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Numerical analysis for viscoelastic fluid flow with distributed/variable order time fractional Maxwell constitutive models*

Yanli QIAO, Xiaoping WANG, Huanying XU, Haitao QI†

School of Mathematics and Statistics, Shandong University,

Weihai 264209, Shandong Province, China

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Abstract Fractional calculus has been widely used to study the flow of viscoelastic fluids recently, and fractional differential equations have attracted a lot of attention. However, the research has shown that the fractional equation with constant order operators has certain limitations in characterizing some physical phenomena. In this paper, the viscoelastic fluid flow of generalized Maxwell fluids in an infinite straight pipe driven by a periodic pressure gradient is investigated systematically. Consider the complexity of the material structure and multi-scale effects in the viscoelastic fluid flow. The modified time fractional Maxwell models and the corresponding governing equations with distributed/variable order time fractional derivatives are proposed. Based on the L_1-approximation formula of Caputo fractional derivatives, the implicit finite difference schemes for the distributed/variable order time fractional governing equations are presented, and the numerical solutions are derived. In order to test the correctness and availability of numerical schemes, two numerical examples are established to give the exact solutions. The comparisons between the numerical solutions and the exact solutions have been made, and their high consistency indicates that the present numerical methods are effective. Then, this paper analyzes the velocity distributions of the distributed/variable order fractional Maxwell governing equations under specific conditions, and discusses the effects of the weight coefficient $\varpi(\alpha)$ in distributed order time fractional derivatives, the order $\alpha(r,t)$ in variable fractional order derivatives, the relaxation time λ, and the frequency ω of the periodic pressure gradient on the fluid flow velocity. Finally, the flow rates of the distributed/variable order fractional Maxwell governing equations are also studied.

Key words distributed order time fractional derivative, variable order time fractional derivative, finite difference scheme, viscoelastic fluid

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† Corresponding author, E-mail: htqi@sdu.edu.cn

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1 Introduction

Fluids as a basic form of matter are widespread in nature and play an important role in industry and social life. The research on fluids has drawn great interest of numerous scholars and is successfully applied to many fields, such as oil transportation, sewage treatment, environmental protection, electricity industry, and blood circulation. It has been found that the constitutive relation of fluids is crucial. The first constitutive equation known as the Newtonian model is proposed to analyze the flow mechanism of Newtonian fluids. However, it has revealed some limitations for the complex fluid flow in practical engineering applications, which shows the characteristics of non-Newtonian fluids. Thus, it is necessary to seek a more suitable constitutive equation to explain the non-Newtonian behaviors of viscoelastic fluids which always exhibit viscosity and elasticity. Taking fully into account the viscoelasticity of fluids, many classical constitutive models have been proposed, including Maxwell, Kelvin, Voigt, Zener, and Burgers. Based on this, the non-Newtonian fluid mechanics have developed rapidly during the past few years. However, with the further studies of viscoelastic fluids, an increasing body of evidence suggests that the classical constitutive models are unable to characterize some complex phenomena, while the fractional models can well describe the viscoelastic behaviors of these materials[1–5]. Therefore, it is natural to introduce fractional calculus to characterize the flow processes of the viscoelastic fluids[3, 6].

There are two methods for the generalization of the classical models to the fractional models. The simplest one is to replace integral derivatives with fractional derivatives directly. The other is based on the fractional element model proposed by Scott-Blair[7], which is easy to understand physically. According to the fractional element model, the fractional Maxwell model, the fractional Zener model, and the fractional Kelvin-Voigt model[2, 5, 8] have been presented. With the help of these fractional models, researchers were devoted to studying the rheological process of viscoelastic fluids in different environments[9–10] and began to promote the applications of viscoelastic fluids in various fields. In recent years, the classical fractional viscoelastic model is still a concerned subject. For example, in view of the fractional Maxwell model, Zhang et al.[11] analyzed the unsteady two-dimensional magnetohydrodynamic flow of fractional Maxwell fluids with the variable pressure gradient in a rectangular pipe. Wang et al.[12] studied the rotating electro-osmotic flow of fractional Maxwell fluids in a parallel plate microchannel with high zeta potentials and proposed an effective numerical algorithm to solve the fractional Maxwell governing equation with the finite difference method. In addition, there are also many investigations for other fractional models[13–16]. All these works make the applications of non-Newtonian fluid mechanics to a higher degree.

In order to describe practical physical problems accurately, the flow of viscoelastic fluids in various pipes has also been continuously investigated with classical fractional models. Yang and Zhu[17] examined the start-up flow of viscoelastic fluids in an infinite long straight pipe based on the fractional Maxwell model. Wang and Zhao[18] studied the transient electro-osmotic flow of generalized Maxwell fluids with a fractional derivative in a narrow capillary tube. Nevertheless, in the wake of developments in industrial technology and biotechnology, viscoelastic materials that we often have to deal with are more complicated and diverse. In these processes, there are still some special situations wherein the nonlinear behaviors of many viscoelastic materials cannot be depicted by constant order operators correctly, e.g., the relaxation processes and reaction kinetics of proteins[19] and the anomalous diffusion and relaxation phenomena in inhomogeneous media[20–21]. Then, the modified fractional models with distributed/variable order derivatives were presented based on the physical mechanism of materials, and the related works have been carried out[22–24]. However, it is difficult to obtain analytical solutions of distributed/variable order fractional models. Therefore, numerical methods have been proposed to analyze many complicated physical processes[25–28]. Moreover, we have noticed that the model with distributed/variable order fractional operators was used to study the flow of viscoelastic fluids. Liu et al.[26] discussed the fluid boundary flow and heat transfer for generalized Maxwell fluids with the distributed order time fractional equations. Yang et al.[27] studied the flow and heat transfer in the boundary layer with the distributed order space constitution equations.

Recently, the viscoelastic fluid flow has been a hot topic with the development of the engineering technology[29] and has a major application in hemorheology[30], whose objective is to study the changes of flow parameters characterizing the blood circulation system under physiological and pathological conditions. As we all know, blood is a kind of non-Newtonian fluids consisting of plasma, blood cells, and platelets, and blood cells have different shapes and complex structures. Blood flow is an important factor affecting the life activities of human body. Therefore, it is of great significance to study the blood flow and its changes through capillary vessels, which are produced by a period pressure gradient. However, considering the inherent properties of blood, it is very difficult and inappropriate to describe the blood flow in blood vessels with fractional models. Moreover, the typical feature of blood flow is regarded as the viscoelastic fluid flow which depends on time, space, pressure gradient, or some other variables. This shows that the blood flow problems would be better described by distributed/variable order operators. To the best of our knowledge, the current research on this is still relatively rare.

