h_{(4)}h'_{(3)}\\ &= \Phi\left(b\#_{\sigma^\chi}h\right)\Phi\left(b'\#_{\sigma^\chi}h'\right),\end{aligned}$$

where the second equality follows by the fact that χ^{c} is the convolution inverse of χ and the counit axiom, and the third equality follows by property (c) in Definition 4.1. This proves that Φ is an algebra map, hence the crossed product algebras $B\#_{\sigma^\chi}\mathscr{L}$ and $B\#_\sigma\mathscr{L}$ are mutually equivalent. □

Next we establish what is meant by an invertible cocycle in this generalised context.

Definition 4.8 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and $\iota : L \to B$ an L-ring, measured by $\mathscr{L}$. A B-valued 2-cocycle σ on $\mathscr{L}$ is *invertible* if there exists a k-linear map $\tilde\sigma : H \otimes_L H \to B$ (where the right and left L-module structures on H are given by right and left multiplication by $s(l)$, respectively) satisfying

(a) $\tilde\sigma(s(l)h, k) = \iota(l)\tilde\sigma(h,k)$ and $\tilde\sigma(t(l)h, k) = \tilde\sigma(h,k)\iota(l)$;
(b) $\tilde\sigma(h_{(1)}, k)(h_{(2)} \cdot \iota(l)) = \tilde\sigma(h, t(l)k)$;

(c) $\sigma(h_{(1)}, k_{(1)})\,\tilde{\sigma}(h_{(2)}, k_{(2)}) = h \cdot (k \cdot 1_B)$ and $\tilde{\sigma}(h_{(1)}, k_{(1)})\,\sigma(h_{(2)}, k_{(2)}) = hk \cdot 1_B$,

for all $h, k \in H$ and $l \in L$. A map $\tilde{\sigma}$ is called an *inverse* of σ.

Again, conditions (a) and (b) are needed so that the inverse property (c) can be stated. In the case $\mathscr{L}$ is a bialgebra over a ring $L = k$, conditions (a) and (b) are satisfied automatically. The following two lemmas explore the nature of cocycles and their inverses.

Lemma 4.9 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an$L$-ring with unit $\iota : L \to B$. Assume that $\mathscr{L}$ measures B and σ is an invertible B-valued 2-cocycle on $\mathscr{L}$. Then an inverse $\tilde{\sigma}$ of σ is unique and normalised, i.e., for all $h \in H$,*

$$\tilde{\sigma}\,(1_H, h) = \iota\,(\pi(h)) = \tilde{\sigma}\,(h, 1_H)\,.$$

Proof Note that, if $\tilde{\sigma}$ is an inverse of σ, then, for all $h, k \in H$,

$$\iota\big(\pi(h_{(1)}k_{(1)})\big)\tilde{\sigma}(h_{(2)}, k_{(2)}) = \tilde{\sigma}(h, k). \tag{4.15}$$

Using this identity and Definition 4.8 (c), one finds that

$$\tilde{\sigma}(h, k) = \tilde{\sigma}\,\big(h_{(1)}, k_{(1)}\big)\,\big[h_{(2)} \cdot \big(k_{(2)} \cdot 1_B\big)\big]. \tag{4.16}$$

Now suppose that $\hat{\sigma}$ is another inverse of σ. Then replacing the expression in square brackets in Eq. 4.16 by the first of equations in Definition 4.8 (c) for $\hat{\sigma}$, using the second of equations Definition 4.8 (c) for $\tilde{\sigma}$, and finally using Eq. 4.15 for $\hat{\sigma}$, one finds that $\tilde{\sigma} = \hat{\sigma}$. Hence the inverse of a cocycle is unique.

Use Eq. 4.15, Definition 4.2 and Definition 4.8 (c) to compute, for all $h \in H$,

$$\begin{aligned}\tilde{\sigma}(1_H, h) &= \iota\big(\pi(h_{(1)})\big)\tilde{\sigma}(1_H, h_{(2)}) = \sigma(1_H, h_{(1)})\tilde{\sigma}(1_H, h_{(2)})\\ &= [1_H \cdot (h \cdot 1_B)]\sigma(1_H, 1_H) = \sigma(1_H, s(\pi(h))) = \iota\big(\pi(h)\big).\end{aligned}$$

The proof of the other identity is similar. □

Lemma 4.10 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid, B an L-ring, measured by $\mathscr{L}$, and σ an invertible B-valued 2-cocycle on $\mathscr{L}$ with the inverse $\tilde{\sigma}$. For all $h, k, m \in H$,*

(a) $h \cdot \sigma(k, m) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m_{(1)})\tilde{\sigma}(h_{(3)}, k_{(3)}m_{(2)})$,
(b) $h \cdot \tilde{\sigma}(k, m) = \sigma(h_{(1)}, k_{(1)}m_{(1)})\tilde{\sigma}(h_{(2)}k_{(2)}, m_{(2)})\tilde{\sigma}(h_{(3)}, k_{(3)})$.

Proof

(a) Denote the unit map of the L-ring B by $\iota : L \to B$. In view of Eq. 4.5 and with the help of properties (c) and (a) in Definition 4.1 and (c) in Definition 4.8, we can compute, for all $h, k, m \in H$,

$$\begin{aligned}h \cdot \sigma(k, m) &= h \cdot \big[\sigma\,\big(k_{(1)}, m_{(1)}\big)\,\iota\,\big(\pi\,\big(k_{(2)}m_{(2)}\big)\big)\big]\\ &= \big[h_{(1)} \cdot \sigma\,\big(k_{(1)}, m_{(1)}\big)\big]\,\big[h_{(2)} \cdot \big(k_{(2)}m_{(2)} \cdot 1_B\big)\big]\\ &= \big[h_{(1)} \cdot \sigma\,\big(k_{(1)}, m_{(1)}\big)\big]\,\sigma\,\big(h_{(2)}, k_{(2)}m_{(2)}\big)\,\tilde{\sigma}\,\big(h_{(3)}, k_{(3)}m_{(3)}\big)\\ &= \sigma\,\big(h_{(1)}, k_{(1)}\big)\,\sigma\,\big(h_{(2)}k_{(2)}, m_{(1)}\big)\,\tilde{\sigma}\,\big(h_{(3)}, k_{(3)}m_{(2)}\big)\,,\end{aligned}$$

where the last equality follows by property (d) in Definition 4.2.

