Appl. Math. Mech. -Engl. Ed., **42**(1), 109–126 (2021)

APPLIED MATHEMATICS AND MECHANICS (ENGLISH EDITION)

https://doi.org/10.1007/s10483-021-2690-6

Asymptotics for filtration of polydisperse suspension with small impurities*

L. I. KUZMINA[1], Y. V. OSIPOV[2,†], T. N. GORBUNOVA[2]

1. Department of Applied Mathematics, National Research University Higher School of Economics, Moscow 101000, Russia;

2. Department of Applied Mathematics, Moscow State University of Civil Engineering (National Research University), Moscow 129337, Russia

(Received May 4, 2020 / Revised Sept. 21, 2020)

Abstract A model for deep bed filtration of a polydisperse suspension with small impurities in a porous medium is considered. Different suspended particles move with the same velocity as the carrier water and get blocked in the pore throats due to the size-exclusion mechanism of particle retention. A solution of the model in the form of a traveling wave is obtained. The global exact solution for a multiparticle filtration with one high concentration and several low concentrations of suspended particles is obtained in an explicit form. The analytic solutions for a bidisperse suspension with large and small particles are constructed. The profiles of the retained small particles change monotony with time. The global asymptotics for the filtration of a polydisperse suspension with small kinetic rates is constructed in the whole filtration zone.

Key words deep bed filtration, suspension, colloid, porous medium, particle size distribution, analytical model

Chinese Library Classification O368
2010 Mathematics Subject Classification 74N15, 82D80

1 Introduction

The filtration of colloids, suspensions, and nanoparticles in a porous medium is important in many areas of science and technology. The movement of small particles in a porous rock can significantly reduce the productivity of oil and artesian wells[1–2]. The disposal of industrial wastes in aquifers can lead to environment contamination[3–4]. The filtration of liquid industrial waste and municipal wastewater serves to purify water[5–6].

In a classical filtration model, the identical solid particles are transported by a carrier fluid in a porous medium. The capture of particles by pores is due to electrical and gravitational forces, diffusion, viscosity and so on[7]. The long-term filtration process, which takes place in the entire porous medium, not just in its surface layer, is called deep bed filtration[8]. If the particle and pore sizes are close, the main retention mechanism is size-exclusion[2,4,9]. The

* Citation: KUZMINA, L. I., OSIPOV, Y. V., and GORBUNOVA, T. N. Asymptotics for filtration of polydisperse suspension with small impurities. *Applied Mathematics and Mechanics* (*English Edition*), **42**(1), 109–126 (2021) https://doi.org/10.1007/s10483-021-2690-6

† Corresponding author, E-mail: yuri-osipov@mail.ru

particles pass freely through large pores, and get stuck in the pore throats, which are smaller than the diameter of the particles. It is assumed that a retained particle cannot be detached from the pore wall by other particles or a fluid flux[10–11].

The simplest one-dimensional (1D) macroscopic model for the filtration of a monodisperse suspension in a porous medium includes the equation for the mass balance of suspended and retained particles and the kinetic equation of deposit growth[12–14]. At a low concentration of particles in the suspension, the deposit growth rate is proportional to the first degree of the suspended particle concentration. The proportionality coefficient in the kinetic equation is called the filtration function. In the simplest case, it is assumed to be constant, and is called the filtration coefficient. In the general case, the filtration function varies during the particle retention, and depends on the concentration of the retained particles. In most cases, a linear filtration function is considered, which is called Langmuir's blocking function[15–16].

Many complex population balance models taking into account size distributions of pores and particles are formulated in Refs. [9] and [17]. The random walk equations and modified Boltzmann models represent other stochastic approaches to deep bed filtration[18–19]. In many practical applications, the suspension contains solid particles of various types and sizes, which interact differently with the porous medium. If different particles do not interact with each other and have constant filtration coefficients, the problem can be reduced to a model of a monodisperse suspension with a variable filtration function[20]. The competition of different suspended particles for the deposition in small pores determines the nonlinear interaction of the solid particles of the suspension[4,21]. Experiments show that size distribution significantly affects the particle transport and the formation of deposits[22–23].

To describe the simultaneous action of several mechanisms of particle capture in a porous medium, e.g., size-exclusion, attachment, and bridging, various macroscopic models are used with one or several filtration functions and a linear or nonlinear concentration function[2,24–25]. Books[16,26] and papers[8,12,19,27–31] contain numerous exact solutions for deep bed filtration problems. If the exact solutions are unknown, asymptotics is constructed[32–33]. In Refs. [34] and [35], a bidisperse suspension filtration model with proportional linear filtration functions was studied. However, the problem with various filtration functions was not considered.

In this paper, a new filtration model for the polydisperse suspension in a porous medium is considered. The phenomenological model for the filtration of a polydisperse suspension is proposed in Section 2. The governing equations for the filtration problem are given in Section 3. The traveling wave solution is obtained in Section 4, while the exact solution for the filtration of a monodisperse suspension with small impurities is obtained in Section 5. Section 6 is devoted to the local exact solutions and the asymptotics for a model of bidisperse suspension. In Section 7, the global asymptotics for the filtration of an aqueous solution with small impurities and a polydisperse suspension with small kinetic rates are constructed. Examples and discussion are presented in Section 8, and the main results are summarized in Section 9 finally.

2 Model assumptions

In the model, we assume that the fluid transporting suspension particles is incompressible and Newtonian and the main mechanism of particle capture is size-exclusion. The flow is single-phase, and the injected suspension contains the same water that is initially in the porous medium. In large-scale approximation, we ignore the diffusion and dispersion of the suspended particles[1,36–37]. The mathematical model of deep bed filtration is a nonlinear hyperbolic system of two first-order equations as follows[2,16,19,37]:

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0, \tag{1}$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C, \tag{2}$$

where $\Lambda(S)$ is the filtration function, $C(x,t)$ is the dimensionless concentration of the suspended particles, and $S(x,t)$ is the dimensionless concentration of the retained particles.

Initially, the porous medium does not contain any suspended or retained particles. Then, a suspension with a constant concentration of suspended particles is injected at the inlet of the porous medium. The boundary and initial conditions are as follows:

$$x = 0:\ C = 1; \quad t = 0:\ C = 0, \quad S = 0. \tag{3}$$

In many practical problems, the suspension contains different types of particles. Figure 1 shows the diagram for the size exclusion (straining), where small particles pass through larger pores without being captured while large particles are captured in the throats of smaller pores. Consider a 1D macroscopic size-exclusion model for filtering a suspension with particles of n different types moving in a porous medium with the same velocity. The particles can vary in shape, size, density, etc. Each type of particles satisfies the mass balance equation and the equation for the kinetic rate of the deposits. Suppose that all particles can clog small pores, and the filtration functions depend on a linear combination S of partial deposits $\Lambda_i(S) = \Lambda_i(\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \cdots + \alpha_n S_n)$ $(i = 1, 2, 3, \cdots, n)$. This model determines the transfer and retention of various types of solid particles and their competition for small pores.

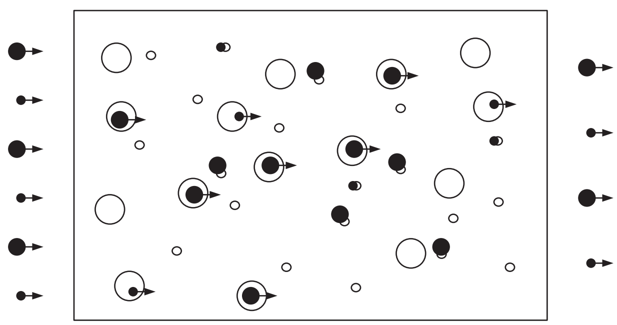


Fig. 1 Diagram for the suspension-colloidal transport and retention of particles in a porous medium

3 Governing equations

In the domain $\Omega = \{0 < x < 1, t > 0\}$, n different types of suspended and retained particle concentrations $C_i(x,t)$ and $S_i(x,t)$ satisfy the following system[26–27]:

$$\frac{\partial C_i}{\partial t} + \frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial t} = 0, \tag{4}$$

$$\frac{\partial S_i}{\partial t} = \Lambda_i(S)C_i, \tag{5}$$

where the total deposit is

$$S = \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \cdots + \alpha_n S_n, \tag{6}$$

and the filtration functions $\Lambda_i(S)$ and the constants α_i are positive.