In this paper, the purpose is to investigate the flow of viscoelastic fluids induced by a periodic pressure gradient, such as blood flow. The flow is studied in an infinite straight pipe. The modified fractional Maxwell models are considered to characterize the viscoelastic fluid, and the time fractional constitutive equations with distributed/variable order derivatives are proposed based on the fractional Maxwell equation presented by Friedrich[8],

$$\sigma + \lambda^{\alpha}\, {}_0\mathrm{D}_t^{\alpha}\sigma = \eta\frac{\partial \varepsilon}{\partial t}, \quad 0 < \alpha < 1, \tag{1}$$

where σ represents the shear stress, λ refers to the relaxation time, η denotes the viscosity coefficient of the fluid, ε represents the shear strain, α is the order, and the operator ${}_0\mathrm{D}_t^{\alpha}$ is the Caputo fractional derivative with respect to t , defined as[31]

$$ {}_0\mathrm{D}_t^{\alpha}\sigma = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial \sigma}{\partial \tau}\mathrm{d}\tau. \tag{2}$$

The distributed/variable order time fractional Maxwell governing equations are given, and the numerical solutions are calculated with the finite difference method. Meanwhile, two numerical examples are constructed to prove the effectiveness of the numerical methods. Then, we discuss the effects of parameters on the velocity distributions of viscoelastic fluids and analyze the variations of flow rates with time. Finally, some conclusions are given.

2 Governing equation for generalized Maxwell fluids

The flow of the generalized fractional Maxwell fluid driven by the periodic pressure gradient in an infinite straight pipe with the radius of R is investigated. The cylindrical coordinate system (r, θ, z) is built, and the z-axis is taken along the axis of the tube. Assume that the fluid is stationary at the initial moment and has only the velocity component of z-direction.

For incompressible viscoelastic fluids, the continuity equation satisfies $\operatorname{div}\boldsymbol{V} = 0$, where $\boldsymbol{V}$ is the velocity vector. Given the above assumptions, the velocity vector can be defined as $\boldsymbol{V} = (0, 0, u(r, t))$. In the absence of body forces, the momentum equation in the cylindrical coordinate system is reduced to

$$\rho\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial(r\sigma_{rz})}{\partial r}-\frac{\partial p}{\partial z},\tag{3}$$

where σ_{rz} is the shear stress, and ρ and p represent the density and the pressure, respectively. Here, we consider a pressure gradient changing periodically with time[10], i.e., $\frac{\partial p}{\partial z}=-P_0\cos(\omega t)$, where P_0 refers to the amplitude of the pressure gradient, and ω denotes the frequency.

2.1 Distributed order time fractional governing equation

Here, the distributed order time fractional operator $\mathbb{D}_{t:[0,1]}^{\varpi(\alpha)}u(r,t)$ is introduced to characterize the features of complex inhomogeneous viscoelastic fluids in an infinite straight pipe, which is defined as[32]

$$\mathbb{D}_{t:[0,1]}^{\varpi(\alpha)}u(r,t)=\int_0^1\varpi(\alpha)\,{}_0\mathrm{D}_t^{\alpha}u(r,t)\mathrm{d}\alpha,\tag{4}$$

where $\varpi(\alpha)$ refers to the continuous weight function and satisfies $\varpi(\alpha)\geqslant 0$, $\int_0^1\varpi(\alpha)\mathrm{d}\alpha=W$, and W is a positive constant. For generalized fractional Maxwell fluids, the modified constitutive equation with distributed order fractional derivatives is proposed based on Eq. (1),

$$\sigma_{rz}+\int_0^1\varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{\alpha}\sigma_{rz}\mathrm{d}\alpha=\eta\frac{\partial u}{\partial r}.\tag{5}$$

Combined with Eq. (3), the distributed order time fractional governing equation for the flow of generalized Maxwell fluids produced by the periodic pressure gradient is obtained as

$$\begin{aligned}&\rho\int_0^1\varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{1+\alpha}u\mathrm{d}\alpha+\rho\frac{\partial u}{\partial t}-\frac{\eta}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)\\&=P_0\int_0^1\varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{\alpha}\cos(\omega t)\mathrm{d}\alpha+P_0\cos(\omega t),\end{aligned}\tag{6}$$

and the initial and boundary conditions are

$$u(r,0)=\frac{\partial u(r,0)}{\partial t}=0,\quad 0\leqslant r\leqslant R,\tag{7}$$

$$\frac{\partial u(0,t)}{\partial r}=0,\quad u(R,t)=0,\quad t>0.\tag{8}$$

For $\varpi(\alpha)=\delta(\beta-\alpha)$, $0\leqslant\beta\leqslant 1$, Eq. (6) will reduce to the fractional Maxwell governing equation with constant order operators.

Introduce the following non-dimensional variables:

$$r^*=\frac{r}{R},\quad u^*=\frac{\eta}{P_0R^2}u,\quad t^*=\frac{\eta}{\rho R^2}t,\quad \lambda^*=\frac{\eta}{\rho R^2}\lambda,\quad \omega^*=\frac{\rho R^2}{\eta}\omega.\tag{9}$$

Then, substitute them into Eqs. (6)–(8). Thus, we obtain the non-dimensional distributed order time fractional governing equation and the initial-boundary conditions (for convenience, the symbol "*" is omitted).

$$\int_0^1\varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{1+\alpha}u\mathrm{d}\alpha+\frac{\partial u}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)=\int_0^1\varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{\alpha}\cos(\omega t)\mathrm{d}\alpha+\cos(\omega t),\tag{10}$$

$$u(r,0)=\frac{\partial u(r,0)}{\partial t}=0,\quad 0\leqslant r\leqslant 1,\tag{11}$$

$$\frac{\partial u(0,t)}{\partial r}=0,\quad u(1,t)=0,\quad t>0.\tag{12}$$

2.2 Variable order time fractional governing equation

Consider the complexity of material structure and the multi-scale effect of viscoelastic fluids in an infinite straight pipe. The generalized Maxwell fluid flow is studied, and a variable fractional order operator is used to depict the effects of time and space variables on the flow process. For generalized Maxwell fluids, the corresponding variable fractional order constitutive relationship is given according to Eq. (1),

$$\sigma_{rz} + \lambda^{\alpha(r,t)}\, {}_0\mathrm{D}_t^{\alpha(r,t)}\sigma_{rz} = \eta\frac{\partial u}{\partial r}, \quad 0 < \alpha(r,t) < 1, \tag{13}$$

where ${}_0\mathrm{D}_t^{\alpha(r,t)}$ is the Caputo variable order fractional derivative operator[33],

$${}_0\mathrm{D}_t^{\alpha(r,t)}u(r,t) = \frac{1}{\Gamma(1-\alpha(r,t))}\int_0^t (t-\tau)^{-\alpha(r,t)}\frac{\partial u(r,\tau)}{\partial \tau}\mathrm{d}\tau, \tag{14}$$

and $\alpha(r,t)$ is the order with respect to t and r.