(b) Use part (a), Eq. 4.5 and Definition 4.8 (c) to find that, for all $h, k, m \in H$,

$$\begin{aligned}&\left[h_{(1)} \cdot \sigma\left(k_{(1)}, m_{(1)}\right)\right] \sigma\left(h_{(2)}, k_{(2)} m_{(2)}\right) \tilde{\sigma}\left(h_{(3)} k_{(3)}, m_{(3)}\right) \tilde{\sigma}\left(h_{(4)}, k_{(4)}\right) \\ &\quad= \sigma\left(h_{(1)}, k_{(1)}\right)\left[h_{(2)} k_{(2)} \cdot\left(m \cdot 1_B\right)\right] \tilde{\sigma}\left(h_{(3)}, k_{(3)}\right).\end{aligned}$$

By Definitions 4.1 and 4.2, $\sigma(h, s(\pi(m))) = h \cdot (m \cdot 1_B))$, for all $h, m \in H$. Hence conditions (d) in Definition 4.2 and (c) in Definition 4.8 allow us to develop the right hand side of the above equality further to arrive at the equation

$$\left[h_{(1)} \cdot \sigma\left(k_{(1)}, m_{(1)}\right)\right] \sigma\left(h_{(2)}, k_{(2)} m_{(2)}\right) \tilde{\sigma}\left(h_{(3)} k_{(3)}, m_{(3)}\right) \tilde{\sigma}\left(h_{(4)}, k_{(4)}\right) = h \cdot\left[k \cdot (m \cdot 1_B)\right]. \tag{4.17}$$

Therefore

$$\begin{aligned} h \cdot \tilde{\sigma}(k, m) &= h \cdot \left\{\tilde{\sigma}\left(k_{(1)}, m_{(1)}\right)\left[k_{(2)} \cdot\left(m_{(2)} \cdot 1_B\right)\right]\right\} \\ &= \left[h_{(1)} \cdot \tilde{\sigma}\left(k_{(1)}, m_{(1)}\right)\right]\left\{h_{(2)} \cdot\left[k_{(2)} \cdot\left(m_{(2)} \cdot 1_B\right)\right]\right\} \\ &= \left[h_{(1)} \cdot \tilde{\sigma}\left(k_{(1)}, m_{(1)}\right)\right]\left[h_{(2)} \cdot \sigma\left(k_{(2)}, m_{(2)}\right)\right] \\ &\quad \times \sigma\left(h_{(3)}, k_{(3)} m_{(3)}\right) \tilde{\sigma}\left(h_{(4)} k_{(4)}, m_{(4)}\right) \tilde{\sigma}\left(h_{(5)}, k_{(5)}\right) \\ &= \left[h_{(1)} \cdot\left(k_{(1)} m_{(1)} \cdot 1_B\right)\right] \sigma\left(h_{(2)}, k_{(2)} m_{(2)}\right) \tilde{\sigma}\left(h_{(3)} k_{(3)}, m_{(3)}\right) \tilde{\sigma}\left(h_{(4)}, k_{(4)}\right) \\ &= \sigma\left(h_{(1)}, k_{(1)} m_{(1)}\right) \tilde{\sigma}\left(h_{(2)} k_{(2)}, m_{(2)}\right) \tilde{\sigma}\left(h_{(3)}, k_{(3)}\right), \end{aligned}$$

where the first equality follows by Eq. 4.16, the second one by property (c) in Definition 4.1 and the third one by Eq. 4.17. The penultimate equality follows by property (c) in Definition 4.1 and(c) in Definition 4.8. The last equality follows by conditions (a) in Definition 4.1 and (b) in Definition 4.2. □

Take a Hopf algebroid $\mathscr{H}$, an $\mathscr{H}_L$-measured L-ring B and a cocycle σ. Then the crossed product $B\#_\sigma \mathscr{H}_L$ is an R-ring with unit map $\eta_R : r \mapsto 1_B \# s_R(r)$ and an L-ring with unit map $\eta_L : l \mapsto 1_B \# s_L(l) = \iota(l)\#1_H$, where $\iota : L \to B$ denotes the unit map of the L-ring B. It is also a right $\mathscr{H}$-comodule with $\mathscr{H}_R$-coaction $B \otimes_L \gamma_R$ (and corresponding right $\mathscr{H}_L$-coaction $B \otimes_L \gamma_L$). The coinvariants are the elements of the form $b\#1_H$ for $b \in B$, hence they form an L-subring, isomorphic to B. Therefore, $B \subseteq B\#_\sigma \mathscr{H}_L$ is a right $\mathscr{H}$-extension, and it is natural to ask whether this extension is cleft.

Theorem 4.11 *Let $\mathscr{H}$ be a Hopf algebroid and $B\#_\sigma \mathscr{H}_L$ a crossed product of an $\mathscr{H}_L$-measured L-ring B. If the cocycle σ is invertible, then the extension $B \subseteq B\#_\sigma \mathscr{H}_L$ is $\mathscr{H}$-cleft.*

Proof We claim that the map $j : H \to B\#_\sigma \mathscr{H}_L$, $h \mapsto 1_B\#h$ is a cleaving map with the convolution inverse

$$j^{\mathrm{c}}(h) = \tilde{\sigma}\left(S(h_{(1)})_{(1)}, h_{(2)}\right) \# S\left(h_{(1)}\right)_{(2)} = \tilde{\sigma}\left(S\left(h^{(2)}{}_{(1)}\right), h^{(2)}{}_{(2)}\right) \# S\left(h^{(1)}\right).$$

The two forms of j^{c} are equivalent by the anti-comultiplicativity of S and the left $\mathscr{H}_R$-colinearity of γ_L. Using the definitions of a cocycle and its inverse, and in particular, the module and normalisation properties of σ and $\tilde{\sigma}$, one verifies that j and j^{c} have the required L-, R-module properties (3.3) and (3.4). Next, take any $h \in H$ and compute

$$\begin{aligned} j^{\mathsf{c}}\left(h_{(1)}\right) j\left(h_{(2)}\right) &= \tilde{\sigma}\left(S(h_{(1)})_{(1)}, h_{(2)}\right) \sigma\left(S\left(h_{(1)}\right)_{(2)}, h_{(3)}\right) \# S\left(h_{(1)}\right)_{(3)} h_{(4)} \\ &= 1_B \# S\left(h_{(1)}\right) h_{(2)} = 1_B \# s_R\left(\pi_R(h)\right) = \eta_R\left(\pi_R(h)\right), \end{aligned}$$

where the second equality follows by condition (c) in Definition 4.8, condition (a) in Definition 4.1 and the counit property of π_L. The third equality follows by the antipode axiom (2.9). The proof of the identity (3.5) is slightly more involved:

$$\begin{aligned} & j\left(h^{(1)}\right) j^{\mathsf{c}}\left(h^{(2)}\right) \\ &= \left[h^{(1)}{}_{(1)} \cdot \tilde{\sigma}\left(S\left(h^{(2)}{}_{(1)}\right)_{(1)}, h^{(2)}{}_{(2)}\right)\right] \sigma\left(h^{(1)}{}_{(2)}, S\left(h^{(2)}{}_{(1)}\right)_{(2)}\right) \# h^{(1)}{}_{(3)} S\left(h^{(2)}{}_{(1)}\right)_{(3)} \\ &= \sigma\left(h^{(1)}{}_{(1)}, S\left(h^{(2)}{}_{(1)}\right)_{(1)} h^{(2)}{}_{(2)}\right) \tilde{\sigma}\left(h^{(1)}{}_{(2)} S\left(h^{(2)}{}_{(1)}\right)_{(2)}, h^{(2)}{}_{(3)}\right) \# h^{(1)}{}_{(3)} S\left(h^{(2)}{}_{(1)}\right)_{(3)} \\ &= \sigma\left(h^{(1)}{}_{(1)}, S\left(h^{(4)}{}_{(1)}\right) h^{(4)}{}_{(2)}\right) \tilde{\sigma}\left(h^{(1)}{}_{(2)} S\left(h^{(3)}\right), h^{(4)}{}_{(3)}\right) \# h^{(1)}{}_{(3)} S\left(h^{(2)}\right) \\ &= \tilde{\sigma}\left(s_L\left(\pi_L\left(h^{(1)}{}_{(1)} s_R\left(\pi_R\left(h^{(4)}{}_{(1)}\right)\right)\right)\right) h^{(1)}{}_{(2)} S\left(h^{(3)}\right), h^{(4)}{}_{(2)}\right) \# h^{(1)}{}_{(3)} S\left(h^{(2)}\right) \\ &= \tilde{\sigma}\left(h^{(1)}{}_{(1)} S\left(h^{(3)} s_R\left(\pi_R\left(h^{(4)}{}_{(1)}\right)\right)\right), h^{(4)}{}_{(2)}\right) \# h^{(1)}{}_{(2)} S\left(h^{(2)}\right) \\ &= \tilde{\sigma}\left(h^{(1)}{}_{(1)} S\left(h^{(3)}{}_{(1)}\right), h^{(3)}{}_{(2)}\right) \# h^{(1)}{}_{(2)} S\left(h^{(2)}\right) \\ &= \tilde{\sigma}\left(h_{(1)} S\left(h_{(2)}{}^{(3)}\right), h_{(3)}\right) \# h_{(2)}{}^{(1)} S\left(h_{(2)}{}^{(2)}\right) \\ &= \tilde{\sigma}\left(t_L\left(\pi_L\left(h_{(2)}{}^{(1)}\right)\right) h_{(1)} S\left(h_{(2)}{}^{(2)}\right), h_{(3)}\right) \# 1_H \\ &= \tilde{\sigma}\left(h_{(1)}{}^{(1)} S\left(h_{(1)}{}^{(2)}\right), h_{(2)}\right) \# 1_H = \tilde{\sigma}(1_H, h) \# 1_H = \eta_L\left(\pi_L(h)\right), \end{aligned}$$

where the second equality follows by Lemma 4.10 (b), condition (c) in Definition 4.8, condition (a) in Definition 4.1, condition (a) in Definition 4.8 and the counit property of π_L. The third equality follows by the anti-comultiplicativity of S and Eq. 2.7. The fourth one follows by the antipode axiom (2.9), the fact that the domain of σ is $H \otimes_{L^{op}} H$ (i.e. $\sigma(ht_L(l), k) = \sigma(h, t_L(l)k)$ for $h, k \in H$, $l \in L$), the normalisation of cocycles (condition (c) in Definition 4.2) and the left L-linearity of $\tilde{\sigma}$ in the first argument (condition (a) in Definition 4.8). In the fifth step the Hopf algebroid identity $\pi_L(hs_R(r)) = \pi_L(hS(s_R(r)))$, implying $s_L\big(\pi_L(h_{(1)}s_R(r))\big)h_{(2)} = hS\big(s_R(r)\big)$, for $h \in H$ and $r \in R$, has been used together with the anti-multiplicativity of S. The sixth and seventh equalities follow by the coassociativity and $\mathscr{H}_L$-colinearity of γ_R and the counit property of π_R. The eighth equality follows by the antipode axiom (2.9) and the right L-linearity of $\tilde{\sigma}$. The ninth one follows by axiom (2.7) and the counit property of π_L. The penultimate equality follows by axiom (2.9) and the fact that the domain of $\tilde{\sigma}$ is $H \otimes_L H$ (i.e. $\tilde{\sigma}(hs_L(l), k) = \tilde{\sigma}(h, s_L(l)k)$ for $h, k \in H, l \in L$). The last equality follows by Lemma 4.9. □

The final aim of this section is to prove that any cleft extension is necessarily isomorphic to a crossed product with an invertible cocycle.

Theorem 4.12 *If $B \subseteq A$ is an $\mathscr{H}$-cleft extension, then there exists an invertible cocycle σ and a left B-module right $\mathscr{H}$-comodule algebra isomorphism $A \to B \#_\sigma \mathscr{H}_L$.*

Proof For an $\mathscr{H}$-cleft extension $B \subseteq A$ the cleaving map j takes the unit element of H to an invertible element of B (with the inverse $j^{\mathsf{c}}(1_H)$). Thus, without the loss of generality, we can assume that a cleaving map j is normalised, i.e. $j(1_H) = 1_A = j^{\mathsf{c}}(1_H)$. By Theorem 3.12, A is isomorphic to $B \otimes_L H$ as a left B-module and a right $\mathscr{H}$-comodule. We can use this isomorphism to induce a comodule algebra structure on $B \otimes_L H$. By Theorem 4.4, the induced algebra structure is necessarily a crossed product $B\#_\sigma \mathscr{H}_L$. In view of the definitions of the map κ and its inverse in the proof of Theorem 3.12 (1) $\Rightarrow$ (2)(b), the measuring and cocycle can be read off Eqs. 4.8 and 4.10, respectively, and come out as

$$h \cdot b = j\left(h^{(1)}\right) b j^{\mathsf{c}}\left(h^{(2)}\right), \qquad \sigma(h,k) = j\left(h^{(1)}\right)\, j\left(k^{(1)}\right)\, j^{\mathsf{c}}\left(h^{(2)}k^{(2)}\right). \tag{4.18}$$

We only need to prove that the cocycle σ is invertible. Define

$$\tilde{\sigma} : H \underset{L}{\otimes} H \to B, \qquad h \underset{L}{\otimes} k \mapsto j\left(h^{(1)}k^{(1)}\right)\, j^{\mathsf{c}}\left(k^{(2)}\right)\, j^{\mathsf{c}}\left(h^{(2)}\right). \tag{4.19}$$

The map (4.19) is well defined by Eqs. 3.3 and 3.4, on one hand, and by Eqs. 3.4, 3.8 and the property that the range of the coproduct of a right bialgebroid is in the Takeuchi product, on the other hand. The proof that $\tilde{\sigma}$ is the inverse of the cocycle σ is done by a routine calculation and is left to the reader. □

Combining Theorem 4.7 with Theorem 4.12, we can fully describe the relationship between different cleaving maps for the same cleft extension.