The initial and boundary conditions at the inlet $x = 0$ of the porous medium and at the initial time $t = 0$ provide a unique solution in the domain $\Omega = \{0 \leqslant x \leqslant 1,\ t \geqslant 0\}$, i.e.,

$$x = 0:\ C_i = p_i, \quad p_i > 0; \tag{7}$$

$$t = 0:\ C_i = 0, \quad S_i = 0. \tag{8}$$

The concentration front of the suspended and retained particles moves with $v = 1$ and divides the domain Ω into two subdomains, i.e., $\Omega = \{0 < x < 1,\ t > x\}$ and $\Omega = \{0 < x < 1,\ 0 < t < x\}$. In Ω_0, the system (4)–(8) has a zero solution. In Ω_S, the solution is positive. Since Eqs. (7) and (8) do not match at the origin, $C_i(x,t)$ has a strong discontinuity on the concentration front, which is the characteristic straight line $t = x$, while $S_i(x,t)$ is continuous in the whole Ω and has a weak discontinuity on the concentration front.

On the concentration front,

$$S_i|_{t=x} = 0, \quad i = 1, 2, 3, \cdots, n. \tag{9}$$

In Ω_S, substitute Eq. (9) into Eq. (8). Then, the problem (4)–(8) is equivalent to the Goursat problem (4)–(7), (9).

With the characteristic variables (Riemann variables)

$$\tau = t - x, \quad x = x \tag{10}$$

in the domain $\Omega'_S = \{0 < x < 1,\ \tau > 0\}$, the system (4)–(7), (9) takes the form as follows:

$$\frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial \tau} = 0, \tag{11}$$

$$\frac{\partial S_i}{\partial \tau} = \Lambda_i(S) C_i \tag{12}$$

with the conditions

$$x = 0: \ C_i = p_i; \tag{13}$$

$$\tau = 0: \ S_i = 0. \tag{14}$$

4 Traveling wave solution

Consider the traveling wave solution for the system (4) and (5). Let $C_i = C_i(w)$, $S_i = S_i(w)$, and $w = x - ut$, where u is the unknown wave velocity. The boundary conditions for the system (4) and (5) at infinity are set as follows:

$$w \to +\infty: \ C_i \to 0, \quad S_i \to 0, \tag{15}$$

$$w \to -\infty: \ C_i \to p_i, \quad S_i \to S_0, \quad S_0 > 0. \tag{16}$$

Then, Eq. (4) takes the form

$$-uC'_i + C'_i - uS'_i = 0. \tag{17}$$

The integration of Eq. (17) over w and the use of the condition (15) yield

$$-uC_i + C_i - uS_i = 0. \tag{18}$$

From Eq. (18), we have

$$C_i = \frac{uS_i}{1-u}. \tag{19}$$

Substituting Eq. (18) into Eq. (6) yields

$$S = \frac{1-u}{u} \sum_{i=1}^{n} \alpha_i C_i. \tag{20}$$

Substituting Eq. (16) into Eq. (20) yields

$$S_0 = \frac{1-u}{u}P, \quad P = \sum_{i=1}^{n} \alpha_i p_i. \tag{21}$$

From Eq. (21), we have

$$u = \frac{P}{S_0 + P}. \tag{22}$$

The traveling wave velocity u satisfies Eq. (4) for the jump from the traveling wave state at the minus infinity C_i^- and S_i^- to the state at the plus infinity C_i^+ and S_i^+, i.e.,

$$[C_i + S_i]u = [C_i], \tag{23}$$

where the symbol $[\Delta]$ corresponds to the jump of the value from the state ahead of the shock Δ^+ to that behind Δ^-, and $[\Delta] = \Delta^+ - \Delta^-$. Substituting Eqs. (15) and (16) into Eq. (23) yields Eq. (22) for the jump speed.

From Eqs. (16), (18), and (22), we have

$$w \to \infty : \ S_i \to S_{i,0} = \frac{p_i}{P} S_i. \tag{24}$$

Then, Eq. (5) takes the form

$$-uS_i' = \Lambda_i(S)C_i. \tag{25}$$

With Eqs. (19) and (22), we can transform Eq. (25) as follows:

$$S_i' + \frac{S_0 + P}{S_0}\Lambda_i(S)S_i = 0. \tag{26}$$

If all filtration functions $\Lambda_i(S)$ are positive on the interval $[0, S_0]$, Eq. (26) shows that the solutions S_i monotonously decrease. Under the additional conditions $S_i|_{w=w_0} = S_i^0$ $(0 < S_i^0 < \frac{p_i}{P}S_0)$, a unique solution of the system (26) can be determined at an arbitrary set point.

If some of the filtration functions have positive roots S_i^* smaller than S_0 and $\Lambda_i(S) = 0$ for $S > S_i^*$, the corresponding solutions are constant, i.e., $S_i = S_{i,0}$, for $w < w_{i,0}$ and decrease to zero for $w > w_{i,0}$, where $S(w_{i,0}) = S_i^*$. If at least one of the equations $S(w) = S_i^*$ does not have a solution, the corresponding solution to Eq. (5) is constant and Eqs. (15) and (16) cannot be satisfied.

Similar to Ref. [38], some transport and deposit characteristics of the particles can be approximately described by the traveling waves. If the capture mechanisms of all types of particles are engaged, all partial deposits S_i $(i = 1, 2, 3, \cdots, n)$ grow over time. If, at a certain level of the total deposit $S = S_0$, the capture of some types of particles ceases, the total deposit continues to grow due to the growth of other partial deposits. The particles lacking a capture mechanism are freely transported by the carrier fluid and do not precipitate.

5 Exact solution for the filtration of a monodisperse suspension with small impurities

A suspension with the identical suspended particles prepared in the laboratory inevitably contains various impurities. Assume that the filtration function of the main particles is linear. Since the content of impurities in the suspension is small, their filtration functions can be considered constant.

Consider the linear-constant model of deep bed filtration for n-size particles with filtration functions, i.e.,

$$\Lambda_1(S) = \alpha - bS, \quad S = S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \cdots + \alpha_n S_n; \quad \Lambda_i = \lambda_i, \quad i = 2, 3, 4, \cdots, n, \tag{27}$$

where a, b, α_i, and λ_i are positive constants.

In the domain Ω'_S, Eqs. (11) and (12) take the forms

$$\frac{\partial C_1}{\partial x} + \frac{\partial S_1}{\partial \tau} = 0, \tag{28}$$

$$\frac{\partial S_1}{\partial \tau} = (a - b(S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \cdots + \alpha_n S_n))C_1, \tag{29}$$

$$\frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial \tau} = 0, \tag{30}$$

$$\frac{\partial S_i}{\partial \tau} = \lambda_i C_i \tag{31}$$

with the conditions in Eqs. (13) and (14).