Consider Eqs. (3) and (13) and eliminate σ_{rz}. Then, we obtain the following governing equation with variable fractional order derivatives:

$$\rho\lambda^{\alpha(r,t)}\, {}_0\mathrm{D}_t^{1+\alpha(r,t)}u + \rho\frac{\partial u}{\partial t} - \frac{\eta}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) = P_0\lambda^{\alpha(r,t)}\, {}_0\mathrm{D}_t^{\alpha(r,t)}\cos(\omega t) + P_0\cos(\omega t), \tag{15}$$

where for $\alpha(r,t) \equiv 1$, the governing equation reduces to a classical Maxwell fluid model, while for $\alpha(r,t) \equiv 0$, the governing equation will change into a Newtonian fluid model. Furthermore, the fractional Maxwell fluid model is obtained when $\alpha(r,t)$ equals a constant.

Similarly, the initial and boundary conditions also satisfy Eqs. (7) and (8). Substitute Eq. (9) into Eqs. (15), (7), and (8). Then, the dimensionless variable order time fractional Maxwell governing equation is given (for convenience, the symbol "*" is omitted),

$$\lambda^{\alpha(r,t)}\, {}_0\mathrm{D}_t^{1+\alpha(r,t)}u + \frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \lambda^{\alpha(r,t)}\, {}_0\mathrm{D}_t^{\alpha(r,t)}\cos(\omega t) + \cos(\omega t), \tag{16}$$

and the dimensionless initial and boundary conditions are also obtained as Eqs. (11) and (12).

Based on the velocity distributions derived by the dimensionless distributed/variable order fractional governing equations, the dimensionless flow rate $Q(t)$ in a pipe can be expressed as

$$Q(t) = 2\pi\int_0^1 u(r,t)r\mathrm{d}r. \tag{17}$$

3 Numerical schemes

In order to investigate the viscoelastic fluid flow of generalized Maxwell fluids driven by the periodic pressure gradient in an infinite straight pipe, we propose the numerical schemes of the governing equations with distributted/variable order time fractional derivatives with the finite difference method. Define $f(t) = \cos(\omega t)$, and assume that the numerical solution $u(r,t)$ and the function $f(t)$ at the mesh grid (r_i, t_n) are defined as u_i^n and f^n, respectively. Let $t_n = n\tau\,(n = 0,1,2,\cdots,N)$ and $r_i = ih\,(i = 0,1,2,\cdots,M)$, where N and M are positive integers, $\tau = T/N$ and $h = 1/M$ refer to the time step and space step, respectively, and T is sample time. The following difference schemes are introduced:

$$\frac{\partial u(r_i,t_n)}{\partial r} = \frac{u_i^n - u_{i-1}^n}{h}, \quad \frac{\partial^2 u(r_i,t_n)}{\partial r^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \quad \frac{\partial u(r_i,t_n)}{\partial t} = \frac{u_i^n - u_i^{n-1}}{\tau}, \tag{18}$$

and for Caputo fractional derivatives ${}_0\mathrm{D}_t^{\alpha+1}u(r,t)$ and ${}_0\mathrm{D}_t^{\alpha}f(t)$, the discrete schemes are obtained with the L_1-approximation formula[34–35],

$$\begin{aligned}{}_0\mathrm{D}_t^{\alpha+1}u(r,t)\Big|_{r=r_i}^{t=t_n} &= \frac{\tau^{-1-\alpha}}{\Gamma(2-\alpha)}\Big(u_i^n - u_i^{n-1} - \sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha} - a_{n-k}^{\alpha})(u_i^k - u_i^{k-1}) \\ &\quad - a_{n-1}^{\alpha}\tau\frac{\partial u(r_i,0)}{\partial t}\Big), \quad 0<\alpha<1, \end{aligned} \tag{19}$$

$${}_0\mathrm{D}_t^{\alpha}f(t)\Big|_{t=t_n} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}a_k^{\alpha}(f^{n-k}-f^{n-k-1}), \quad 0<\alpha<1, \tag{20}$$

where $a_k^{\alpha} = (k+1)^{1-\alpha} - k^{1-\alpha}$, $k \geqslant 0$.

On the basis of Eq. (17), the discretization of the flow rate $Q(t)$ is presented by using the composite trapezoidal formula,

$$Q(t_n) = 2\pi\sum_{i=1}^{M}\int_{r_{i-1}}^{r_i} ru(r,t_n)\mathrm{d}r = \pi h\sum_{i=1}^{M}(r_{i-1}u(r_{i-1},t_n) + r_i u(r_i,t_n)), \tag{21}$$

and the accuracy of the scheme is second-order with respect to r. Then, according to the distributed and variable order time fractional governing equations mentioned above, the investigation of the numerical schemes will be divided into the following two parts.

3.1 Numerical scheme for distributed order time fractional governing equation

First, we discretize integral terms of the distributed order time fractional governing equation (10). Let mesh grids $\{\xi_i, i=0,1,2,\cdots,q\}$ discretize the change interval $[0,1]$ of fractional order α included in integral terms[36], i.e., $0=\xi_0<\xi_1<\xi_2<\cdots<\xi_q=1$, and then we obtain

$$\int_0^1 \varpi(\alpha)\lambda^{\alpha}\,{}_0\mathrm{D}_t^{\alpha}f(t)\mathrm{d}\alpha = \sum_{s=1}^{q}\varpi(\alpha_s)\lambda^{\alpha_s}\,{}_0\mathrm{D}_t^{\alpha_s}f(t)\Delta\xi_s, \tag{22}$$

where $\Delta\xi_s = \xi_s - \xi_{s-1} = 1/q = \delta$, and $\alpha_s = (\xi_{s-1}+\xi_s)/2 = (2s-1)/(2q)$, $1\leqslant s\leqslant q$.

Substitute Eq. (22) into Eq. (10). The distributed order time fractional Maxwell governing equation will change into the following multi-term time fractional equation:

$$\frac{1}{q}\sum_{s=1}^{q}\varpi(\alpha_s)\lambda^{\alpha_s}\,{}_0\mathrm{D}_t^{1+\alpha_s}u + \frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) = \frac{1}{q}\sum_{s=1}^{q}\varpi(\alpha_s)\lambda^{\alpha_s}\,{}_0\mathrm{D}_t^{\alpha_s}f(t) + f(t). \tag{23}$$