Corollary 4.13 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension with a (non-necessarily unital)cleaving map $j : H \to A$. Then a map $j' : H \to A$ is a cleaving map if and only if there exists a convolution invertible L-L bilinear map $\chi : H \to B$, such that*

$$j'(h) = \chi\left(h_{(1)}\right) j\left(h_{(2)}\right), \qquad \textit{for all } h \in H. \tag{4.20}$$

Proof If j is a cleaving map and $\chi \in \mathrm{Hom}_{L,L}(H, A)$ is convolution invertible, then Eq. 4.20 obviously defines a cleaving map. In order to prove the converse claim, suppose first that both j and j' are normalised as in the proof of Theorem 4.12. By Theorem 4.12, the crossed products corresponding to j and j' are isomorphic to A via left B-module right $\mathscr{H}$-comodule algebra maps, hence they are equivalent to each other. The isomorphism, obtained from combining the maps ν (for j') with κ (for j) in the proof of Theorem 3.12 (1) $\Rightarrow$ (2)(b), explicitly comes out as $\Phi : b \otimes_L h \mapsto b j'(h_{(1)}{}^{(1)}) j^{\mathsf{c}}(h_{(1)}{}^{(2)}) \otimes_L h_{(2)}$. Then, by Theorem 4.7, the existence of Φ is equivalent to the existence of a normalised convolution invertible map $\chi \in \mathrm{Hom}_{L,L}(H, B)$, $\chi(h) = j'(h^{(1)}) j^{\mathsf{c}}(h^{(2)})$. Using the right $\mathscr{H}_L$-colinearity of γ_R and the fact that j^{c} is a left convolution inverse of j, one finds, for all $h \in H$, $\chi(h_{(1)}) j(h_{(2)}) = j'(h)$, i.e. Eq. 4.20 holds. Allowing for j, j' to be non-unital is equivalent to not requiring that χ be normalised. □

5 The Relative Chern–Galois Character for $\mathscr{H}$-Cleft Extensions

The aim of this section is to give a complete description of strong connections in a cleft extension $B \subseteq A$ of a Hopf algebroid $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ (over rings L and R)

and to find criteria for the existence and independence on the strong connection of the corresponding relative Chern–Galois characters introduced and computed in [9].

Begin with a right $\mathscr{H}$-extension $B \subseteq A$ and suppose that T is a subalgebra of B. Then A is called an *$(\mathscr{H}_R, T)$-projective left B-module* provided there exists a left B-linear, right $\mathscr{H}_R$-colinear section α_T of the multiplication map $B \otimes_T A \to A$. To consider the most general case possible, we make no assumptions on a ring T (but, possibly, the most natural choice for T is the base algebra L).

Lemma 5.1 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathscr{H}$-cleft extension. Then A is an $(\mathscr{H}_R, L)$-projective left B-module.*

Proof The map $\tilde{\alpha}_L : B \otimes_L H \to B \otimes_L B \otimes_L H$, $b \otimes_L h \mapsto b \otimes_L 1_B \otimes_L h$ is a left B-linear right $\mathscr{H}$-colinear splitting of the product map $b \otimes_L b' \otimes_L h \mapsto bb' \otimes_L h$. By Theorem 3.12, $A \simeq B \otimes_L H$ as left B-modules and right $\mathscr{H}$-comodules, hence there is a corresponding splitting α_L of the B-product map in A. Explicitly,

$$\alpha_L = \left(B \underset{L}{\otimes} \kappa^{-1}\right) \circ \tilde{\alpha}_L \circ \kappa, \quad a \mapsto a_{[0]}{}^{[0]} j^{\mathsf{c}}\left(a_{[0]}{}^{[1]}\right) \underset{L}{\otimes} j\left(a_{[1]}\right) = a^{[0]} j^{\mathsf{c}}\left(a^{[1]}{}_{(1)}\right) \underset{L}{\otimes} j\left(a^{[1]}{}_{(2)}\right),$$

where κ is the isomorphism (3.12) in the proof of Theorem 3.12 and j is a cleaving map with the convolution inverse j^{c}. □

Any right $\mathscr{H}$-comodule algebra A with $\mathscr{H}_R$-coaction ρ^A gives rise to an entwining map (over R) $\psi : H \otimes_R A \to A \otimes_R H$, $h \otimes_R a \mapsto a^{[0]} \otimes_R ha^{[1]}$. The map ψ is bijective, provided the antipode S is bijective (cf. [8, Lemma 4.1]), and then the corresponding left $\mathscr{H}_R$-coaction on A is

$$^{A}\rho : A \to H \underset{R}{\otimes} A, \qquad a \mapsto S^{-1}\left(a_{[1]}\right) \underset{R}{\otimes} a_{[0]} \tag{5.1}$$

(compare with (2.26)). Thus, following [9, Definition 3.4], if $B \subseteq A$ is a right $\mathscr{H}$-extension and T is a subalgebra of B, then a left and right $\mathscr{H}_R$-comodule map $\ell_T : H \to A \otimes_T A$ is a *strong T-connection* provided that $\widetilde{\mathrm{can}}_T\big(\ell_T(h)\big) = 1_A \otimes_R h$, for all $h \in H$, where the map

$$\widetilde{\mathrm{can}}_T : A \underset{T}{\otimes} A \to A \underset{R}{\otimes} H, \qquad a \underset{T}{\otimes} a' \mapsto aa'^{[0]} \underset{R}{\otimes} a'^{[1]}, \tag{5.2}$$

is well defined by the left T-linearity of ρ^A. The $\mathscr{H}_R$-coactions in $A \otimes_T A$ are $A \otimes_T \rho^A$ and $^{A}\rho \otimes_T A$, with $^{A}\rho$ given in (5.1). The first observation is that a cleft extension comes equipped with a strong L-connection.

Theorem 5.2 *Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode and $B \subseteq A$ an $\mathscr{H}$-cleft extension with a cleaving map j. Then the map*

$$\ell_L : H \to A \underset{L}{\otimes} A, \qquad h \mapsto j^{\mathsf{c}}\left(h_{(1)}\right) \underset{L}{\otimes} j\left(h_{(2)}\right), \tag{5.3}$$

is a strong L-connection.

Proof By Theorem 3.12, $B \subseteq A$ is a Galois $\mathscr{H}_R$-extension, which is $(\mathscr{H}_R, L)$-projective by Lemma 5.1. Thus the existence of a strong connection follows by [9, Theorem 3.7]. Using the explicit forms of the inverse of the canonical $\mathscr{H}_R$-Galois map in the proof of Theorem 3.12 and of α_L in the proof of Lemma 5.1, following the proof of [9, Theorem 3.7] one arrives at the form of a strong L-connection in (5.3). □

The full classification of strong T-connections in a cleft extension is described in the following

Theorem 5.3 *Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode and $B \subseteq A$ an $\mathscr{H}$-cleft extension, and let T be a subalgebra of B. Write μ_B for the multiplication map $B \otimes_T B \to B$. Strong T-connections in $B \subseteq A$ are in bijective correspondence with L-L bilinear maps $f : H \to B \otimes_T B$ such that $\mu_B \circ f = \eta_L \circ \pi_L$.*

Proof Let j be a cleaving map and j^{c} its convolution inverse. By Theorem 2.2, Theorem 3.12 and Corollary 3.13, there is a chain of isomorphisms

$$\mathrm{Hom}^{\mathscr{H}_R,\mathscr{H}_R}\left(H, A\underset{T}{\otimes}A\right) \simeq \mathrm{Hom}^{\mathscr{H}_L,\mathscr{H}_L}\left(H, H\underset{L}{\otimes}B\underset{T}{\otimes}B\underset{L}{\otimes}H\right) \simeq \mathrm{Hom}_{L,L}\left(H, B\underset{T}{\otimes}B\right),$$

where the last isomorphism is $\varphi \mapsto (\pi_L \otimes_L B \otimes_T B \otimes_L \pi_L) \circ \varphi$ (cf. [17, 18.10 (1) and 18.11 (1)]). In view of the explicit form of the isomorphism $\kappa : A \to B \otimes_L H$ in (3.12) and its left-handed version (3.13), we thus obtain:

$$\mathrm{Hom}^{\mathscr{H}_R,\mathscr{H}_R}\left(H, A\underset{T}{\otimes}A\right) \ni \ell_T \mapsto \left[f : h \mapsto j(h^{(1)})\ell_T(h^{(2)})j^{\mathsf{c}}(h^{(3)})\right], \tag{5.4}$$

and its inverse

$$\mathrm{Hom}_{L,L}\left(H, B\underset{T}{\otimes}B\right) \ni f \mapsto \left[\ell_T : h \mapsto j^{\mathsf{c}}\left(h_{(1)}\right) f\left(h_{(2)}\right) j\left(h_{(3)}\right)\right]. \tag{5.5}$$

