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Moore-Gibson-Thompson theory for thermoelastic dielectrics*

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Abstract We consider the system of equations determining the linear thermoelastic deformations of dielectrics within the recently called Moore-Gibson-Thompson (MGT) theory. First, we obtain the system of equations for such a case. Second, we consider the case of a rigid solid and show the existence and the exponential decay of solutions. Third, we consider the thermoelastic case and obtain the existence and the stability of the solutions. Exponential decay of solutions in the one-dimensional case is also recalled.

Key words Moore-Gibson-Thompson (MGT) thermoelastic dielectrics, existence, semi-group, exponential decay

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1 Introduction

The interaction of electromagnetic fields with thermoelastic dielectrics has been investigated for a long time. Several works have been devoted to this theory. During the recent years, major interest has arisen to understand the so-called Moore-Gibson-Thompson (MGT) thermoelasticity, and several contributions have been proposed for this recent theory. Our work is concerned with the linear theory of thermoelastic dielectrics based on the MGT theory. The equations for the heat conduction and electric field are based on the MGT theory. To this end, our initial point is the work of Ciarletta and Ieşan[1] concerning thermoviscoelastic dielectrics which is also based on the idea of the invariance of the entropy under time reversal[2].

The invariance of the infinitesimal entropy production under time reversal was studied by Borghesani and Morro[3–4], but we here start with the equations proposed by Ciarletta and Ieşan[1], also including the elastic deformations. Taking them as the initial point, we obtain the system of equations for the thermoelastic dielectrics of the MGT type. It is worth saying that recently significant interest has been developed to understand the MGT thermoelastic theories[5–14]. However, we focus our attention on the material with a center of symmetry. Therefore, the tensors of odd order are not considered. It is clear that the general case could be also obtained; however, in this note, we want to emphasize the new consequences proposed by the MGT-structure in the case of dielectrics which is different from the usual one. In our case,

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we obtain that the electric intensity is present in the equations of heat and electric fields, but not of the displacement. Nevertheless, the thermo-electric coupling leads to a nice problem. It is the coupling of a hyperbolic partial differential equation with an ordinary differential equation. The contribution of this paper is double. On one side, we extend the dielectric materials to the problems of the MGT type[13]. On the other side, we propose, from the mathematical point of view, an energy for the coupling in such a way that it defines a norm which is equivalent to the classical one in the Sobolev space $W^{1,2}$.

The remainder of this paper is organized as follows. Section 2 is devoted to obtaining the system of equations. The rigid solid case is considered in Section 3. Existence and exponential decay of solutions are obtained. The general system of the MGT thermoelasticity of dielectric materials is studied in Section 4. Existence of solutions and stability are also shown.

2 Basic equations

The system of equations for the thermoviscoelastic dielectrics for materials with a center of symmetry is determined by the following evolution equations[1]:

$$\begin{aligned} &\rho\ddot{u}_i = t_{ij,j},\\ &T_0\dot{\eta} = q_{i,i},\\ &d_{i,i} = 0. \end{aligned}$$

We recall that, in general, the tensor multiplying the history of the electric displacement in the heat flux vector and the tensor multiplying the history of the gradient of temperature in the last constitutive equation are equal except for a constant tensor. However, as a first approximation to this problem, we assume that they agree. Then, the constitutive equations are

$$\begin{aligned} t_{ij} &= \int_{\infty}^{t}\Big(G_{ijmn}(t-s)\dot{u}_{m,n}(s) - B_{ij}(t-s)\dot{\theta}(s)\Big)\mathrm{d}s,\\ \eta &= \int_{\infty}^{t}\Big(B_{ij}(t-s)\dot{u}_{i,j}(s) + A(t-s)\dot{\theta}(s)\Big)\mathrm{d}s,\\ q_i &= \int_{\infty}^{t}\Big(Q_{ji}(t-s)\dot{E}_j(s) + K_{ij}(t-s)\theta_{,j}(s)\Big)\mathrm{d}s,\\ d_i &= \int_{\infty}^{t}\Big(\gamma_{ji}(t-s)\dot{E}_j(s) + Q_{ij}(t-s)\theta_{,j}(s)\Big)\mathrm{d}s, \end{aligned}$$