In Ω'_S, the solution of the system (28)–(31) with the conditions in Eqs. (13) and (14) is

$$C_1 = \frac{p_1 Q(x)}{p_1 + \mathrm{e}^{-bQ(x)\tau}(Q(x)\mathrm{e}^{ax} - p_1)}, \tag{32}$$

$$S_1 = \frac{Q'(x)}{bQ(x)} + \frac{bQ'(x)\tau(Q(x)\mathrm{e}^{ax} - p_1) - \mathrm{e}^{ax}(Q'(x) + aQ(x))}{b(p_1 \mathrm{e}^{bp_1 Q(x)\tau} + Q(x)\mathrm{e}^{ax} - p_1)} + \frac{a}{b} - \tau \sum_{k=2}^{n} \alpha_k \lambda_k p_k \mathrm{e}^{-\lambda_k x}, \tag{33}$$

$$C_i = p_i \mathrm{e}^{-\lambda_i x}, \quad S_i = \lambda_i p_i \mathrm{e}^{-\lambda_i x}\tau, \quad i = 2, 3, 4, \cdots, n. \tag{34}$$

In the domain Ω_S, the solution is

$$C_1 = \frac{p_1 Q(x)}{p_1 + \mathrm{e}^{-bQ(x)(t-x)}(Q(x)\mathrm{e}^{ax} - p_1)}, \tag{35}$$

$$S_1 = \frac{Q'(x)}{bQ(x)} + \frac{bQ'(x)(Q(x)\mathrm{e}^{ax} - p_1)(t-x) - \mathrm{e}^{ax}(Q'(x) + aQ(x))}{b(p_1 \mathrm{e}^{bp_1 Q(x)(t-x)} + Q(x)\mathrm{e}^{ax} - p_1)} + \frac{a}{b} - Q'(x)(t-x), \tag{36}$$

$$C_i = p_i \mathrm{e}^{-\lambda_i x}, \quad S_i = \lambda_i p_i \mathrm{e}^{-\lambda_i x}(t-x), \quad i = 2,\ 3,\ 4, \cdots, n. \tag{37}$$

The suspended concentration C_1 of the main particles increases with time and tends to $Q(x)$. The retained concentrations of small particles grow indefinitely, and the retained concentration of the main particles decreases with a long time. Therefore, Eqs. (35)–(37) are only applicable for a limited time.

Consider the time applicability of the model with one linear filtration function and several constant filtration coefficients. The assumption of non-separation of retained particles from the porous medium frame means an increase in the concentration of the deposit with the increase in t. According to Eq. (29), $\Lambda_1(S) > 0$. The linear function $\Lambda_1(S) = a - bS$ decreases as the total deposit $S(x, t)$ increases. At a certain time $t = t_0$, the linear filtration function $\Lambda_1(S)$ becomes zero. Therefore, the model is not applicable for $t > t_0$.

Since the filtration is the most intense at the entrance of the porous medium, the time boundary t_0 is determined at $x = 0$ by

$$\Lambda_1(S)|_{x=0} = a\mathrm{e}^{-bp_1^2 t} - \frac{A}{p_1}(1 - \mathrm{e}^{-bp_1^2 t}) = 0, \quad A = \sum_{k=2}^{n} \alpha_k p_k \lambda_k. \tag{38}$$

From Eq. (38), we have

$$t_0 = \frac{1}{bp_1^2} \ln\Big(1 + \frac{ap_1}{A}\Big). \tag{39}$$

The linear-constant model is applicable for the period $0 < t < t_0$.

For a monodisperse suspension, the case of a linear-constant filtration function with two independent mechanisms of particle capture is studied in Ref. [25].

6 Exact and asymptotic solutions for bidisperse suspension with large and small particles

6.1 Mathematical model

Consider the filtration of a suspension with particles of two sizes, i.e., large and small. Suppose that the only particle capture mechanism is size-exclusion with the following Langmuir filtration functions:

$$\Lambda_i(S) = \begin{cases} \lambda_i(S_{\max,i} - S), \quad 0 \leqslant S \leqslant S_{\max,i}, \\ 0, \quad S > S_{\max,i}. \end{cases} \tag{40}$$

In the domain $\widetilde{\Omega}_S = \{0 < x < 1,\, \tau > 0\}$, a macroscopic model is set by

$$\frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial \tau} = 0, \tag{41}$$

$$\frac{\partial S_i}{\partial \tau} = \Lambda_i(S) C_i \tag{42}$$

with the following boundary and initial conditions:

$$x = 0:\ C_1 = p_1, \quad C_2 = p_2, \tag{43}$$

$$\tau = 0:\ S_1 = 0, \quad S_2 = 0. \tag{44}$$

In the above equations, $S = S_1 + S_2$, index 1 means large particles, while index 2 means small particles. The ratio of large and small particles is determined by their concentrations at the porous medium inlet. For example, if $p_1 > p_2$, large particles prevail in the suspension. $S_{\max,j}$ corresponds to the maximum amount of retained particles when all pores with sizes smaller than the particle diameter d_j are clogged.

Since larger particles can block more pores, $S_{\max,1} > S_{\max,2}$.

At $t = 0$, $S = 0$, and the capture probability of large particles is greater than that of small particles. Thus, $\lambda_1 S_{\max,1} > \lambda_2 S_{\max,2}$.

For a fixed x, the total deposit $S(x,t)$ increases and tends to $S_{\max,1}$ as t tends to infinity. Therefore, the Langmuir filtration function of large particles is smooth, and can be given by $\Lambda_1(S) = \lambda_1(S_{\max,1} - S)$.

At $\tau = \tau_m(x)$, the total deposit S reaches the value $S_{\max,2}$, and the filtration function $\Lambda_2(S)$ changes from linear dependence to zero in Eq. (40).

Since the particle capture starts from the inlet, the deposit grows faster for smaller x and the function $\tau = \tau_m(x)$ is increasing. Following Ref. [25], the domain $\widetilde{\Omega}_S$ can be divided into three domains, i.e., $\Omega_1 = \{0 < x < 1,\, 0 \leqslant \tau \leqslant \tau_0\}$, $\Omega_2 = \{0 < x < 1,\, \tau_0 \leqslant \tau \leqslant \tau_m(x)\}$, and $\Omega_3 = \{0 < x < 1,\, \tau \geqslant \tau_m(x)\}$, where $\tau_0 = \tau_m(0)$.

In the domains Ω_1 and Ω_2, the filtration function of small particles is linear. In the domain Ω_3, it is zero. Domains Ω_1 and Ω_3 are adjacent to the inlet $x = 0$, while domain Ω_2 is separated from the inlet.

6.2 Exact solution at the inlet $x = 0$

At the porous medium inlet $x = 0$, the filtration function $\Lambda_2(S)$ changes from linear to zero in Eq. (40) at the moment

$$\tau_0 = \frac{1}{\lambda_1 p_1 + \lambda_2 p_2} \ln \frac{\lambda_1 S_{\max,1} p_1 + \lambda_2 S_{\max,2} p_2}{\lambda_1 p_1 (S_{\max,1} - S_{\max,2})}. \tag{45}$$

The total and partial deposit concentrations for $\tau \leqslant \tau_0$ are determined by

$$S(0,\tau) = \frac{\lambda_1 S_{\max,1} p_1 + \lambda_2 S_{\max,2} p_2}{\lambda_1 p_1 + \lambda_2 p_2} (1 - \mathrm{e}^{-(\lambda_1 p_1 + \lambda_2 p_2)\tau}), \tag{46}$$

$$S_1(0,\tau) = \frac{\lambda_1 p_1}{\lambda_1 p_1 + \lambda_2 p_2} \Big(\lambda_2 p_2 (S_{\max,1} - S_{\max,2})\tau + \frac{\lambda_1 S_{\max,1} p_1 + \lambda_2 S_{\max,2} p_2}{\lambda_1 p_1 + \lambda_2 p_2} (1 - \mathrm{e}^{-(\lambda_1 p_1 + \lambda_2 p_2)\tau}) \Big), \tag{47}$$

$$S_2(0,\tau) = \frac{\lambda_2 p_2}{\lambda_1 p_1 + \lambda_2 p_2} \Big(- \lambda_1 p_1 (S_{\max,1} - S_{\max,2})\tau + \frac{\lambda_1 S_{\max,1} p_1 + \lambda_2 S_{\max,2} p_2}{\lambda_1 p_1 + \lambda_2 p_2} (1 - \mathrm{e}^{-(\lambda_1 p_1 + \lambda_2 p_2)\tau}) \Big). \tag{48}$$

For $\tau > \tau_0$, the total and partial deposit concentrations are given by

$$S(0,\tau) = S_{\max,1} - (S_{\max,1} - S_{\max,2})\mathrm{e}^{-\lambda_1 p_1 \tau}, \tag{49}$$

$$S_1(0,\tau) = S_{\max,1} - S_2(0,\tau_0) - (S_{\max,1} - S_{\max,2})\mathrm{e}^{-\lambda_1 p_1 \tau}, \tag{50}$$

$$S_2(0,\tau) = S_2(0,\tau_0). \tag{51}$$

When $\tau \leqslant \tau_0$, both partial deposits grow with time. When $\tau > \tau_0$, the retained concentration S_2 does not change, and thus small particles do not clog the pores at the inlet. The retained concentration S_1 of large particles always grows and tends to $S_{\max,1} - S_2(0,\tau_0)$ at $\tau \to \infty$.