If ℓ_T is a strong T-connection, then $\mu_A \circ \ell_T = \eta_R \circ \pi_R$, where $\mu_A : A \otimes_T A \to A$ is the product map in A. This implies that for the corresponding f in (5.4), $\mu_B \circ f = \eta_L \circ \pi_L$. Conversely, suppose that f has this property, write $f(h) = h^{\{1\}} \otimes_T h^{\{2\}}$ for all $h \in H$, and compute

$$\begin{aligned}
\widetilde{\mathrm{can}}_T\left(\ell_T(h)\right) &= j^{\mathsf{c}}\left(h_{(1)}\right) h_{(2)}{}^{\{1\}} h_{(2)}{}^{\{2\}} j\left(h_{(3)}\right)^{[0]} \underset{R}{\otimes} j\left(h_{(3)}\right)^{[1]} \\
&= j^{\mathsf{c}}\left(h_{(1)}\right) \eta_L\left(\pi_L\left(h_{(2)}\right)\right) j\left(h_{(3)}{}^{(1)}\right) \underset{R}{\otimes} h_{(3)}{}^{(2)} \\
&= j^{\mathsf{c}}\left(h_{(1)}\right) j\left(h_{(2)}{}^{(1)}\right) \underset{R}{\otimes} h_{(2)}{}^{(2)} = j^{\mathsf{c}}\left(h^{(1)}{}_{(1)}\right) j\left(h^{(1)}{}_{(2)}\right) \underset{R}{\otimes} h^{(2)} \\
&= \eta_R\left(\pi_R\left(h^{(1)}\right)\right) \underset{R}{\otimes} h^{(2)} = 1_A \underset{R}{\otimes} h,
\end{aligned}$$

where the first equality follows by (5.2) and the fact that the range of f is in $B \otimes_T B$. The second equality follows by the hypothesis $\mu_B \circ f = \eta_L \circ \pi_L$ and the right $\mathscr{H}$-colinearity of j. The third one follows by the left L-linearity of j (i.e. Eq. 3.3) and the counit property of π_L. The fourth equality follows by the left $\mathscr{H}_L$-colinearity of γ_R. In the penultimate step we used that j^{c} is a left convolution inverse of j, i.e. Eq. 3.6. This means that ℓ_T given in (5.5) is a strong T-connection and completes the proof of the theorem. □

Take a bijective entwining structure $(A, C, \psi)_R$ over a non-commutative base algebra R and consider a T-flat entwined extension $B \subseteq A$ in the sense of [9, Definition 5.2] (T is a subalgebra of B). Given a strong T-connection in $B \subseteq A$, one constructs a family of maps of Abelian groups from the Grothendieck group of C-comodules to the even T-relative cyclic homology groups of B (cf. [9, Theorem 5.4]). This family of maps is termed the *T-relative Chern–Galois character*. Comodule algebras for Hopf algebroids (with bijective antipodes) provide examples of (bijective) entwining structures over non-commutative bases, hence the general theory worked out in [9] can be applied to such algebras. In particular, the components of the T-relative Chern–Galois characters, corresponding to strong T-connections in Theorem 5.3 for a T-flat cleft extension of a Hopf algebroid with bijective antipode, have been computed in [9, Example 5.6].

It is important to note, however, that the T-relative Chern–Galois character, a priori, depends on the choice of a strong T-connection. Its independence is proven in [9, Theorem 5.14], under the assumption that the T-flat entwined extension $B \subseteq A$ enjoys the following properties:

(a) A is a locally projective right T-module;
(b) The extension $B \subseteq A$ splits as a B-T bimodule.

In the remainder of this section we analyse the meaning of these conditions and of the T-flatness in the case of cleft Hopf algebroid extensions. In this way we find sufficient conditions for the existence and the strong-connection-independence of the relative Chern–Galois character computed in [9, Example 5.6].

Definition 5.4 Let $B \subseteq A$ be a right extension of a Hopf algebroid $\mathscr{H}$ with a bijective antipode and let T be a subalgebra of B. View A as a left $\mathscr{H}_R$-comodule with coaction (5.1). A *left total T-integral* is a left $\mathscr{H}_R$-colinear map $\vartheta : H \to A$ such that $\vartheta(H) \subseteq A^T := \{\, a \in A \mid ta = at \quad \forall t \in T \,\}$ and $\vartheta(1_H) = 1_A$.

For example, the convolution inverse of a normalised cleaving map is a left total k-integral by Lemma 3.8.

By arguments similar to those used to prove Lemma 3.10, any left total T-integral ϑ determines a B-T bilinear section of the extension $B \subseteq A$,

$$a \mapsto a^{[0]}\vartheta\left(a^{[1]}\right). \tag{5.6}$$

The next proposition shows that, for a cleft extension of a Hopf algebroid with a bijective antipode, this is a one-to-one correspondence.

Proposition 5.5 *Let $B \subseteq A$ be a cleft extension of a Hopf algebroid $\mathscr{H}$ with a bijective antipode, and let T be a subalgebra of B. Then B-T bilinear sections of the extension $B \subseteq A$ are in bijective correspondence with left total T-integrals in $B \subseteq A$.*

Proof Let j be a cleaving map for $B \subseteq A$. In terms of j we construct the inverse of the map associating to a left total T-integral ϑ the section (5.6). To a B-T bilinear section φ, associate the map

$$\vartheta : H \to A, \qquad h \mapsto j^{\mathsf{c}}\left(h_{(1)}\right)\varphi\left(j\left(h_{(2)}\right)\right). \tag{5.7}$$

Since $j(1_H)$ is an element of B, $\vartheta(1_H) = j^{\mathsf{c}}(1_H)j(1_H)\varphi(1_A) = 1_A$. The left $\mathscr{H}_R$-colinearity of ϑ follows by the left $\mathscr{H}_R$-colinearity of γ_L and j^{c}, and the fact that the range of φ is equal to B. It remains to check that the range of ϑ is in A^T. Note that by its left B-linearity, φ is determined by the left L-linear map $\varphi \circ j : H \to B$. Indeed, for all $a \in A$,

$$\begin{aligned}\varphi(a) &= \varphi\left(a^{[0]} j^{\mathsf{c}}\left(a^{[1]}{}_{(1)}\right) j\left(a^{[1]}{}_{(2)}\right)\right) = \varphi\left(a_{[0]}{}^{[0]} j^{\mathsf{c}}\left(a_{[0]}{}^{[1]}\right) j\left(a_{[1]}\right)\right)\\ &= a_{[0]}{}^{[0]} j^{\mathsf{c}}\left(a_{[0]}{}^{[1]}\right) \varphi\left(j\left(a_{[1]}\right)\right),\end{aligned}$$

where in the last equality Lemma 3.10 has been used. Hence the right T-linearity of φ is equivalent to

$$a^{[0]} t j^{\mathsf{c}}\left(a^{[1]}{}_{(1)}\right) \varphi\left(j\left(a^{[1]}{}_{(2)}\right)\right) = \varphi(a)t, \qquad \text{for all } a \in A \text{ and } t \in T. \tag{5.8}$$