where ρ is the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, T_0 is the reference temperature which is assumed to be equal to one to simplify the calculations, η is the entropy, q_i is the heat flux vector, d_i is the electric displacement, $E_i = -\phi_{,i}$ is the electric intensity, ϕ is the electric potential, θ is the temperature shift, and $G_{ijmn}(\boldsymbol{x}, s)$, $B_{ij}(\boldsymbol{x}, s)$, $A(\boldsymbol{x}, s)$, $Q_{ij}(\boldsymbol{x}, s)$, $K_{ij}(\boldsymbol{x}, s)$, and $\gamma_{ij}(\boldsymbol{x}, s)$ are the constitutive functions. It is known that

$$G_{ijmn} = G_{mnij}, \quad K_{ij} = K_{ji}, \quad \gamma_{ij} = \gamma_{ji}.$$

We consider the following constitutive functions:

$$\begin{aligned} &G_{ijmn}(\boldsymbol{x}, s) = G^*_{ijmn}(\boldsymbol{x}), \quad B_{ij}(\boldsymbol{x}, s) = B^*_{ij}(\boldsymbol{x}), \quad A(\boldsymbol{x}, s) = A^*(\boldsymbol{x}),\\ &K_{ij}(\boldsymbol{x}, s) = K^*_{ij}(\boldsymbol{x}) + (\tau^{-1}K_{ij}(\boldsymbol{x}) - K^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s),\\ &\gamma_{ij}(\boldsymbol{x}, s) = \gamma^*_{ij}(\boldsymbol{x}) + (\tau^{-1}\gamma_{ij}(\boldsymbol{x}) - \gamma^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s),\\ &Q_{ij}(\boldsymbol{x}, s) = Q^*_{ij}(\boldsymbol{x}) + (\tau^{-1}Q_{ij}(\boldsymbol{x}) - Q^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s), \end{aligned}$$

where τ is a positive and constant parameter.

We remark that G^*_{ijmn} is usually called the elasticity tensor, B^*_{ij} is related to the thermo-mechanical expansion, A^* is the thermal capacity, K_{ij} is the thermal conductivity, K^*_{ij} is usually called the rate conductivity, γ_{ij} and γ^*_{ij} are related to the electric permittivity, Q^*_{ij} and Q_{ij} determine the thermo-electric coupling, and τ is a relaxation parameter.

From the previous assumptions, we obtain

$$\begin{aligned}
&\dot{\eta} + \tau\ddot{\eta} = B^*_{ij}(\dot{u}_{i,j} + \tau\ddot{u}_{i,j}) + A^*(\dot{\theta} + \tau\ddot{\theta}),\\
&q_i + \tau\dot{q}_i = Q^*_{ji}E_j + Q_{ji}\dot{E}_j + K^*_{ij}\alpha_{,j} + K_{ij}\theta_{,j},
\end{aligned}$$

where

$$\alpha(\boldsymbol{x}, t) = \alpha_0(\boldsymbol{x}) + \int_0^t \theta(\boldsymbol{x}, s)\mathrm{d}s$$

is the thermal displacement.

In a similar way, we find

$$d_i + \tau\dot{d}_i = \gamma^*_{ji}E_j + \gamma_{ji}\dot{E}_j + Q^*_{ij}\alpha_{,j} + Q_{ij}\theta_{,i}.$$

Substituting these expressions into the evolution equations yields the following system of field equations:

$$\begin{aligned}
&\rho\ddot{u}_i = \big(G^*_{ijmn}u_{m,n} - B^*_{ij}\theta\big)_{,j},\\
&A^*(\dot{\theta} + \tau\ddot{\theta}) = -B^*_{ij}(\dot{u}_{i,j} + \tau\ddot{u}_{i,j}) + \big(Q^*_{ji}E_j + Q_{ji}\dot{E}_j + K^*_{ij}\alpha_{,j} + K_{ij}\theta_{,j}\big)_{,i},\\
&\big(\gamma^*_{ji}E_j + \gamma_{ji}\dot{E}_j + Q^*_{ij}\alpha_{,j} + Q_{ij}\theta_{,j}\big)_{,i} = 0.
\end{aligned}$$