Then, with the help of the discrete schemes (see Eqs. (18)–(20)), the implicit finite difference scheme for Eq. (23) is obtained,

$$\begin{aligned}&-\frac{1}{h^2}u_{i+1}^1 + \Big(\frac{2}{h^2}+\frac{1}{\tau}-\frac{1}{ih^2}+\sum_{s=1}^{q}A_s\Big)u_i^1 + \Big(\frac{1}{ih^2}-\frac{1}{h^2}\Big)u_{i-1}^1 \\ &= \Big(\frac{1}{\tau}+\sum_{s=1}^{q}A_s\Big)u_i^0 + \Big(1+\sum_{s=1}^{q}B_s\Big)f^1 - \sum_{s=1}^{q}B_s f^0, \quad n=1,\end{aligned} \tag{24}$$

$$\begin{aligned}&-\frac{1}{h^2}u_{i+1}^n + \Big(\frac{2}{h^2}+\frac{1}{\tau}-\frac{1}{ih^2}+\sum_{s=1}^{q}A_s\Big)u_i^n + \Big(\frac{1}{ih^2}-\frac{1}{h^2}\Big)u_{i-1}^n \\ &= \Big(\frac{1}{\tau}+\sum_{s=1}^{q}A_s\Big)u_i^{n-1} + \sum_{s=1}^{q}A_s\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_s}-a_{n-k}^{\alpha_s})(u_i^k-u_i^{k-1}) + \Big(1+\sum_{s=1}^{q}B_s\Big)f^n \\ &\quad -\sum_{s=1}^{q}B_s f^{n-1} + \sum_{s=1}^{q}B_s\sum_{k=1}^{n-1}a_k^{\alpha_s}(f^{n-k}-f^{n-k-1}), \quad 1<n\leqslant N,\end{aligned} \tag{25}$$

where $A_s = \tau^{-1-\alpha_s}\lambda^{\alpha_s}\varpi(\alpha_s)/(q\Gamma(2-\alpha_s))$, and $B_s = \tau A_s$. The discrete forms of initial and boundary conditions can be given as

$$u_i^0 = 0, \quad \frac{u_i^1 - u_i^0}{\tau} = 0, \quad 0 \leqslant i \leqslant M, \tag{26}$$

$$\frac{u_1^n - u_0^n}{h} = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. \tag{27}$$

Finally, it should be noted that the numerical scheme (see Eqs. (24)–(27)) is convergent with the order $O(\tau + h + \delta^2)$, and for more details, see Ref. [37].

3.2 Numerical scheme for variable order time fractional governing equation

For the variable order fractional governing equation (16), we assume that $\alpha(r,t)$ at the mesh grid (r_i, t_n) is denoted as α_i^n. The discrete schemes of variable order time fractional derivatives are obtained based on Eqs. (19) and (20), which are expressed as[33]

$$\begin{aligned}{}_0\mathrm{D}_t^{\alpha_i^n+1}u(r,t)\Big|_{r=r_i}^{t=t_n} &= \frac{\tau^{-1-\alpha_i^n}}{\Gamma(2-\alpha_i^n)}\Big(u_i^n - u_i^{n-1} - \sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_i^n} - a_{n-k}^{\alpha_i^n})(u_i^k - u_i^{k-1}) \\ &\quad - a_{n-1}^{\alpha_i^n}\tau\frac{\partial u(r_i,0)}{\partial t}\Big),\end{aligned} \tag{28}$$

$${}_0\mathrm{D}_t^{\alpha_i^n}f(t)\Big|_{t=t_n} = \frac{\tau^{-\alpha_i^n}}{\Gamma(2-\alpha_i^n)}\sum_{k=0}^{n-1}a_k^{\alpha_i^n}(f^{n-k} - f^{n-k-1}), \tag{29}$$

where $a_k^{\alpha_i^n} = (k+1)^{1-\alpha_i^n} - k^{1-\alpha_i^n}, k \geqslant 0$. Substituting Eqs. (18), (28), and (29) into Eq. (16), we get the following implicit finite difference scheme:

$$\begin{aligned}&-\frac{1}{h^2}u_{i+1}^1 + \Big(\frac{2}{h^2} + \frac{1}{\tau} - \frac{1}{ih^2} + V_i^1\Big)u_i^1 + \Big(\frac{1}{ih^2} - \frac{1}{h^2}\Big)u_{i-1}^1 \\ &= \Big(\frac{1}{\tau} + V_i^1\Big)u_i^0 + (1 + S_i^1)f^1 - S_i^1 f^0, \quad n = 1,\end{aligned} \tag{30}$$

$$\begin{aligned}&-\frac{1}{h^2}u_{i+1}^n + \Big(\frac{2}{h^2} + \frac{1}{\tau} - \frac{1}{ih^2} + V_i^n\Big)u_i^n + \Big(\frac{1}{ih^2} - \frac{1}{h^2}\Big)u_{i-1}^n \\ &= \Big(\frac{1}{\tau} + V_i^n\Big)u_i^{n-1} + V_i^n\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_i^k} - a_{n-k}^{\alpha_i^k})(u_i^k - u_i^{k-1}) + (1 + S_i^n)f^n \\ &\quad - S_i^n f^{n-1} + S_i^n\sum_{k=1}^{n-1}a_k^{\alpha_i^k}(f^{n-k} - f^{n-k-1}), \quad 1 < n \leqslant N,\end{aligned} \tag{31}$$

where $V_i^n = \frac{\tau^{-1-\alpha_i^n}\lambda^{\alpha_i^n}}{\Gamma(2-\alpha_i^n)}$, and $S_i^n = \tau V_i^n$. The discretizations of initial and boundary conditions also satisfy Eqs. (26) and (27). Then, by referencing relevant literature[33], it can be proved that the numerical scheme (see Eqs. (26), (27), and (30)–(31)) is also convergent with the order $O(\tau + h)$.

4 Numerical examples

In order to prove the effectiveness and correctness of the numerical methods, two different source terms are introduced, and two examples with the exact solutions are proposed based on the above distributed/variable order time fractional Maxwell governing equation.

Example 1 Let $\varpi(\alpha) = \Gamma(2-\alpha)$, $\lambda = 1$, and we consider the following distributed order governing equation with the initial and boundary conditions (11) and (12):

$$\int_0^1 \varpi(\alpha)\lambda^\alpha\,{}_0\mathrm{D}_t^{1+\alpha}u(r,t)\mathrm{d}\alpha + \frac{\partial u(r,t)}{\partial t} - \Big(\frac{1}{r}+\frac{\partial}{\partial r}\Big)\frac{\partial u(r,t)}{\partial r} = f_1(r,t), \tag{32}$$

and a simple exact solution is defined as $u(r,t) = t^2(1-r^2)$. The source term $f_1(r,t)$ is given and can be expressed as

$$f_1(r,t) = 2(1-r^2)\int_0^1 \varpi(\alpha)\lambda^\alpha \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{d}\alpha + 4t^2 + 2t(1-r^2). \tag{33}$$

Example 2 Assume that the variable order time fractional governing equation with the source term also satisfies the initial and boundary conditions (11) and (12), and thus we have

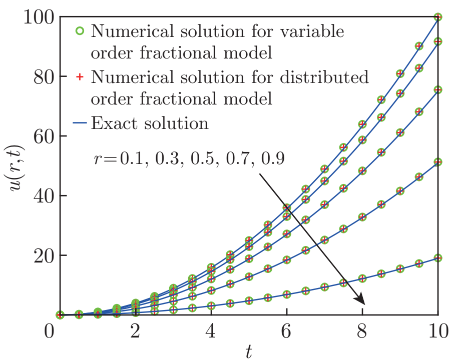
$$\lambda^{\alpha(r,t)}\,{}_0\mathrm{D}_t^{1+\alpha(r,t)}u(r,t) + \frac{\partial u(r,t)}{\partial t} - \Big(\frac{1}{r}+\frac{\partial}{\partial r}\Big)\frac{\partial u(r,t)}{\partial r} = f_2(r,t), \tag{34}$$

where $\lambda = 1$, and $\alpha(r,t) = 0.5 + 0.5rt/T$. Here, the same exact solution is considered, i.e., $u(r,t) = t^2(1-r^2)$, and then the source term is given as