Take any $h \in H$ and apply Eq. 5.8 to $a = j(h)$. By the right $\mathscr{H}_R$-colinearity of j,

$$j\left(h^{(1)}\right) t j^{\mathsf{c}}\left(h^{(2)}{}_{(1)}\right) \varphi\left(h^{(2)}{}_{(2)}\right) = \varphi(j(h))t.$$

Hence

$$\begin{aligned}\vartheta(h)t &= j^{\mathsf{c}}\left(h_{(1)}\right) j\left(h_{(2)}{}^{(1)}\right) t j^{\mathsf{c}}\left(h_{(2)}{}^{(2)}{}_{(1)}\right) \varphi\left(j\left(h_{(2)}{}^{(2)}{}_{(2)}\right)\right)\\ &= \eta_R\left(\pi_R\left(h^{(1)}\right)\right) t\vartheta\left(h^{(2)}\right) = t\vartheta(h),\end{aligned}$$

where the second equality follows by the Hopf algebroid axiom (2.7) and the last one follows by the fact that the elements of B (and hence, in particular, the elements of T) commute with $\eta_R(r)$, for $r \in R$, and by the left R-linearity of ϑ. It is checked by a routine computation that the map, associating to a B-T bilinear section φ of the inclusion $B \subseteq A$ the left total T-integral (5.7), is the inverse of the map, associating to the left total T-integral ϑ the B-T bilinear section (5.6). □

For a cleft extension $B \subseteq A$ of a Hopf algebroid $\mathscr{H}$ with a bijective antipode, consider the B-B bilinear map,

$$A \to A \underset{R}{\otimes} H, \qquad a \mapsto a^{[0]} \underset{R}{\otimes} a^{[1]} - a \underset{R}{\otimes} 1_H,$$

where $A \otimes_R H$ is a B-B bimodule via the first tensorand. For any subalgebra T of B this map projects to the map $\upsilon_T : A/[A, T] \to A/[A, T] \otimes_R H$, where $[A, T] = \left\{\sum_k (a_k t_k - t_k a_k) \mid a_k \in A,\ t_k \in T\right\}$ is a right R-submodule of A. Following [9, Definition 5.2], the extension $B \subseteq A$ is said to be *T-flat* if B and A are flat left and right T-modules and the obvious map

$$B/[B, T] \to \ker \upsilon_T, \qquad [b]_B \mapsto [b]_A, \tag{5.9}$$

(where $[\ ]_B$ denotes the equivalence class in $B/[B, T]$ and $[\ ]_A$ denotes the equivalence class in $A/[A, T]$) is an isomorphism.

Proposition 5.6 *Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode. A cleft $\mathscr{H}$-extension $B \subseteq A$ which splits as a B-T bimodule for some subalgebra T of B, is T-flat if and only if A is a flat left and right T-module.*

Proof Since a direct summand of a flat module is flat, it suffices to prove that the existence of a B-T bimodule splitting of the inclusion, i.e. the existence of a left total T-integral, implies that the map (5.9) is an isomorphism. In order to prove injectivity of the map in (5.9), choose $b \in B$ such that $[b]_A = 0$. This means the existence of finite sets $\{a_k\}$ in A and $\{t_k\}$ in T such that $b = \sum_k (a_k t_k - t_k a_k)$. Applying a B-T bilinear section φ of the extension $B \subseteq A$ to this identity, we obtain $b = \sum_k (\varphi(a_k) t_k - t_k \varphi(a_k))$, hence $[b]_B = 0$. In order to prove the surjectivity of the map (5.9), choose $a \in A$ such that $\upsilon_T([a]_A) = 0$. This means the existence of finite sets $\{a_k\}$ in A, $\{h_k\}$ in H and $\{t_k\}$ in T such that

$$a^{[0]} \underset{R}{\otimes} a^{[1]} - a \underset{R}{\otimes} 1_H = \sum_k \left(a_k t_k \underset{R}{\otimes} h_k - t_k a_k \underset{R}{\otimes} h_k \right). \tag{5.10}$$

By Proposition 5.5, there is a left total T-integral ϑ in $B \subseteq A$. Apply $\mu_A \circ (A \otimes_R \vartheta)$ to Eq. 5.10 to obtain

$$a^{[0]} \vartheta(a^{[1]}) - a = \sum_k (a_k \vartheta(h_k) t_k - t_k a_k \vartheta(h_k)).$$

This proves that $[a]_A = [a^{[0]} \vartheta(a^{[1]})]_A$. Since $a^{[0]} \vartheta(a^{[1]})$ is an element of B, $[a]_A$ belongs to the image of the map (5.9). □

Combining Proposition 5.5 and Proposition 5.6 with [9, Theorem 5.14] we obtain

Corollary 5.7 *Let $B \subseteq A$ be a cleft extension of a Hopf algebroid with a bijective antipode and let T be a subalgebra of B. Assume that:*

(a) *A is a flat left T-module and a locally projective right T-module;*
(b) *There exists a left total T-integral for the extension $B \subseteq A$;*
(c) *There exists a strong T-connection.*

Then there exists a T-relative Chern–Galois character, independent on the choice of the strong T-connection in (c).

We close the section with some examples of Hopf algebroid cleft extensions, in which there exist (strong-connection-independent) relative Chern–Galois characters.

Example 5.8 Let $B \subseteq A$ be a cleft extension of a Hopf algebroid with a bijective antipode and T a *separable* k-subalgebra of B. In light of [9, Proposition 3.2 (1)], since $B \subseteq A$ splits as a left B-module by Lemma 3.11, it splits as a B-T bimodule. The corresponding total T-integral ϑ in $B \subseteq A$ is

$$\vartheta(h) = \sum_i e_i j^{\mathsf{c}}(h) j(1_H) e^i,$$

where $\sum_i e_i \otimes_k e^i \in T \otimes_k T$ is a separability idempotent. Therefore, if A is a flat left T-module and a locally projective right T-module and there exists a strong T-connection ℓ_T, then there exists a corresponding T-relative Chern–Galois character which is independent of ℓ_T.