6.3 Exact solution on the concentration front $\tau = 0$

According to the condition (44), the filtration functions are constant on the concentration front, i.e., $\Lambda_i(S) = \lambda_i S_{\max,i}$. Substitute Eq. (42) into Eq. (41) yields

$$\frac{\partial C_i}{\partial x} + \lambda_i S_{\max,i} C_i = 0. \tag{52}$$

The solution to Eq. (52) with the conditions in Eq. (43) is

$$C_i = p_i \mathrm{e}^{-\lambda_i S_{\max,i} x}, \quad i = 1, 2, \tag{53}$$

which determines the suspended concentrations of large and small particles on the concentration front $\tau = 0$. The concentrations C_i $(i = 1, 2)$ decrease with increasing x, since behind the front, some of the small and large particles are retained in the pores.

6.4 Asymptotics of the filtration problem for a bidisperse suspension

Consider a bidisperse suspension, where $S_{\max,1} \gg S_{\max,2}$. In $\widetilde{\Omega}_S$, the asymptotics of Eqs. (41)–(44) is constructed with respect to a small parameter $S_{\max,2}$. From Eq. (42), it follows that the partial deposit concentration S_2 is much smaller than S_1 and the total deposit concentration $S \approx S_1$. Therefore, the main asymptotic terms can be obtained from

$$\frac{\partial C_1}{\partial x} + \frac{\partial S_1}{\partial \tau} = 0, \quad \frac{\partial S_1}{\partial \tau} = \lambda_1 (S_{\max,1} - S_1) C_1. \tag{54}$$

The solution to Eq. (54) with the conditions in Eqs. (43) and (44) is

$$C_1(x,\tau) = \frac{p_1 \mathrm{e}^{\lambda_1 p_1 \tau}}{\mathrm{e}^{\lambda_1 p_1 \tau} + \mathrm{e}^{\lambda_1 S_{\max,1} x} - 1}, \quad S_1(x,\tau) = \frac{S_{\max,1}(\mathrm{e}^{\lambda_1 p_1 \tau} - 1)}{\mathrm{e}^{\lambda_1 p_1 \tau} + \mathrm{e}^{\lambda_1 S_{\max,1} x} - 1}. \tag{55}$$

The main asymptotic term of the boundary $\tau = \tau_m(x)$ between Ω_2 and Ω_3 is determined by $S_1(x,\tau) = S_{\max,2}$. From the second formula in Eq. (55), we have

$$\tau_m(x) = \frac{1}{\lambda_1 p_1} \ln \frac{S_{\max,1} + S_{\max,2}(\mathrm{e}^{\lambda_1 S_{\max,1} x} - 1)}{S_{\max,1} - S_{\max,2}}. \tag{56}$$

From the system (41) and (42) and the conditions (43) and (44), we have

$$C_2(x,\tau) = p_2 + O(S_{\max,2}), \tag{57}$$

$$S_2(x,\tau) = \begin{cases} \dfrac{\lambda_2 p_2}{\lambda_1 p_1} \dfrac{S_{\max,1}\mathrm{e}^{\lambda_1 S_{\max,1} x}}{\mathrm{e}^{\lambda_1 S_{\max,1} x} - 1} \ln \dfrac{\mathrm{e}^{\lambda_1 p_1 \tau}\mathrm{e}^{\lambda_1 S_{\max,1} x}}{\mathrm{e}^{\lambda_1 p_1 \tau} + \mathrm{e}^{\lambda_1 S_{\max,1} x} - 1} - \lambda_2 p_2 (S_{\max,1} - S_{\max,2})\tau \\ \quad + O(S_{\max,2})^3, \quad \tau \leqslant \tau_m(x), \\ S_2(x, \tau_m(x)) = \dfrac{\lambda_2 p_2}{\lambda_1 p_1} \dfrac{(S_{\max,2})^2}{S_{\max,1}} \mathrm{e}^{\lambda_1 S_{\max,1} x} + O(S_{\max,2})^3, \quad \tau > \tau_m(x). \end{cases} \tag{58}$$

According to Eq. (58), the retained concentration of small particles stabilizes and does not depend on time at $\tau > \tau_m(x)$. Near the porous medium inlet, large particles intensively block the pores and prevent small particles from precipitating. With the increase in the distance x to the inlet, the suspended concentration C_1 of large particles decreases rapidly, and the small particle deposit S_2 increases.

7 Asymptotic solutions

7.1 Asymptotics for the filtration of an aqueous solution with small impurities

Consider the filtration of water with a small portion of different suspended impurities in a porous medium.

Suppose that the functions $\Lambda_i(S)$ are smooth and can be expanded in powers of S, i.e.,

$$\Lambda_i(S) = \lambda_i^0 + \lambda_i^1 S + \lambda_i^2 S^2 + O(S^3), \tag{59}$$

where

$$\lambda_i^0 = \Lambda_i(0), \quad \lambda_i^1 = \frac{\partial \Lambda_i(0)}{\partial S}, \quad \lambda_i^2 = \frac{1}{2}\frac{\partial^2 \Lambda_i(0)}{\partial S^2}, \quad i = 1,2,3,\cdots,n. \tag{60}$$

Consider the condition (13) with small injected concentrations at the inlet $x = 0$,

$$p_i = \varepsilon q_i, \quad q_i > 0, \tag{61}$$

where ε is a small positive parameter.

The asymptotics of the system (11)–(14) is constructed as follows:

$$S_i = \varepsilon s_i^1 + \varepsilon^2 s_i^2 + \varepsilon^3 s_i^3 + O(\varepsilon^4), \quad C_i = \varepsilon c_i^1 + \varepsilon^2 c_i^2 + \varepsilon^3 c_i^3 + O(\varepsilon^4), \quad i = 1,2,3,\cdots,n. \tag{62}$$

Denote

$$S^1 = \sum_{m=1}^{n} \alpha_m s_m^1, \quad S^2 = \sum_{m=1}^{n} \alpha_m s_m^2.$$

The appropriate expansions of the total deposit and the filtration functions in powers of ε are

$$S = \varepsilon S^1 + \varepsilon^2 S^2 + O(\varepsilon^3), \tag{63}$$

$$\Lambda_i(S) = \lambda_i^0 + \varepsilon \lambda_i^1 S^1 + \varepsilon^2 (\lambda_i^1 S^2 + \lambda_i^2 (S^1)^2) + O(\varepsilon^3). \tag{64}$$

Substituting Eqs. (62)–(64) into Eqs. (11) and (12) and the equation of the terms with identical powers of the small parameter ε yields the recurrent system of linear ordinary differential equations as follows:

$$\frac{\partial c_i^1}{\partial x} + \lambda_i^0 c_i^1 = 0, \quad \frac{\partial s_i^1}{\partial \tau} = \lambda_i^0 c_i^1, \tag{65}$$

$$\frac{\partial c_i^2}{\partial x} + \lambda_i^0 c_i^2 + \lambda_i^1 S^1 c_i^1 = 0, \quad \frac{\partial s_i^2}{\partial \tau} = \lambda_i^0 c_i^2 + \lambda_i^1 S^1 c_i^1, \tag{66}$$

$$\begin{cases} \dfrac{\partial c_i^3}{\partial x} + \lambda_i^0 c_i^3 + \lambda_i^1 S^1 c_i^2 + c_i^1(\lambda_i^1 S^2 + \lambda_i^2 (S^1)^2) = 0, \\ \dfrac{\partial s_i^3}{\partial \tau} = \lambda_i^0 c_i^3 + \lambda_i^1 S^1 c_i^2 + c_i^1(\lambda_i^1 S^2 + \lambda_i^2 (S^1)^2). \end{cases} \tag{67}$$