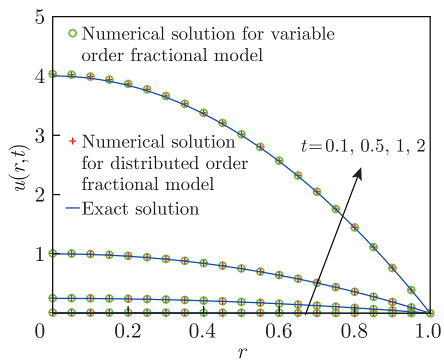
$$f_2(r,t) = 2(1-r^2)\frac{\lambda^{\alpha(r,t)}t^{1-\alpha(r,t)}}{\Gamma(2-\alpha(r,t))} + 4t^2 + 2t(1-r^2). \tag{35}$$

5 Results and discussion

In this section, some results on the viscoelastic fluid flow with the generalized Maxwell model in the infinite straight pipe are given and discussed in detail. First, the correctness and validity of the numerical methods for the distributed order and variable order fractional governing equations are tested and verified. Based on the two numerical examples mentioned in Section 4, the comparisons between numerical solutions and exact solutions of the distributed order and variable order time fractional equations are made, as shown in Fig. 1, where Fig. 1(a) shows the variations of the flow velocity with time, and Fig. 1(b) gives the profiles of the velocity with r. We can see from the two figures that the numerical solutions agree very well with the exact solutions, in which the time step and space step are set as $\tau = h = 0.01$. This shows that the numerical methods we have proposed are precise and effective. Then, the velocity distributions of the distributed order and variable order time fractional Maxwell governing equations are studied numerically.



(a) Velocity changing with time



(b) Velocity changing with space

Fig. 1 Dimensionless velocity profiles for numerical solutions and exact solutions of Eqs. (32) and (34) (color online)

In recent years, the fractional models with distributed/variable order derivatives have been considered to depict complex motion processes. Here, based on viscoelastic Maxwell fluids, we analyze the differences among the fractional Maxwell models with constant order, distributed order, and variable order derivatives. Assume that the weight coefficient in distributed order fractional derivatives satisfies $\varpi(\alpha) = \Gamma(2-\alpha)$, and the order of variable order fractional derivatives changes with a simple function, defined as $\alpha(r,t) = 0.5+0.5rt/T$. Then, the velocity distributions of the fractional Maxwell fluid flow produced by a periodic pressure gradient with time are illustrated in Fig. 2, where $\lambda = 1$, $r = 0.5$, and $\omega = 2\pi$, and for constant fractional derivatives, the fractional order α equals 0.3, 0.5, and 0.8, respectively. From this figure, we can find that the flow velocity is oscillatory and reaches a series of peaks and troughs. That is because the flow is driven by the periodic pressure gradient. In addition, we can also observe that the amplitudes of velocity among these curves show distinct variations. The velocity distribution for the distributed order fractional model is very close to that of the fractional model with the order $\alpha = 0.5$, although a little distinction exists, especially for a long time. For the variable order fractional model, we can see that the amplitudes grow with t and have a larger value at longer time compared with the fractional model with the order $\alpha = 0.3$, 0.5, and 0.8, which indicates that the changes of the peaks and troughs are related to the form of $\alpha(r,t)$. These results show that the distributed order and variable order fractional models can more accurately characterize some nonlinear behaviours that are unable to be explained by the fractional model with constant order operators.

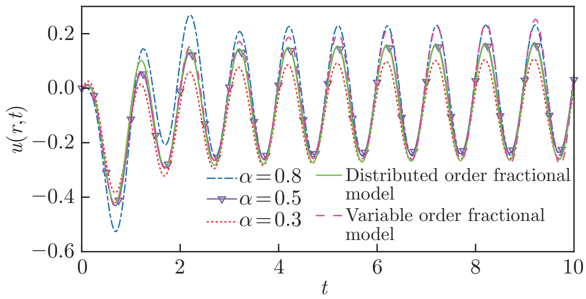


Fig. 2 Dimensionless velocities for fractional Maxwell model with constant order, distributed order, and variable order fractional derivatives are plotted at $r = 0.5$. For fractional model with constant order derivatives, different orders are considered, i.e., $\alpha = 0.3$, 0.5, and 0.8. For distributed order fractional model, $\varpi(\alpha) = \Gamma(2-\alpha)$. For variable fractional order model, $\alpha(r,t) = 0.5 + 0.5rt/T$. Relaxation time satisfies $\lambda = 1$, and for periodic pressure gradient, frequency $\omega = 2\pi$ (color online)

Then, the effects of parameters on the velocity distributions for the distributed order time fractional Maxwell model (see Eqs. (10)–(12)) are investigated. At first, we study the effects of weight coefficient on the velocity distribution. In this work, four forms of the weight coefficient $\varpi(\alpha)$ are selected, i.e., $\varpi_1(\alpha) = 1$, $\varpi_2(\alpha) = \Gamma(2-\alpha)$, $\varpi_3(\alpha) = 2\alpha$, and $\varpi_4(\alpha) = \delta(\alpha - 0.8)$. For the first case in which the weight coefficient is one[26], the uniform distributed order time fractional Maxwell governing equation is considered. It indicates that the distributed order fractional equation under the situation can be used to depict the fluid flow processes where the memory characteristic follows the uniform distribution for the fractional order α from zero to one. Next, we analyze the second instance $\varpi_2(\alpha)$ in which the weight function decreases at a smaller fractional order α and then increases with the order. It shows that the memory effect is related to the order. For the power-law weight coefficient which has been used in Refs. [24] and [38], i.e., $\varpi(\alpha) = \nu\alpha^{\nu-1}$, only a special case where $\varpi(\alpha)$ is a linear increasing function is investigated by taking $\nu = 2$. It is suitable to characterize some physical phenomena followed by the ultraslow kinetics. For the last one, the Dirac delta function is constructed and defined as a weight coefficient of distributed order time fractional derivatives, and then the distributed order time fractional Maxwell governing equations will be reduced to the fractional Maxwell equation with constant order operators. The choice for the weight coefficient is also proposed to interpret the propagation of stress waves in viscoelastic media[38]. Based on this, in order to study the influence of weight function on the flow of viscoelastic fluids more accurately, the normalized weight function $\varpi_i^*(\alpha)$ has been obtained by $\varpi_i^*(\alpha) = \varpi_i(\alpha)/W$ $(i = 1, 2, 3, 4)$ and is used to analyze the velocity distributions. However, in the following discussion, the symbol "*" is omitted for convenience.