In the case that the electric potential vanishes on the boundary, the system is written as follows:

$$\begin{aligned}
&\rho\ddot{u}_i = \big(G^*_{ijmn}u_{m,n} - B^*_{ij}(\theta + \tau\dot{\theta})\big)_{,j},\\
&A^*(\dot{\theta} + \tau\ddot{\theta}) = -B^*_{ij}\dot{u}_{i,j} + \big(K^*_{ij}\alpha_{,j} + K_{ij}\theta_{,j} - Q^*_{ji}\phi_{,j} - Q_{ji}\dot{\phi}_{,j}\big)_{,i},\\
&\dot{\phi} = \Phi^{-1}\big(\big(Q^*_{ij}\alpha_{,j} + Q_{ij}\theta_{,j}\big)_{,i} - \gamma^*_{ji}\phi_j\big)_{,i}\big),
\end{aligned}$$

where Φ is the isomorphism between $W_0^{1,2} \cap W^{2,2}$ and L^2 determined by $\Phi(f) = (\gamma_{ij}f_{,j})_{,i}$. The existence of this isomorphism is guaranteed whenever γ_{ij} is positive definite and suitable boundary conditions are assumed.

Working on this general case is a little bit cumbersome. Therefore, in order to make the analysis clear and transparent, we focus on the isotropic and homogeneous case, but we want to emphasize that the analysis could be done in a similar way. In this situation, our system of equations can be written as

$$\rho\ddot{u}_i = \mu^* u_{i,jj} + (\lambda^* + \mu^*)u_{j,ji} - \beta^*(\theta_{,i} + \tau\dot{\theta}_{,i}), \tag{1}$$

$$A^*(\dot{\theta} + \tau\ddot{\theta}) = -\beta^*\dot{u}_{i,i} + k^*\Delta\alpha + k\Delta\theta - Q^*\Delta\phi - Q\Delta F(\alpha, \theta, \phi), \tag{2}$$

$$\dot{\phi} = F(\alpha, \theta, \phi), \tag{3}$$

where

$$F(\alpha, \theta, \phi) = \gamma^{-1}(Q^*\alpha + Q\theta - \gamma^*\phi).$$

It is worth noting that the energy equation in this case is

$$E(t) + \int_0^t D(s)\mathrm{d}s = E(0),$$

where

$$\begin{aligned}
E(t) =& \frac{1}{2}\int_B \big(\rho \dot{u}_i \dot{u}_i + \mu^* u_{i,j}u_{i,j} + (\lambda^* + \mu^*)u_{i,i}u_{j,j}\big)\mathrm{d}v \\
&+ \frac{1}{2}\int_B \big(A^*(\theta + \tau\dot{\theta})^2 + k^*|\nabla(\alpha+\tau\theta)|^2 + \tau\bar{k}|\nabla\theta|^2 + \gamma^*|\nabla(\phi+\tau F)|^2 + \tau\bar{\gamma}|\nabla F|^2\big)\mathrm{d}v \\
&- \int_B \big(Q^*\nabla(\alpha+\tau\theta)\nabla(\phi+\tau F) + \tau\overline{Q}\nabla\theta\nabla F\big)\mathrm{d}v,
\end{aligned}$$

and

$$D(t) = \int_B \big(\bar{k}|\nabla\theta|^2 + \overline{\gamma}|\nabla F|^2 - 2\overline{Q}\nabla\theta\nabla F\big)\mathrm{d}v.$$

Here, we have used the notations

$$\overline{k} = k - \tau k^* > 0, \quad \overline{\gamma} = \gamma - \tau\gamma^* > 0, \quad \overline{Q} = Q - \tau Q^*.$$