The boundary and initial conditions for this system with Eqs. (13) and (14) are

$$c_i^1|_{x=0} = q_i, \quad c_i^2|_{x=0} = c_i^3|_{x=0} = 0, \quad s_i^j|_{\tau=0} = 0, \quad j = 1,2,3, \quad i = 1,2,3,\cdots,n. \tag{68}$$

The terms of the asymptotic expansions obtained by the successive solution to Eqs. (65)–(67) with the conditions in Eq. (68) are

$$c_i^1 = q_i \mathrm{e}^{-\lambda_i^0 x}, \quad s_i^1 = q_i \lambda_i^0 \mathrm{e}^{-\lambda_i^0 x}\tau, \tag{69}$$

$$\begin{cases} c_i^2 = q_i \lambda_i^1 \mathrm{e}^{-\lambda_i^0 x} \displaystyle\sum_{m=1}^{n} \alpha_m q_m (\mathrm{e}^{-\lambda_m^0 x} - 1)\tau, \\ s_i^2 = \lambda_i^1 q_i \mathrm{e}^{-\lambda_i^0 x} \displaystyle\sum_{m=1}^{n} \alpha_m q_m ((\lambda_i^0 + \lambda_m^0)\mathrm{e}^{-\lambda_m^0 x} - \lambda_i^0)\frac{\tau^2}{2}, \end{cases} \tag{70}$$

$$c_i^3 = q_i \mathrm{e}^{-\lambda_i^0 x}\tau^2 F(x), \quad s_i^3 = q_i \mathrm{e}^{-\lambda_i^0 x}\frac{\tau^3}{3}(\lambda_i^0 F(x) - F'(x)), \tag{71}$$

where

$$F(x) = \sum_{k,l=1}^{n} \alpha_k \alpha_l q_k q_l \Big(\Big((\lambda_i^1)^2 \lambda_l^0 + \frac{1}{2}\lambda_i^1 \lambda_l^1 (\lambda_k^0 + \lambda_l^0) + \lambda_i^2 \lambda_k^0 \lambda_l^0 \Big) \frac{\mathrm{e}^{-(\lambda_k^0 + \lambda_l^0)x} - 1}{\lambda_k^0 + \lambda_l^0}$$
$$+ \lambda_i^1 \Big(\lambda_i^1 + \frac{1}{2}\lambda_l^1 \Big)(1 - \mathrm{e}^{-\lambda_l^0 x}) \Big).$$

The substitution of Eqs. (69)–(71) into Eq. (62) and the inverse change of variables in Eq. (10) give the asymptotics of the problem (4)–(8) in Ω_S. The asymptotic solution is applicable when $\varepsilon\tau \ll 1$. For small positive ε, the asymptotics is close to the exact solution on a large time interval.

7.2 Asymptotic solution for suspension in a porous medium with small kinetic rates

If the pore sizes are generally larger than the particle size, the retention of the suspended particles is rare and the filtration functions are small.

Consider the system (4)–(8) with small filtration functions as follows:

$$\Lambda_i(S) = \varepsilon \Lambda_i'(S), \quad i = 1,2,3,\cdots,n. \tag{72}$$

Assume that the functions $\Lambda_i'(S)$ have the form (59).

In the domain Ω_S, the asymptotics is constructed as follows:

$$S_i = \varepsilon s_i^1 + \varepsilon^2 s_i^2 + \varepsilon^3 s_i^3 + O(\varepsilon^4), \quad C_i = p_i + \varepsilon c_i^1 + \varepsilon^2 c_i^2 + \varepsilon^3 c_i^3 + O(\varepsilon^4), \quad i = 1,2,3,\cdots,n. \tag{73}$$

An asymptotic solution in powers of ε to the system (4)–(8) with the filtration functions of Eq. (72) in Ω_S is

$$\begin{aligned} C_i(x,t) &= p_i + p_i x\varepsilon\Big(-\lambda_i^0 + \Big((\lambda_i^0)^2\frac{x}{2} - \lambda_i^1 A_0(t-x)\Big)\varepsilon + \Big(-(\lambda_i^0)^3\frac{x^2}{6} \\ &\quad + \lambda_i^1(A_2 + 2\lambda_i^0 A_0)\frac{x}{2}(t-x) - A_0(\lambda_i^1 A_1 + 2\lambda_i^2 A_0)\frac{(t-x)^2}{2}\Big)\varepsilon^2\Big) + O(\varepsilon^4), \end{aligned} \tag{74}$$

$$\begin{aligned} S_i(x,t) &= p_i(t-x)\varepsilon\Big(\lambda_i^0 + \Big(\lambda_i^1 A_0\frac{t-x}{2} - (\lambda_i^0)^2 x\Big)\varepsilon + \Big((\lambda_i^0)^3\frac{x^2}{2} - \lambda_i^1(A_2 + 2\lambda_i^0 A_0)x\frac{t-x}{2} \\ &\quad + A_0(\lambda_i^1 A_1 + 2\lambda_i^2 A_0)\frac{t-x^2}{6}\Big)\varepsilon^2\Big) + O(\varepsilon^4), \end{aligned} \tag{75}$$

where $i = 1, 2, 3, \cdots, n$, and

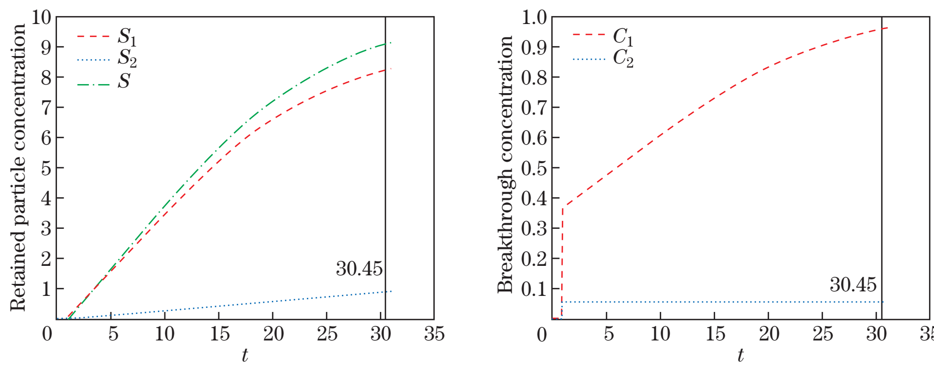
$$A_0 = \sum_{m=1}^{n}\alpha_m\lambda_m^0 p_m, \quad A_1 = \sum_{m=1}^{n}\alpha_m\lambda_m^1 p_m, \quad A_2 = \sum_{m=1}^{n}\alpha_m(\lambda_m^0)^2 p_m.$$

With a small epsilon, the suspended and retained particle concentrations are first linear and then nonlinear when time goes on.

8 Examples and discussion

8.1 Monodisperse suspension with small impurities (Section 5)

Consider the model with linear and constant filtration functions $\Lambda_1 = 1 - 0.1S$ and $\Lambda_2 = 0.5$. The show that the deposit of the main particles increases nonlinearly with time, while the impurity deposit grows linearly and slowly (see Fig. 2). The total deposit tends to $S_{\max}$. According to Eq. (39), the linear-constant model is applicable up to approximately $t_0 = 30$ and $0 \leqslant x \leqslant 1$, and the result of the numerical calculation is $t_0 = 30.45$.