Example 5.9 The base algebra R of a Hopf algebroid $\mathscr{H}$ is a right $\mathscr{H}$-comodule algebra with $\mathscr{H}_R$-coaction s_R. It follows by [7, Theorem 3.2] that a Hopf algebroid $\mathscr{H}$ with a bijective antipode is coseparable (as an L- or, equivalently, as an R-coring) if and only if there exists a left total k-integral λ for the $\mathscr{H}$-extension $I \subseteq R$, where I is the $\mathscr{H}$-coinvariant subalgebra of R, $I = \{ r \in R \mid s_R(r) = t_R(r) \}$. Note that if $\mathscr{H}$ is a coseparable Hopf algebroid, then any right $\mathscr{H}$-extension $B \subseteq A$ is split as a B-B bimodule by the map

$$A \to B, \qquad a \mapsto a^{[0]} \eta_R \left(\lambda \left(a^{[1]} \right) \right).$$

Let $B \subseteq A$ be a cleft extension of a *coseparable* Hopf algebroid with a bijective antipode and T a subalgebra of B. Then there is a left total T-integral $\vartheta = \eta_R \circ \lambda$. If A is a locally projective right T-module and a flat left T-module and there exists a strong T-connection ℓ_T, then the corresponding T-relative Chern–Galois character is independent of ℓ_T.

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Appendix: Weak Cleft Extensions and Weak Crossed Products

Many of the results described in Sections 3, 4 and 5 (e.g. Lemma 3.11, Corollary 3.14, Corollary 3.15 or Theorem 5.2, Corollary 5.7) remain valid if the right $\mathscr{H}$-extension $B \subseteq A$ is an $\mathscr{H}_R$-Galois extension but, instead Theorem 3.12 (2)(b), A is only a direct summand of $B \otimes_L H$ as a left B-module and as a right $\mathscr{H}$-comodule. Such extensions can be studied along the same lines as in Sections 3, 4 and 5. In this appendix we present the results of such studies; we give no proofs as these are very similar to the proofs of corresponding results in preceding sections.

Motivated by the forthcoming analogue of Theorem 3.12 (Theorem A.2), we introduce the following weakening of Definition 3.5.

Definition A.1 Let $\mathscr{H}$ be a Hopf algebroid. A right $\mathscr{H}$-extension $B \subseteq A$ is *weak cleft* if

(a) in addition to its canonical R-ring structure, A possesses an L-ring structure and B is an L-subring of A;
(b) there exists a left L-linear right $\mathscr{H}$-colinear morphism $j : H \to A$, with left convolution inverse j^{w}, which is right $\mathscr{H}$-colinear in the sense of identities (3.8) and (3.9).

A map j, satisfying condition (b), is called a *weak cleaving map*.

Note that in the situation described in Definition A.1, the assumption that j^{w} satisfies Eq. 3.9 implies that the image of the map $A \to A$, $a \mapsto a^{[0]} j^{\mathsf{w}}(a^{[1]})$ is contained in B. Hence a weak $\mathscr{H}$-cleft extension $B \subseteq A$ is split by the left B-linear map (3.11) after replacing j^{c} with j^{w}.

Theorem A.2 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ a right $\mathscr{H}$-extension. Then the following statements are equivalent:*

(1) *$B \subseteq A$ is a weak $\mathscr{H}$-cleft extension.*
(2) (a) *The extension $B \subseteq A$ is $\mathscr{H}_R$-Galois;*
    (b) *A is a direct summand of $B \otimes_L H$ as a left B-module and right $\mathscr{H}$-comodule.*

In particular, Theorem A.2 implies that if $\mathscr{H}$ is projective as a left L-module, then, for any weak cleft $\mathscr{H}$-extension $B \subseteq A$, A is a faithfully flat left B-module.

Recall that in Section 4 we applied a (unnormalised) gauge transformation to a general cleaving map in order to normalise it as $j(1_H) = 1_B = j^{\mathsf{c}}(1_H)$. However, in the case when j possesses a left convolution inverse j^{w} only, there is no guarantee for $j(1_H)$ to be an invertible element of B. Hence it can not be gauge transformed to the unit element in general. The need to describe this more general situation leads to the following generalisations of Definitions 4.1 and 4.2.

Definition A.3 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring. $\mathscr{L}$ *weakly measures* B if there exits a k-linear map, termed a *weak measuring*, $H \otimes_k B \to B$, $h \otimes_k b \mapsto h \cdot b$ that satisfies properties (b) and (c) in Definition 4.1.

Definition A.4 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring, weakly measured by $\mathscr{L}$. A B-valued *weak 2-cocycle* σ on $\mathscr{L}$ is a k-linear map $H \otimes_{L^{op}} H \to B$ (where the right and left L^{op}-module structures on H are given by right and left multiplication by $t(l)$, respectively) satisfying properties (a), (b) and (d) in Definition 4.2 and, in addition, for all $h, k \in H$,

$$\sigma(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \cdot 1_B) = \sigma(h, k). \tag{5.11}$$

A weakly $\mathscr{L}$-measured L-ring B is called a *σ-twisted left $\mathscr{L}$-module* if a weak 2-cocycle σ satisfies property (f) in Definition 4.2 and there exist elements x and $\tilde{x}$ in B such that, for all $b \in B$ and $h \in H$,

$$\tilde{x}x = 1_B \qquad \text{and} \qquad xb\tilde{x} = 1_H \cdot b,$$
$$\sigma(1_H, h) = x(h \cdot 1_B) \qquad \text{and} \qquad \sigma(h, 1_H) = h \cdot x.$$

It is easy to see that a B-valued 2-cocycle is also a weak 2-cocycle. If the L-ring B is a σ-twisted $\mathscr{L}$-module for a 2-cocycle σ, then it is a σ-twisted $\mathscr{L}$-module also in the weaker sense of Definition A.4 with $x = 1_B = \tilde{x}$.

Recall from [18, p. 39] that, for a non-unital ring A, an element $e \in A$ such that, for all $a \in A$, $ea = ae = ae^2$ is called a *preunit*. Proposition 4.3 can be extended to the case of weak cocycles as follows.

Proposition A.5 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring, weakly measured by $\mathscr{L}$. Let $\sigma : H \otimes_{L^{op}} H \to B$ be a map that satisfies properties* (a) *and* (b) *in Definition* 4.2 *and condition* (5.11) *in Definition* A.4. *Consider the k-module $B \otimes_L H$ in Proposition* 4.3, *and the following assertions.*

(a) *$B \otimes_L H$ is an associative (possibly non-unital)algebra with multiplication* (4.1).
(b) *There exists $\tilde{y} \in B$ such that $\tilde{y}\sigma(1_H, h) = h \cdot 1_B$, for all $h \in H$, and $\tilde{y} \otimes_L 1_H$ is a preunit for the algebra in part* (a) *(hence $A := \{(b \otimes_L h)(\tilde{y} \otimes_L 1_H) \mid b \otimes_L h \in B \otimes_L H\}$ is a right $\mathscr{L}$-comodule algebra with coinvariant subalgebra $\{(b \otimes_L 1_H)(\tilde{y} \otimes_L 1_H) \mid b \in B\}$).*
(c) *The map $B \to A^{co\mathscr{L}}$, $b \mapsto (b\,\tilde{y} \otimes_L 1_H)(\tilde{y} \otimes_L 1_H)$ is an algebra isomorphism.*

These assertions hold if and only if σ is a weak 2-cocycle and B is a σ-twisted left $\mathscr{L}$-module. In this case A is called a weak crossed product *of B with $\mathscr{L}$ with respect to the weak 2-cocycle σ.*

Our next task is to characterise equivalent weak crossed products, in analogy with Theorem 4.7.