From now on, we assume

$$\begin{aligned}
&\rho > 0, \quad \mu^* > 0, \quad \lambda^* + \mu^* > 0, \quad k^* > 0, \quad \gamma^* > 0, \quad \overline{k} > 0, \quad \overline{\gamma} > 0, \\
&k^*\gamma^* > (Q^*)^2, \quad \overline{k}\overline{\gamma} > (\overline{Q})^2.
\end{aligned}$$

3 Rigid solid

In this section, we study the problem determined on a rigid solid. Our system of equations is

$$\begin{aligned}
&A^*(\dot{\theta} + \tau\ddot{\theta}) = (k^* - \gamma^{-1}QQ^*)\Delta\alpha + (k - \gamma^{-1}Q^2)\Delta\theta - (Q^* - \gamma^{-1}Q\gamma)\Delta\phi, \\
&\dot{\phi} = \gamma^{-1}(Q^*\alpha + Q\theta - \gamma^*\phi).
\end{aligned}$$

Assume

$$\alpha(\boldsymbol{x},t) = \phi(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial B, \quad t > 0, \tag{4}$$
$$\alpha(\boldsymbol{x},0) = \alpha_0(\boldsymbol{x}), \quad \theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}), \quad \boldsymbol{x} \in B, \tag{5}$$
$$\dot{\theta}(\boldsymbol{x},0) = \xi_0(\boldsymbol{x}), \quad \phi(\boldsymbol{x},0) = \phi_0(\boldsymbol{x}), \quad \boldsymbol{x} \in B. \tag{6}$$

We consider our problem on a suitable Hilbert space

$$\mathcal{H} = W_0^{1,2}(B) \times W_0^{1,2}(B) \times L^2(B) \times W_0^{1,2}(B),$$

and for every $(\alpha,\theta,\xi,\phi), (\alpha^*,\theta^*,\xi^*,\phi^*) \in \mathcal{H}$, we define the inner product

$$\begin{aligned}
\langle(\alpha,\theta,\xi,\phi),(\alpha^*,\theta^*,\xi^*,\phi^*)\rangle =& \frac{1}{2}\int_B \big(A^*(\theta+\tau\xi)\overline{(\theta^*+\tau\xi^*)} + k^*\nabla(\alpha+\tau\theta)\nabla\overline{(\alpha^*+\tau\theta^*)} \\
&+ \tau\overline{k}\nabla\theta\nabla\overline{\theta^*} + \gamma^*\nabla(\phi+\tau G)\nabla\overline{(\phi^*+\tau G^*)} + \tau\overline{\gamma}\nabla G\nabla\overline{G^*} \\
&- Q^*(\nabla(\alpha+\tau\theta)\nabla\overline{(\phi^*+\tau G^*)} + \nabla\overline{(\alpha^*+\tau\theta^*)}\nabla(\phi^*+\tau G^*)) \\
&- \tau\overline{Q}(\nabla\theta\nabla\overline{G^*} + \nabla\overline{\theta^*}\nabla G)\big)\mathrm{d}v,
\end{aligned}$$

where the overline over the elements of the Hilbert space means the conjugated complex, and

$$G(\alpha,\theta,\phi) = \gamma^{-1}(Q^*\alpha + Q\theta - \gamma^*\phi).$$

Under the assumptions proposed at the end of the previous section, the norm induced by the above inner product is equivalent to the classical one defined on the Hilbert space $\mathcal{H}$. We can write our problem as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}U, \quad U(0) = U_0, \tag{7}$$

where $U = (\alpha, \theta, \xi, \phi)$, $U_0 = (\alpha_0, \theta_0, \xi_0, \phi_0)$, and the matrix operator is

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ (\tau A^*)^{-1}(k^* - \gamma^{-1}QQ^*)\Delta & (\tau A^*)^{-1}(k - \gamma^{-1}Q^2)\Delta & -\tau^{-1} & (\tau A^*)^{-1}(\gamma^{-1}Q\gamma^* - Q^*)\Delta \\ \gamma^{-1}Q^* & \gamma^{-1}Q & 0 & -\gamma^{-1}\gamma^* \end{pmatrix}.$$