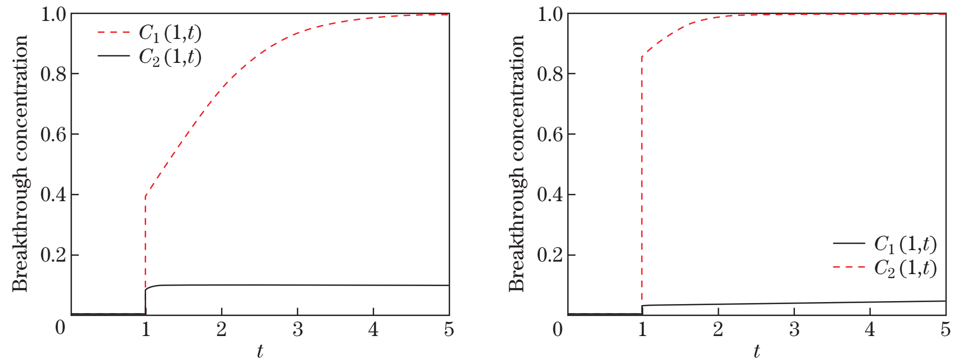
(a) Partial and total retained particle concentrations (b) Breakthrough concentration at the effluent

Fig. 2 Exact solution for a monodisperse suspension with small impurities at the outlet $x = 1$ (color online)

8.2 Bidisperse suspension with large and small particles (Section 6)

In this subsection, the retention of large and small particles is determined by the filtration functions $\Lambda_1 = 1 - S$ and $\Lambda_2 = 0.25 - S$, respectively. Two cases are considered, i.e., a flow of large particle suspension with impurities of small particles ($p_1 = 1$ and $p_2 = 0.1$) and a flow of small particles with impurities of large particles ($p_1 = 0.1$ and $p_2 = 1$). The obtained breakthrough concentrations $C_i(1,t)$ at the outlet and the deposit profiles $S_i(x,t_0)$ and $S(x,t_0)$ are shown in Figs. 3–7.

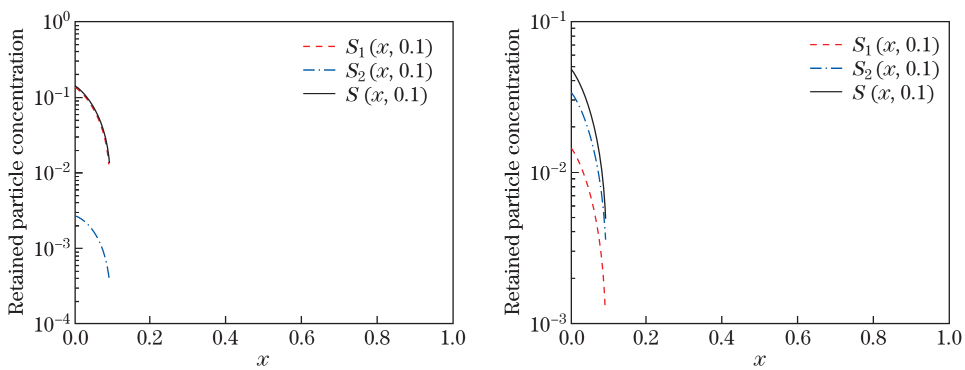
It is seen in Fig. 3 that the filtration for the case of large particle suspension with impurities of small particles stops quickly after the breakthrough concentrations become equal to the



(a) $p_1 = 1, p_2 = 0.1$ (b) $p_1 = 0.1, p_2 = 1$

Fig. 3 Results of the breakthrough concentrations $C_1(1,t)$ and $C_2(1,t)$ (color online)

injection concentration $p_2 = 0.1$, and the suspended concentration of the main large particles slowly approaches the limiting value $p_1 = 1$. Meanwhile, the breakthrough concentration of the main small particles quickly reaches the limiting value $p_2 = 1$ for the case of small particles with impurities of large particles, and the suspended concentration of large particle impurities increases almost linearly. Figure 4 shows that for a short time $t = 0.1$, the retention profiles of large and small particles decrease behind the concentration front, and the concentrations are zero ahead of the front when $x > 0.1$.



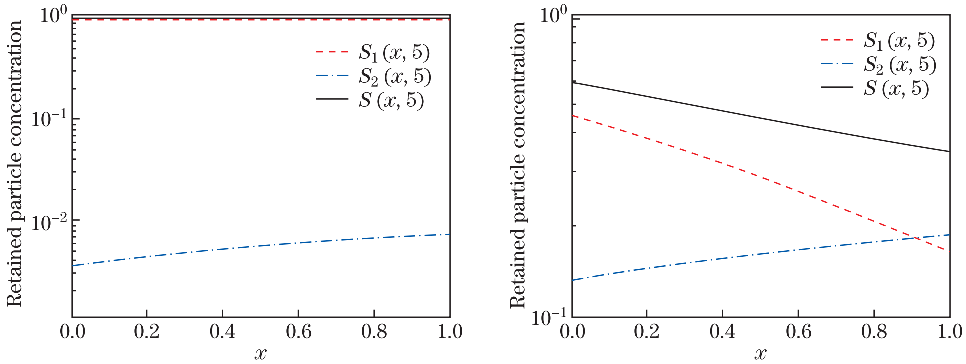
(a) $p_1 = 1, p_2 = 0.1$ (b) $p_1 = 0.1, p_2 = 1$

Fig. 4 Retention profiles of $S_1(x, 0.1)$, $S_2(x, 0.1)$, and $S(x, 0.1)$ (color online)

Figure 5 shows the retention profiles of $S_1(x, t_0)$, $S_2(x, t_0)$, and $S(x, t_0)$ at $t_0 = 5$. It can be seen that, for the case of large particle suspension with impurities of small particles, the retention profile of the main large particles is practically constant near the limiting value $p_1 = 1$, while for the case of small particles with impurities of large particles, the retention profile of large particle impurities decreases rapidly.

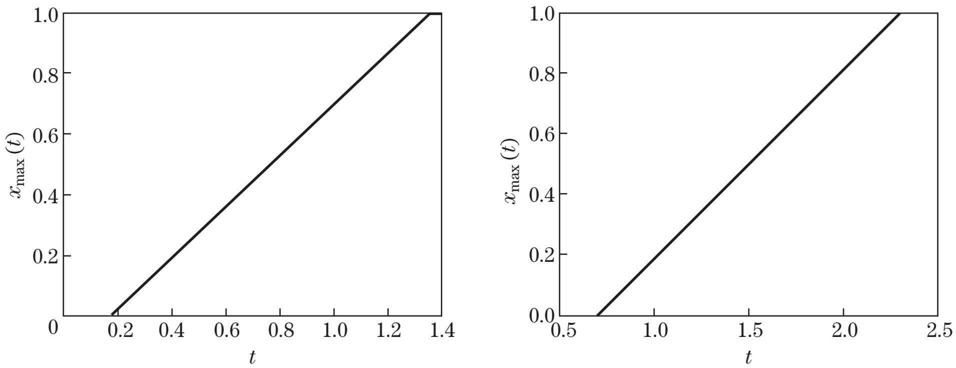
Figure 6 shows the maximum point $x_{\max}(t)$ of the retention profile $S_2(x, t)$. It can be seen that $S_1(x, t_0)$ monotonously decreases at all t_0. Similar to Ref. [35], $S_2(x, t_0)$ monotonically decreases at small t_0 and monotonically increases at large time. At intermediate time, the profiles have a maximum point $x_{\max}(t)$, which moves from the inlet $x = 0$ to the outlet $x = 1$ with an "almost constant" velocity (see Fig. 6). Moreover, the velocity of the profile maximum point depends on the injected concentrations. The velocity $v_1 = 0.83$ in Fig. 6(a) while $v_2 = 0.45$ in Fig. 6(b). If small particles are the main part of the suspension, the velocity of the maximum point is less than that when they form an impurity.