Figure 3 plots the velocity profiles versus time at $r = 0.5$, where different values for the weight coefficient $\varpi(\alpha)$ and the relation time λ are considered. As shown in Fig. 3, the velocity distributions have a roughly similar tendency during the entire time period, when different weight functions $\varpi(\alpha)$ are chosen, and the rest parameters are fixed, while the difference of velocity amplitudes is more obvious. For $\varpi_1(\alpha)$ and $\varpi_2(\alpha)$, it is clear that the amplitudes of velocity distributions increase steadily with time, while the amplitudes for $\varpi_3(\alpha)$ and $\varpi_4(\alpha)$ could be slightly up and down during the whole time domain when $\lambda = 1$ and 2. Thus, different forms of the weight function can be used to describe various nonlinear behaviors. This shows that the weight function reflecting the memory effect of viscoelastic materials plays an important role in modeling the fluid flow.

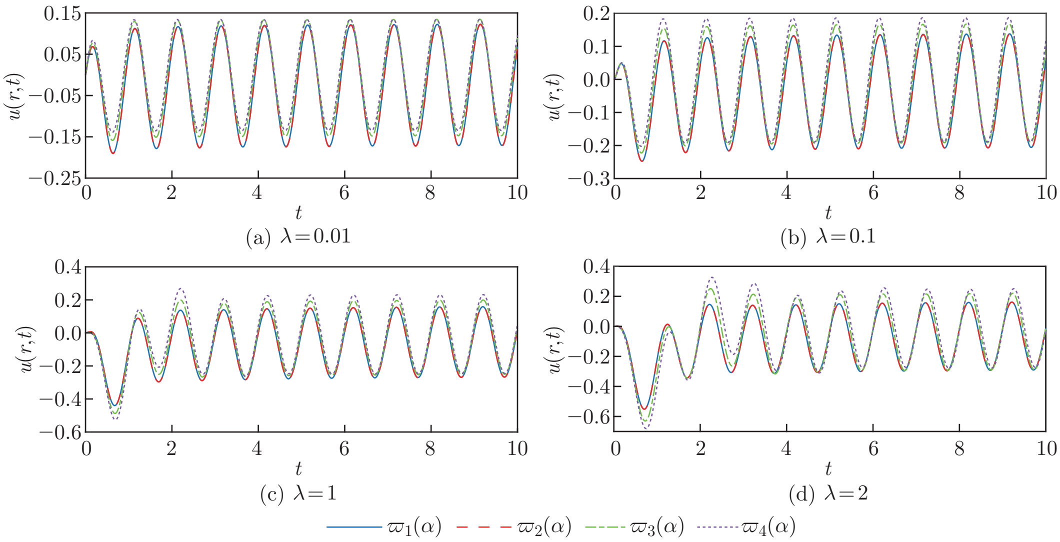


Fig. 3 Dimensionless velocity distributions of distributed order fractional Maxwell model with time at $r = 0.5$ when weight coefficient $\varpi(\alpha)$ and relaxation time λ take different values, where for periodic pressure gradient, frequency $\omega = 2\pi$ (color online)

Then, we discuss the effects of the relaxation time λ on the fluid flow velocity for the distributed order time fractional Maxwell model. Figures 3(a)–3(d) represent the velocity variations versus time when the relaxation time is 0.01, 0.1, 1, and 2, respectively. From these figures, we can see that the peaks and troughs of the velocity distributions become larger with the increase of λ. Meanwhile, it is noted that the peak values of the velocity for the other two weight functions $\varpi_1(\alpha)$ and $\varpi_2(\alpha)$ are relatively stable compared with the conditions $\varpi_3(\alpha)$ and $\varpi_4(\alpha)$, in which the peak values exhibit instability with the increase of the relaxation time, and the larger the relaxation time λ, the greater the fluctuation of the peaks at smaller t. The results indicate that the effects of the relaxation time on the velocity are related to the weight function.

The variable order fractional model is a generalization of the fractional model with constant order derivatives and can capture some important physical behaviors. Here, we define the variable fractional order $\alpha(r, t)$ as the following common functions with respect to time and space:

$$\alpha_1(r,t) = a + br\frac{t}{T}, \quad \alpha_2(r,t) = \begin{cases} 1, & 0 \leqslant r \leqslant 0.1, \\ 0.5, & 0.1 < r < 0.9, \\ 0, & 0.9 \leqslant r \leqslant 1, \end{cases} \quad \alpha_3(r,t) = c + d\sin(wt), \tag{36}$$

where a, b, c, and d are constants, and ω is the frequency. For these variable orders $\alpha_1(r,t)$, $\alpha_2(r,t)$, and $\alpha_3(r,t)$, we think that the variable fractional order equations can be used to characterize the following three phenomena. For the first one $\alpha_1(r,t)$, it refers to the flow that the memory properties of viscoelastic materials are positively correlated with time and tube radius because of the non-homogeneity of the material structure. For the second one $\alpha_2(r,t)$, it can be used to depict the stratified fluid flow in the pipe, where the fluid is regarded as the fractional Maxwell fluid inside the pipe, while the Newtonian fluid and the classical Maxwell fluid are found near the wall and the centre of the pipe, respectively. The last one $\alpha_3(r,t)$ represents the flow that the memory behavior of the viscoelastic material is affected by the external pressure. In this work, w is assumed to be equal to the frequency ω of the periodic pressure gradient.

Then, we investigate the effects of the variable order $\alpha(r,t)$ on the velocity. First, a simple situation in which the variable order $\alpha(r,t) = a+brt/T$ is discussed, where different values for a and b of the order $\alpha_1(r,t)$ are considered, as shown in Fig. 4. We can observe that the velocities are equal at smaller t when a is the same and b is different. With the increase of time, the larger the value b, the larger the peaks and troughs of the velocity distributions. It indicates that a plays a major role in velocity distributions at smaller t, and b brings an increasing role at larger t. Next, considering various forms of variable order $\alpha(r,t)$, we analyze the variations of the velocity distributions with time when the other parameters are fixed. As shown in Fig. 5, we note that the flow velocities of fluids are similar at smaller t as the orders for the three variable order functions are very close and even equal at that time. Moreover, the values of the velocity exhibit apparent differences with the increase of t. The amplitudes of the velocity distributions increase gradually over the whole time range and are related to the variable order function. The results indicate that different $\alpha(r,t)$ will lead to different velocity profiles, and the greater the variations of the order, the greater the changes of velocities.