We note that the domain of the operator is

$$\{(\alpha, \theta, \xi, \phi) \in \mathcal{H}, \xi \in W_0^{1,2}, (k^* - \gamma^{-1}QQ^*)\Delta\alpha + (k - \gamma^{-1}Q^2)\Delta\theta - (Q^* - \gamma^{-1}Q\gamma^*)\Delta\phi \in L^2\}.$$

Obviously, this is a dense subspace. On the other hand, for every $U = (\alpha, \theta, \xi, \phi)$ in the domain of the operator, we have

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\frac{1}{2}\int_B \big(\overline{k}|\nabla\theta|^2 + \overline{\gamma}|\nabla G|^2 - \overline{Q}(\nabla\theta\nabla\overline{G} + \nabla\overline{\theta}\nabla G)\big)\mathrm{d}v. \tag{8}$$

In view of the assumptions, we see that this is equal to or less than zero.

Our next step is to prove that zero belongs to the resolvent of the operator. To this end, let us consider $L = (l_1, l_2, l_3, l_4) \in \mathcal{H}$. We will prove that there exists $U = (\alpha, \theta, \xi, \phi)$ in the domain of the operator such that $\mathcal{A}U = L$. Writing this equation in coordinates, we obtain

$$\theta = l_1, \quad \xi = l_2, \quad Q^*\alpha + Q\theta - \gamma^*\phi = \gamma l_4,$$

and

$$(k^* - \gamma^{-1}QQ^*)\Delta\alpha + (k - \gamma^{-1}Q^2)\Delta\theta - A^*\xi - (Q^* - \gamma^{-1}Q\gamma^*)\Delta\phi = \tau A^* l_3.$$

We obtain the expressions for θ and ξ. We also have

$$\phi = (\gamma^*)^{-1}(Q^*\alpha + Ql_1 - \gamma l_4).$$

It then follows that we obtain an equation for the variable α which can be easily solved because $k^*\gamma^* > (Q^*)^2$. Moreover, we can obtain the regularity conditions, and the following result is found.

Theorem 1 *The operator $\mathcal{A}$ produces a contractive semigroup.*

We note that, using the above result, we conclude the existence, uniqueness, and continuous dependence of the solutions to our problem.

In the rest of the section, we will prove the exponential decay of the energy under some additional conditions. In order to show it, we recall the following characterization[15].

Theorem 2 *Let $S(t) = \{\mathrm{e}^{\mathcal{A}t}\}_{t\geqslant 0}$ be a C_0-semigroup of contractions defined in a Hilbert space. Therefore, $S(t)$ is exponentially stable if and only if the imaginary axis is contained in the resolvent of $\mathcal{A}$ and*

$$\overline{\lim_{|\lambda|\to\infty}} \|(\mathrm{i}\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{9}$$

Now, we follow the arguments already used in the book of Liu and Zheng[15]. First, we assume that the imaginary axis and the spectrum have a non-empty intersection. We conclude that there exist a sequence of real numbers (of course converging to a real number) λ_n with

$\lambda_n \to \varpi$ and $|\lambda_n| < |\varpi|$ and a sequence of corresponding vectors $U_n = (\alpha_n, \theta_n, \xi_n, \phi_n)$ in the domain of $\mathcal{A}$ and with unit norm, such that

$$\|(\mathrm{i}\lambda_n \mathcal{I} - \mathcal{A})U_n\| \to 0.$$

It then follows

$$\mathrm{i}\lambda_n \alpha_n - \theta_n \to 0 \quad \text{in} \quad W^{1,2}, \tag{10}$$

$$\mathrm{i}\lambda_n \theta_n - \xi_n \to 0 \quad \text{in} \quad W^{1,2}, \tag{11}$$

$$\begin{aligned} &\mathrm{i}\tau A^* \lambda_n \xi_n - (k^* - \gamma^{-1} Q Q^*)\Delta\alpha_n - (k - \gamma^{-1}Q^2)\Delta + A^*\xi_n \\ &- (\tau A^*)^{-1}(\gamma^{-1} Q\gamma^* - Q^*)\Delta\phi_n \to 0 \quad \text{in} \quad L^2, \end{aligned} \tag{12}$$

$$\mathrm{i}\gamma\lambda_n \phi_n - Q^*\alpha_n - Q\theta_n + \gamma^*\phi_n \to 0 \quad \text{in} \quad W^{1,2}. \tag{13}$$