Figure 7 shows the boundary between Ω_2 and Ω_3 and its asymptotics. The filtration of small particles of the suspension stops at $\tau = \tau_m(x)$. For a longer time, small particles are transported through the porous medium without retention, while large particles are retained



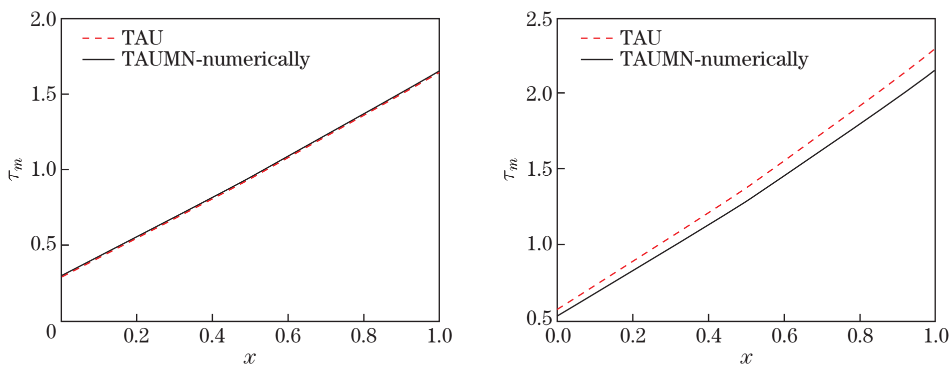
(a) $p_1 = 1, p_2 = 0.1$ (b) $p_1 = 0.1, p_2 = 1$

Fig. 5 Retention profiles of $S_1(x, 5)$, $S_2(x, 5)$, and $S(x, 5)$ (color online)



(a) $p_1 = 1, p_2 = 0.1$ (b) $p_1 = 0.1, p_2 = 1$

Fig. 6 Maximum point $x_{\max}(t)$ of the retention profile $S_2(x, t)$



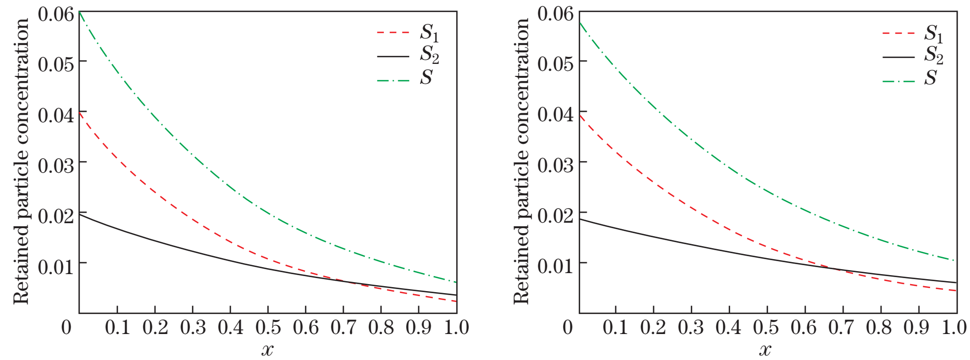
(a) $p_1 = 1, p_2 = 0.1$ (b) $p_1 = 0.5, p_2 = 0.5$

Fig. 7 Boundary $\tau = \tau_m(x)$ between Ω_2 and Ω_3 (color online)

at any time. In both Figs. 7(a) and 7(b), the asymptotic formulae for the boundary are close to the numerical solution.

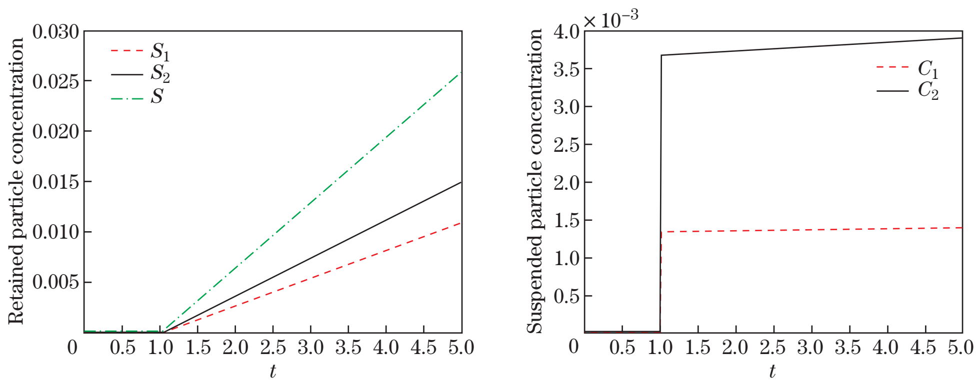
8.3 Aqueous solution with small impurities (Subsection 7.1)

In this example, the filtration functions are $\Lambda_1 = 2 - 0.5S$ and $\Lambda_2 = 1 - S$, and the parameters are identical, i.e., $\alpha_1 = \alpha_2 = 1$ and $q_1 = q_2 = 1$. The results are shown in Figs. 8 and 9. Figure 8 shows the retention profiles S_1, S_2, and S for the filtration of water with small impurities at $t = 2$ and $t = 5$. It can be seen that in the area adjacent to the porous medium inlet, large



(a) At $t = 2$ (b) At $t = 5$

Fig. 8 Retention profiles S, S_1, and S_2 for the filtration of water with small impurities (color online)



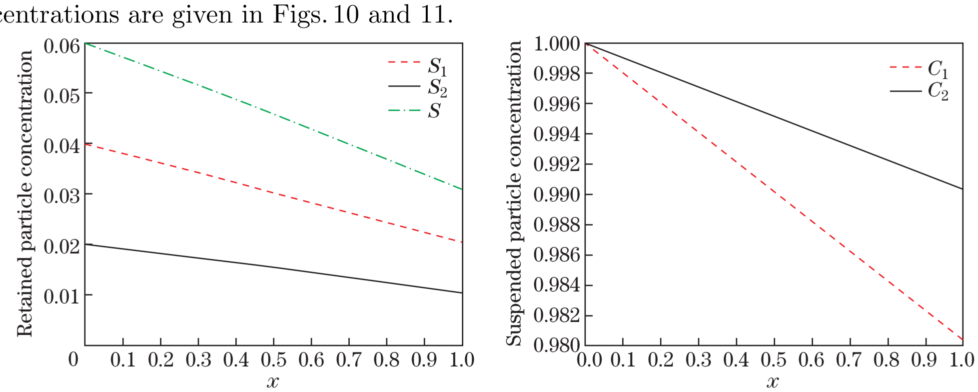
(a) Retained particle concentration (b) Suspended concentration

Fig. 9 Dynamics of the retained and suspended particle concentrations at the outlet of the porous medium (color online)

particles prevail in the deposit, while in the area near the outlet, the retained concentration of small particles becomes higher. Moreover, at the outlet of the porous medium $x = 1$, the filtration of small particles is more intensive, and the suspended and retained concentrations of small particles grow faster than the large ones.

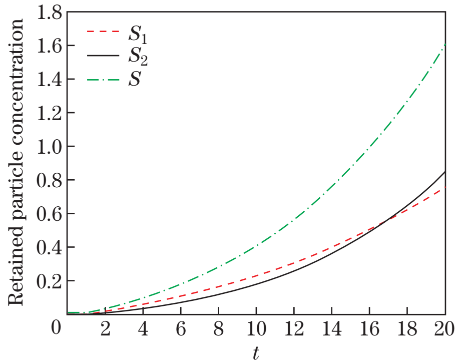
8.4 Suspension in a porous medium with small kinetic rates (Subsection 7.2)

The solution is calculated for the filtration functions $\tilde{\Lambda}_1 = 2 - 0.5S$, $\tilde{\Lambda}_2 = 1 - S$ and the parameters $\alpha_1 = \alpha_2 = 1$, $p_1 = p_2 = 1$. The profiles and dynamics of large and small particle concentrations are given in Figs. 10 and 11.

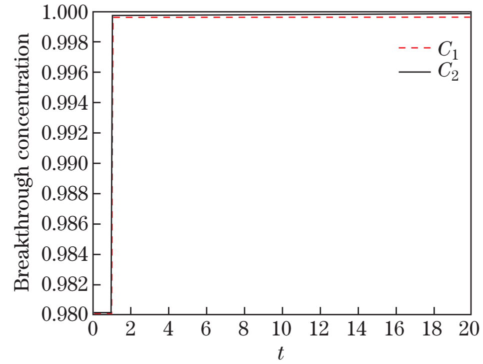


(a) Retention particle profiles (b) Suspended particle profiles

Fig. 10 Profiles for the particles with small kinetic rates at $t = 2$ (color online)



(a) Retained particle concentration



(b) Breakthrough concentration at the outlet

Fig. 11 Dynamics of the retained and suspended particles at $x = 1$ (color online)

From Fig. 10, it can be seen that at $t = 2$, the retained concentration of large particles exceeds the retained concentration of small particles, while the excess decreases with the increase in the distance from the porous medium inlet. Figure 11 shows that when $t > 17$, the retained concentration of small particles becomes higher than the retained concentration of large particles near the outlet $x = 1$. Moreover, the asymptotics approximates well the solution for small t. With increasing time, the discrepancy between the numerical and asymptotic solutions increases, and the asymptotics becomes inappropriate. To increase the availability interval for an asymptotic solution, it is necessary to construct a higher order asymptotics.