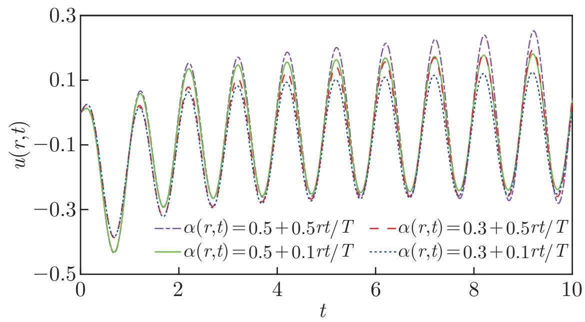


Fig. 4 Dimensionless velocity distributions for variable fractional order Maxwell governing equation at $r = 0.5$, $\lambda = 1$, and variable order $\alpha(r,t) = a + brt/T$, where $a = 0.5$, 0.3, $b = 0.5$, 0.1, and frequency $\omega = 2\pi$ for periodic pressure gradient (color online)

Here, we study the relationship between the velocity distributions of the variable order fractional Maxwell model and relaxation time λ. At first, the velocity distributions are given in Figs. 5(a)–5(d) when the relaxation time equals 0.01, 0.1, 1, and 2, respectively. As shown in the figures, the amplitudes of the velocity increase with the increase of the relaxation time from 0.01 to 1, while for $\lambda = 2$, the amplitudes decrease slightly compared with those for $\lambda = 1$. For

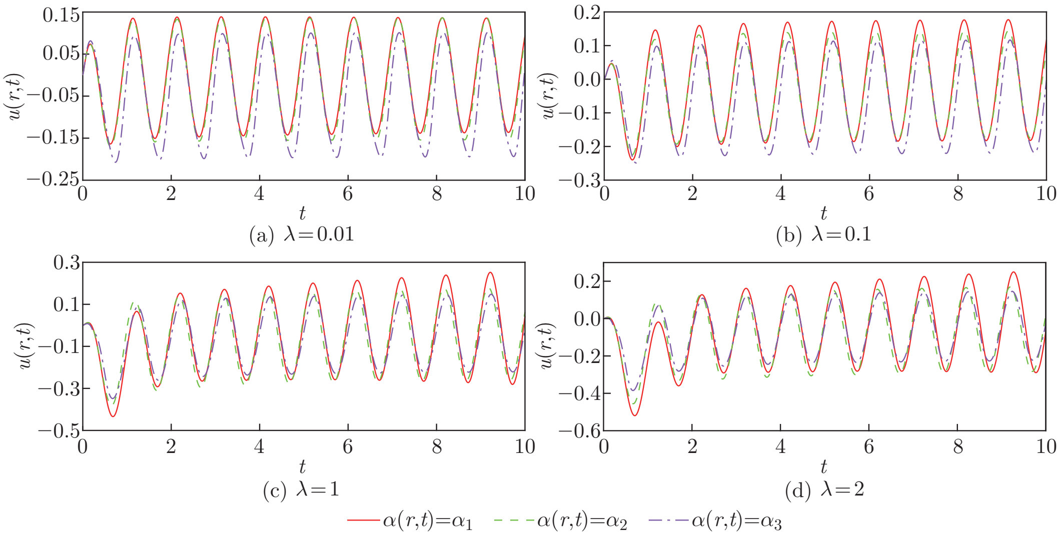


Fig. 5 Dimensionless velocity distributions of variable fractional order Maxwell model with time at $r = 0.5$ when variable order $\alpha(r,t)$ and relaxation time λ all take different values, where $\alpha_1 = 0.5 + 0.5rt/T$, $\alpha_2 = \alpha_2(r,t)$, $\alpha_3 = 0.5 + 0.5\sin(\omega t)$, and $\omega = 2\pi$ for periodic pressure gradient (color online)

the order $\alpha_1(r,t)$, the amplitude variations are larger compared with the changes for the order $\alpha_3(r,t)$, especially for larger relaxation time. It shows that the relaxation time has significant effects on the flow velocity. Thus, in order to explain the actual phenomenon accurately, it is very important to analyze the properties of the material and the processes of the material motion.

Further, the effects of the frequency ω of the periodic pressure gradient on the velocity distributions are also investigated. Here, different values of the frequency ω, namely, $\omega = 0, 0.5\pi, \pi,$ and 1.5π, are considered when the relaxation time $\lambda = 1$. In this case, the flow velocities of the distributed order and variable order fractional Maxwell fluids driven by the periodic pressure gradient at $r = 0.5$ are plotted in Figs. 6 and 7, respectively. From Fig. 6, we find that the velocities of the fractional Maxwell model with distributed order derivatives for $\omega = 0$ increase first and then decrease, finally tending to a steady-state with time. The velocity profiles, as shown in Fig. 6(a), are obviously different from the other three cases owing to the pressure gradient that reduces to a constant pressure gradient when $\omega = 0$. For these conditions of $\omega = 0.5\pi$, π, and 1.5π, we can see from Figs. 6(b)–6(d) that the velocity distributions also exhibit periodic responses on the basis of the periodicity of pressure gradient, and the amplitudes of velocity decrease with the growth of ω when the weight coefficient $\varpi(\alpha) = \varpi_1(\alpha)$, $\varpi_2(\alpha)$. However, for $\varpi_3(\alpha)$ and $\varpi_4(\alpha)$, the amplitudes of velocity fluctuate with ω. These results suggest that the frequency ω also has a significant effect on the velocity, but it is possible that the influence varies with the weight functions.

Then, the velocity distributions of the fractional Maxwell model with variable order derivatives are illustrated in Fig. 7. Similarly, for the case $\omega = 0$, the velocity is stable at larger time, while the velocities for $\omega = 0.5\pi$, π, and 1.5π oscillate with time. An apparent feature is that the oscillation period changes with frequency. The amplitudes of velocity profiles decrease gradually with the increase of ω. It reveals that the velocity distributions depend on the frequency ω and the variable order function. However, the same trends of velocity amplitudes with ω can be found based on these variable order functions that we consider.

Finally, Fig. 8 depicts the flow rate $Q(t)$ of generalized fractional Maxwell fluids produced by

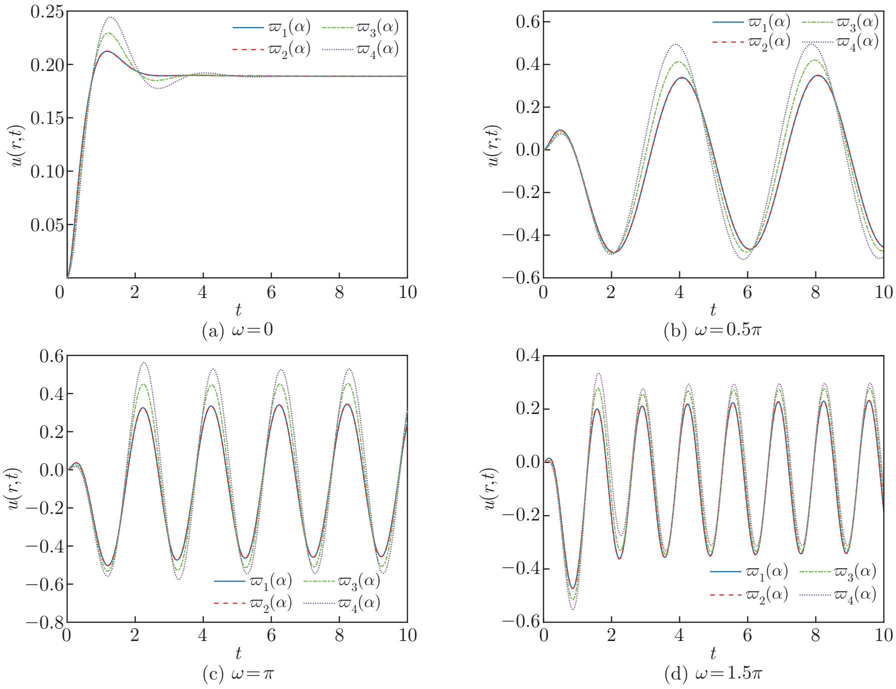


Fig. 6 Dimensionless velocity distributions of distributed order fractional Maxwell model with time at $r = 0.5$ when weight coefficient $\varpi(\alpha)$ takes different values, $\lambda = 1$, and rest parameters are fixed (color online)