Definition A.6 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring. Weak crossed product algebras of B with $\mathscr{L}$ are said to be *equivalent* if they are isomorphic via a left B-linear isomorphism of right $\mathscr{L}$-comodule algebras.

Note that a (left B-linear right $\mathscr{L}$-colinear) isomorphism of weak crossed product algebras of B with $\mathscr{L}$ in Definition A.5 needs not extend to the (non-unital) algebra $B \otimes_L H$. The following lemma extends Corollary 4.5.

Lemma A.7 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring, weakly measured by $\mathscr{L}$. Let σ be a weak 2-cocycle, such that B is a σ-twisted left $\mathscr{L}$-module. Let χ and $\tilde{\chi}$ be morphisms in* $\mathrm{Hom}_{L,L}(H, B)$ *such that, for all $h \in H$,*

$$\tilde{\chi}(h_{(1)})\chi(h_{(2)}) = h \cdot 1_B \qquad \textit{and} \tag{5.12}$$

$$\tilde{\chi}(h_{(1)})\chi(h_{(2)})\tilde{\chi}(h_{(3)}) = \tilde{\chi}(h), \quad \chi(h_{(1)})\tilde{\chi}(h_{(2)})\chi(h_{(3)}) = \chi(h). \tag{5.13}$$

Then Eq. 4.12 defines a weak measuring of $\mathscr{L}$ on B and Eq. 4.13 defines a weak 2-cocycle σ^{χ}, such that B is a σ^{χ}-twisted left $\mathscr{L}$-module.

A pair $\chi, \tilde{\chi} \in \mathrm{Hom}_{L,L}(H, B)$, satisfying Eqs. 5.12 and 5.13, is called a *gauge transformation* of the weak crossed product of B with $\mathscr{L}$. Gauge transformations form a groupoid, with multiplication, the convolution product $\diamond$ in the first component, and its opposite in the second one. The left unit of a gauge transformation $(\chi, \tilde{\chi})$ is $(\chi \diamond \tilde{\chi}, \chi \diamond \tilde{\chi})$ and its right unit is $(\tilde{\chi} \diamond \chi, \tilde{\chi} \diamond \chi)$. The inverse of $(\chi, \tilde{\chi})$ is $(\tilde{\chi}, \chi)$.

Theorem A.8 *Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring. Two weak crossed products of B with $\mathscr{L}$ are equivalent if and only if they are related by a gauge transformation.*

In order to make connection between weak crossed products and weak cleft extensions, the notion of invertible weak 2-cocycles is needed.

Definition A.9 Let $\mathscr{L} = (H, L, s, t, \gamma, \pi)$ be a left bialgebroid and B an L-ring weakly measured by $\mathscr{L}$. An *inverse* of a B-valued weak 2-cocycle σ on $\mathscr{L}$ is a k-linear map, $\tilde{\sigma} : H \otimes_L H \to B$ (where the right and left L-module structures on H are given by right and left multiplication by $s(l)$, respectively) satisfying properties (a), (b) and (c) in Definition 4.8 and, in addition, for all $h, k \in H$,

$$\left(h_{(1)}k_{(1)} \cdot 1_B\right) \tilde{\sigma}\left(h_{(2)}, k_{(2)}\right) = \tilde{\sigma}(h, k).$$

If σ is a 2-cocycle in the sense of Definition 4.2 (in particular the measuring satisfies also property (a) in Definition 4.1), then Definition A.9 is equivalent to Definition 4.8. By an argument similar to the proof of Lemma 4.9, the convolution inverse of a weak 2-cocycle is unique, provided it exists. A generalisation of Theorems 4.11 and 4.12 is given in the following

Theorem A.10 *Let $\mathscr{H}$ be a Hopf algebroid. A right $\mathscr{H}$-extension $B \subseteq A$ is weak $\mathscr{H}$-cleft if and only if A is isomorphic to a weak crossed product of B with the constituent left bialgebroid $\mathscr{H}_L$ of $\mathscr{H}$, with respect to an invertible weak 2-cocycle.*

Analogously to Corollary 4.13, Theorems A.10 and A.8 lead to the following

Corollary A.11 *Let $\mathscr{H}$ be a Hopf algebroid and $B \subseteq A$ a weak$\mathscr{H}$-cleft extension. Let $j : H \to A$ be a weak cleaving map with left convolution inverse j^{w}, satisfying conditions (3.8) and (3.9). Then the map $h \cdot b :\ = j(h^{(1)}) b\, j^{\mathsf{w}}(h^{(2)})$, for all $b \in B$ and $h \in H$, is a weak measuring of the constituent left bialgebroid $\mathscr{H}_L$ on B. A map $j' : H \to A$ is a weak cleaving map if and only if there exist morphisms $\chi, \tilde{\chi} \in$ $\mathrm{Hom}_{L,L}(H, B)$, satisfying Eqs. 5.12 and 5.13, in terms of which $j' : h \mapsto \chi(h_{(1)})\, j(h_{(2)})$.*

Let $\mathscr{H}$ be a Hopf algebroid with a bijective antipode and let $B \subseteq A$ be a weak $\mathscr{H}$-cleft extension and T a k-subalgebra of B. Let j be a weak cleaving map with a left convolution inverse j^{w}, satisfying conditions (3.8) and (3.9). Any morphism $f \in \mathrm{Hom}_{L,L}(H, B \otimes_T B)$ such that, for all $h \in H$, $\mu_B\big(f(h)\big) = j(h^{(1)})\, j^{\mathsf{w}}(h^{(2)})$, determines a strong$T$-connection via (5.5). Conversely, any strong T-connection is of this form (though the correspondence (5.5) is not bijective in the weak case).

Any B-T bimodule section of a weak cleft Hopf algebroid extension $B \subseteq A$, for a subalgebra$T$ of B, corresponds to a left total T-integral via (5.6) (although the correspondence between B-T sections and total integrals is not bijective in the weak case). Hence Corollary 5.7 is valid without modification for weak cleft extensions of Hopf algebroids with bijective antipode.

A weak Hopf algebra (W, Δ, ϵ, S) determines a Hopf algebroid $\mathscr{W}$ with constituent left bialgebroid $\mathscr{W}_L$ over the 'left' subalgebra W^L of W, right bialgebroid $\mathscr{W}_R$ over the 'right' subalgebra W^R, and antipode S. The category of right comodules for the coalgebra (W, Δ, ϵ) is isomorphic to the category of right $\mathscr{W}$-comodules as a monoidal category. As a consequence, also the respective notions of comodule algebras and of coinvariants are equivalent (cf. [14]). Let A be a right W- (or, equivalently, $\mathscr{W}$-) comodule algebra with coinvariants B. By [3, Theorem 2.11], the extension $B \subseteq A$ is W-cleft (i.e. the corresponding weak entwining structure is cleft in the sense of [2, Definition 1.9]) if and only if it is W-Galois and A is a direct summand of $B \otimes_k W$ as a left B module right W-comodule. By [8, Example 3.5] the$W$-Galois property is equivalent to the $\mathscr{W}_R$-Galois property, hence Theorem A.2 implies that any weak $\mathscr{W}$-cleft extension is weak W-cleft (but not conversely).

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