In view of the dissipation, we see $\theta_n, \phi_n \to 0$ in $W^{1,2}$. Therefore, we also have $\alpha_n \to 0$ in $W^{1,2}$. If we now consider the convergence (12) multiplied by $\lambda_n^{-1}\xi_n$, after the use of the integration by parts, we obtain that $\xi_n \to 0$ in L^2. This contradicts the condition that the elements of the sequence have unit norm. Therefore, we can conclude that $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$.

Now, we want to prove that the asymptotic condition (9) also holds. In the case that this condition does not hold, there exist a sequence of real numbers λ_n with $|\lambda_n| \to \infty$ and another sequence of unit norm vectors $U_n = (\alpha_n, \theta_n, \xi_n, \phi_n)$ in $\mathcal{D}(\mathcal{A})$ in such a way that (10)–(13) hold. Therefore, we can proceed in an analogous way as we show that the imaginary axis is contained in the resolvent of the operator, because the key point is to note that the sequence λ_n does not tend to zero. Thus, it leads to a contradiction, and the condition (9) is also true.

We have proved the following.

Theorem 3 *Let us assume that the previous conditions hold. Then, the operator $\mathcal{A}$ produces an exponentially stable semigroup; that is, we can find two positive constants M and ω such that*

$$||U(t)|| \leqslant M \exp(-\omega t)||U(0)||$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$.

4 Thermoelastic case

In this section, we prove the existence of solutions to the problem determined by the general system (1)–(3). Apart from the initial and boundary conditions (4)–(5), we also impose in this section that

$$u_i(\boldsymbol{x}, 0) = u_{i0}(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_{i0}(\boldsymbol{x}), \quad \boldsymbol{x} \in B, \tag{14}$$

and

$$u_i(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial B, \quad t > 0. \tag{15}$$

In what follows, we will show an existence theorem for the solutions to the problem determined by the system (1)–(3) with conditions (4)–(5) and (14)–(15). The existence will be shown in a suitable Hilbert space. In this section, we will work with the space

$$\mathcal{H} = \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{L}^2(B) \times W_0^{1,2}(B) \times W_0^{1,2}(B) \times L^2(B) \times W_0^{1,2}(B),$$

and, for every $(\boldsymbol{u}, \boldsymbol{v}, \alpha, \theta, \xi, \phi), (\boldsymbol{u}^*, \boldsymbol{v}^*, \alpha^*, \theta^*, \xi^*, \phi^*) \in \mathcal{H}$, we define the inner product

$$\begin{aligned} \langle(\boldsymbol{u}, \boldsymbol{v}, \alpha, \theta, \xi, \phi), (\boldsymbol{u}^*, \boldsymbol{v}^*, \alpha^*, \theta^*, \xi^*, \phi^*)\rangle = \frac{1}{2}\int_B \big(&\rho v_i v_i^* + \mu^* u_{i,j}\overline{u}_{i,j}^* + (\lambda^* + \mu^*)u_{i,i}\overline{u}_{j,j}^* \\ &+ A^*(\theta + \tau\xi)\overline{(\theta^* + \tau\xi^*)} + k^*\nabla(\alpha + \tau\theta)\nabla\overline{(\alpha^* + \tau\theta^*)} \end{aligned}$$

$$+\tau\overline{k}\nabla\theta\nabla\overline{\theta^*}+\gamma^*\nabla(\phi+\tau G)\nabla\overline{(\phi^*+\tau G^*)}+\tau\overline{\gamma}\nabla G\nabla\overline{G^*}$$
$$-Q^*(\nabla(\alpha+\tau\theta)\nabla\overline{(\phi^*+\tau G^*)}+\nabla\overline{(\alpha^*+\tau\theta^*)}\nabla(\phi^*+\tau G^*))$$
$$-\tau\overline{Q}(\nabla\theta\nabla\overline{G^*}+\nabla\overline{\theta^*}\nabla G)\big)\mathrm{d}v.$$