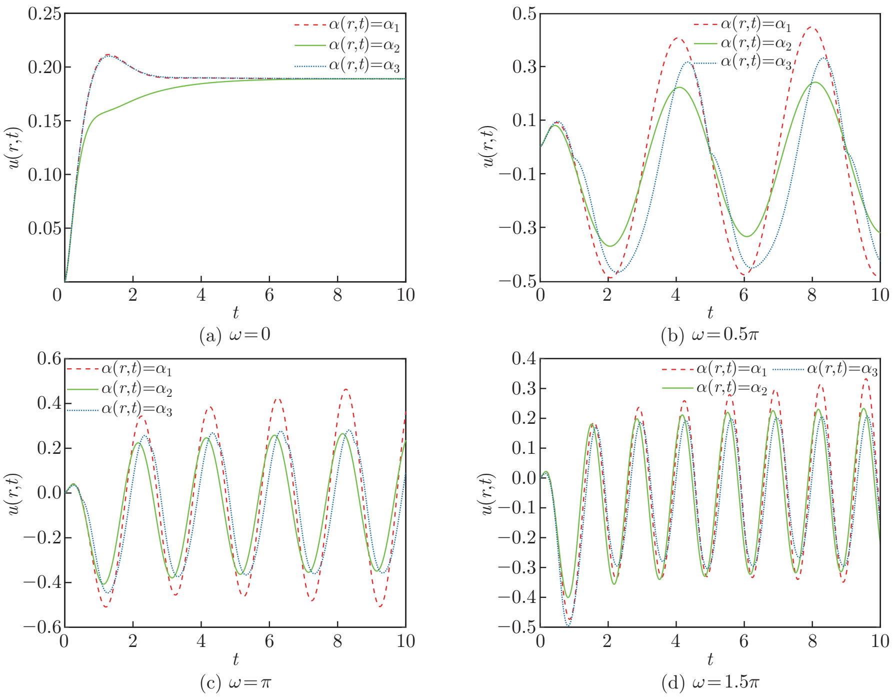
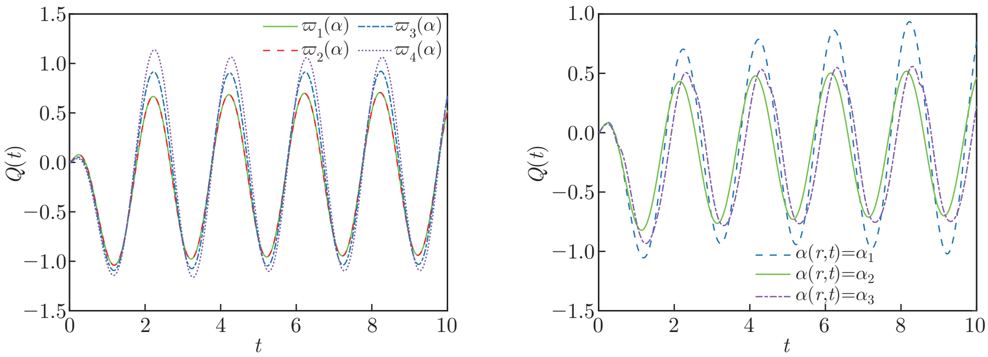


Fig. 7 Dimensionless velocity distributions of variable order fractional Maxwell model with time at $r = 0.5$ when variable order $\alpha(r, t)$ takes different forms, $\lambda = 1$, and rest parameters are fixed (color online)

the periodic pressure gradient in an infinite straight pipe with time for $\lambda = 1$ and $\omega = \pi$, where Fig. 8(a) shows the flow rate variation of the distributed order fractional Maxwell model for different forms of weight coefficient $\varpi(\alpha)$, and Fig. 8(b) presents the flow rate of the variable order fractional Maxwell model for different forms of variable order function $\alpha(r, t)$. It is noted that the flow rates display the characteristics of periodic oscillation, and a reasonable explanation for this behavior is that the periodic pressure gradient plays a critical role in the flow. In addition, an interesting phenomenon observed is that the flow rate profiles of the distributed order and variable order fractional models with time are related to the weight coefficient and the variable order function, respectively. It indicates that the flow is affected by the memory characteristics of viscoelastic fluids.



(a) Distributed order (b) Variable order

Fig. 8 Dimensionless flow rate $Q(t)$ of distributed order and variable order fractional Maxwell models with time, where $\lambda = 1$, and $\omega = \pi$ (color online)

6 Conclusions

The generalized Maxwell fluid flow driven by the periodic pressure gradient in the infinite straight pipe is considered and analyzed. The time fractional Maxwell model with distributed/variable order derivatives is proposed and solved numerically. Based on the L_1-formula of the Caputo fractional derivative, the implicit finite difference schemes for the distributed order and variable order time fractional Maxwell governing equations are given. On the basis of the distributed order and variable order time fractional Maxwell equations, two numerical examples are established to prove the effectiveness and correctness of the numerical algorithms. Thus, the comparisons between the numerical and exact solutions of the two numerical examples are made. A high consistency of them shows that our numerical methods are available. Then, the effects of the weight coefficient $\varpi(\alpha)$ for distributed order fractional derivatives, the order $\alpha(r, t)$ for variable fractional order derivatives, the relaxation time λ, and the frequency ω of periodic pressure gradient on the velocity distributions of the flow are discussed. These results indicate that the flow velocity oscillates during the whole time domain owing to the periodicity of pressure gradient. For the distributed order fractional Maxwell model, four weight functions are presented, and the profiles of the velocity are plotted. We realize that the amplitudes of the velocity distributions are distinct, when different $\varpi(\alpha)$ are taken, and there are different changes for various $\varpi(\alpha)$ with the increase of λ. For the variable order fractional Maxwell model, three variable order functions are chosen, and then the effects of relaxation time λ on the velocity are discussed. The results show that the amplitudes of the velocity distributions are different for different values of variable order function $\alpha(r, t)$ and relaxation time λ. The amplitudes increase with time when the other parameters are fixed, and the variations of the amplitude are related to the form of $\alpha(r, t)$. Then, the discussion about frequency ω indicates that ω will change the amplitudes and periodicity of velocity distributions. Finally, the variation of the flow rate $Q(t)$ with time is studied, and these profiles also show the features of oscillation. All of these make researchers have a systematic understanding for the distributed/variable order time fractional Maxwell equations. Therefore, we believe that our work is important and will provide an interesting thought for further research.

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