Again, our problem can be written in the form of system (7), where $U=(\boldsymbol{u},\boldsymbol{v},\alpha,\theta,\xi,\phi)$ and $U_0=(\boldsymbol{u}_0,\boldsymbol{v}_0,\alpha_0,\theta_0,\xi_0,\phi_0)$, whenever we define the operator

$$\mathcal{A}\begin{pmatrix}u_i\\ v_i\\ \alpha\\ \theta\\ \xi\\ \phi\end{pmatrix}=\begin{pmatrix}v_i\\ \rho^{-1}(\mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}-\beta^*(\theta_{,i}+\tau\xi_{,i}))\\ \theta\\ \xi\\ (A^*)^{-1}(-\beta^*v_{i,i}+\tau^{-1}(M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi))-\tau^{-1}\xi\\ \gamma^{-1}(Q^*\alpha+Q\beta-\gamma^*\phi)\end{pmatrix},$$

where

$$M_1=k^*-\gamma^{-1}QQ^*,\quad M_2=k-\gamma^{-1}Q^2,\quad M_3=-(Q^*-\gamma^{-1}Q\gamma^*).$$

The domain of the operator is given by the elements in the Hilbert space $\mathcal{H}$ such that

$$\boldsymbol{u}\in\boldsymbol{W}^{2,2},\quad \boldsymbol{v}\in\boldsymbol{W}_0^{1,2},\quad \xi\in W_0^{1,2},\quad M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi\in L^2.$$

Therefore, it is a dense subspace. We have that the relation (8) also holds in this case. That is, we find

$$\mathrm{Re}\langle\mathcal{A}U,U\rangle=-\frac{1}{2}\int_B\big(\overline{k}|\nabla\theta|^2+\overline{\gamma}|\nabla G|^2-\overline{Q}(\nabla\theta\nabla\overline{G}+\nabla\overline{\theta}\nabla G)\big)\mathrm{d}v.$$

Thus, to prove the existence of a semigroup of linear operators, it is sufficient to show that zero belongs to the resolvent of the operator. We consider $L=(\boldsymbol{n}_1,\boldsymbol{n}_2,l_1,l_2,l_3,l_4)$ in the Hilbert space, and we need to show the existence of an element in the domain of the operator such that $\mathcal{A}U=L$. It leads to the following system:

$$\boldsymbol{v}=\boldsymbol{n}_1,\quad \theta=l_1,\quad \xi=l_2,\quad Q^*\alpha+Q\theta-\gamma\phi=\gamma l_4,$$
$$-\beta^*v_{i,i}+M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi-A^*\xi=\tau A^*l_3,$$
$$\mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}-\beta^*(\theta_{,i}+\tau\xi_{,i})=\rho n_{2i}.$$

As in the case of the rigid solid, we can also obtain ϕ.

We can find the expressions of $\boldsymbol{v},\theta$, and ξ, and our system reduces to

$$\Big(k^*-\frac{(Q^*)^2}{\gamma^*}\Big)\Delta\alpha=F_1,\quad \mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}=F_{2i}.$$

This system admits a solution in the domain of the operator and we obtain the following.

Theorem 4 *The operator $\mathcal{A}$ generates a contractive semigroup.*

We may conclude the stability of solutions as well as the well-posedness in the three-dimensional case.

The exponential decay of solutions in the general case cannot be expected. We should find that the behavior is similar to the usual one for the MGT thermoelasticity; however, it is obvious that the combination of the arguments proposed in this section, with those used in the previous one, would allow us to prove, in the one-dimensional setting, the exponential decay of solutions. Anyway, we do not give the details in order to shorten the length of the paper.

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