The coupling of the flow field with the particle deposit is achieved by the introduction of the accessible porosity fraction, where the particles can penetrate the corresponding fractional flow through the accessible area[39–40] and the particle-free water flows through the inaccessible area. The coupling significantly complicates the model of particle capture. The upscaled equations contain additional accessibility and flux-reduction factors depending on the concentration of the total deposit. For a monodisperse suspension, the analytic solutions were obtained in Ref. [31]. A similar problem for a polydisperse suspension requires a separate study.

The introduction of velocity-dependent maximum retention concentration reflects instant particle detachment by the increased flow rate[11,41]. The velocity-dependent maximum retention function is determined by the torque balance of electrostatic, drag, and lift forces exerting the attached fine particle. This model exhibits an excellent match with the laboratory data[42–44].

The present paper deals with the filtration of a suspension with small impurities. For a monodisperse suspension, the effect of impurities is insignificant, and their filtration functions can be considered constant. The exact solution makes it possible to study the effect of additives in the preparation of a mixture and how random impurities affect the filtration of the suspension in the porous medium.

The numerical calculations show that during deep bed filtration of a bidisperse suspension in a porous medium, the retained profile of large particles monotonously decreases at all time. The profile of small retained particles decreases at the beginning of the filtration process and increases at large time. At intermediate time, the profile has a maximum point moving with an "almost constant" speed. Currently, there are no analytical formulae of retained profiles.

The asymptotics constructed here corresponds to the filtration of an aqueous solution with small impurities and a polydisperse suspension with small kinetic rates. The asymptotic solutions enable us to study the dependence of small filtration functions on the total deposit concentration.

This paper considers the so-called deep bed filtration in clean bed, corresponding to the zero initial conditions in Eq. (3). The same system (1) and (2) with non-zero initial conditions and the zero boundary condition at $x = 0$ describes fine migration in porous media and natural reservoirs, where the injected clean water lifts naturally-attached particles from the rock[42–44]. Those problems occur in plant irrigation and Vadoze zone dynamics. The asymptotic method developed in this work can be applied for the prediction of suspended and retained concentrations during migration of the natural reservoir fines.

Both asymptotic solutions of injection into clean bed and fine migration allow direct extension to the case of two-phase colloidal flows, where the splitting mapping degenerates the two-phase system to auxiliary single-phase flow equations[45–46].

9 Conclusions

A new model for the filtration of a polydisperse suspension in a porous medium is proposed, which generalizes the macroscopic model of the long-term deep bed filtration of a suspension with the identical suspended solid particles and size-exclusion particle capture mechanism. The assumption of the dependence of all filtration functions on the total retained concentration merges the separate equations for each type of particles into a unified hyperbolic system of equations.

The exact solution for the suspension flow of one high and several low concentrations of suspended particles and the global asymptotics for the filtration of an aqueous solution with small impurities and a polydisperse suspension with small kinetic rates lead us to the following conclusions:

(i) There are no suspended and retained particles in the zone ahead of the front.

(ii) There is a break in the concentrations of suspended particles of all types on the front.

(iii) The exact and asymptotic solutions determine the dependence of the suspended and retained particle concentrations on the model parameters, which is important for solving the inverse problem.

(iv) The retained profile of small particles of a bidisperse suspension changes its monotonic behavior with time.

(v) During long-time filtration, the large particles deposit near the inlet of the porous medium, and the small particles deposit near the outlet.

References

[1] BEDRIKOVETSKY, P. *Mathematical Theory of Oil and Gas Recovery with Applications to Ex-USSR Oil and Gas Fields*, Springer Science and Business Media, Des Moines (2013)

[2] CIVAN, F. *Reservoir Formation Damage* (*Fundamentals*, *Modeling*, *Assessment*, *and Mitigation*), Gulf Professional Publishing, Houston (2007)

[3] LU, B., LIU, X., DONG, P., TICK, G. R., ZHENG, C., ZHANG, Y., MAHMOOD-UI-HASSAN, M., BAI, H., and LAMY, E. Quantifying fate and transport of nitrate in saturated soil systems using fractional derivative model. *Applied Mathematical Modelling*, **81**, 279–295 (2020)

[4] ELIMELECH, M., GREGORY, J., JIA, X., and WILLIAMS, R. *Particle Deposition and Aggregation*: *Measurement*, *Modelling*, *and Simulation*, Butterworth-Heinemann, New York (2013)

[5] DOMGA, R., TCHATCHUENG, J. B., TOGUE-KAMGA, F., and NOUBACTEP, C. Discussing porosity loss of FeO packed water filters at ground level. *Chemical Engineering Journal*, **263**, 127–134 (2014)

[6] ZEMAN, L. J. and ZYDNEY, A. L. *Microfiltration and Ultrafiltration*: *Principles and Applications*, Marcel Dekker, New York (1996)

[7] BRADFORD, S. A., YATES, S. R., BETTAHAR, M., and SIMUNEK, J. Physical factors affecting the transport and fate of colloids in saturated porous media. *Water Resources Research*, **38**, 1327–1334 (2002)

[8] GITIS, V., RUBINSTEIN, I., LIVSHTS, M., and ZISKIND, G. Deep-bed filtration model with multistage deposition kinetics. *Chemical Engineering Journal*, **163**, 78–85 (2010)

[9] BEDRIKOVETSKY, P. Upscaling of stochastic micro model for suspension transport in porous media. *Transport in Porous Media*, **75**, 335–369 (2008)

[10] TIEN, C. *Principles of Filtration*, Elsevier, Oxford (2012)

[11] BEDRIKOVETSKY, P., SIQUEIRA, F. D., FURTADO, C., and SOUZA, A. L. S. Modified particle detachment model for colloidal transport in porous media. *Transport in Porous Media*, **86**, 353–383 (2011)

[12] HERZIG, J. P., LECLERC, D. M., and GOFF, P. L. Flow of suspensions through porous media—application to deep filtration. *Industrial and Engineering Chemistry Research*, **62**, 8–35 (1970)

[13] POLYAKOV, Y. S. and ZYDNEY, A. L. Ultrafiltration membrane performance: effects of pore blockage/constriction. *Journal of Membrane Science*, **434**, 106–120 (2013)

[14] TUFENKJI, N. Colloid and microbe migration in granular environments: a discussion of modelling methods. *Colloidal Transport in Porous Media*, Springer, Berlin (2007)

[15] TUFENKJI, N. and ELIMELECH, M. Correlation equation for predicting single-collector efficiency in physicochemical filtration in saturated porous media. *Environmental Science and Technology*, **38**, 529–536 (2004)

[16] POLYANIN, A. D. and MANZHIROV, A. V. *Handbook of Mathematics for Engineers and Scientists*, Chapman and Hall/CRC Press, Boca Raton (2006)

[17] SHARMA, M. and YORTSOS, Y. Transport of particulate suspensions in porous media: model formulation. *AIChE Journal*, **33**, 1636–1643 (1987)

[18] SHAPIRO, A. Elliptic equation for random walks: application to transport in microporous media. *Physica A*: *Statistical Mechanics and Its Applications*, **375**, 81–96 (2007)

[19] YUAN, H. and SHAPIRO, A. Colloid transport and retention: recent advances in colloids filtration theory. *Colloids*: *Classification*, *Properties and Applications*, Nova Science Publishers, New York (2013)

[20] MACKIE, R. I. and ZHAO, Q. A framework for modeling removal in the filtration of polydisperse suspensions. *Water Research*, **33**, 794–806 (1999)

[21] HARRIS, T. C., HOGG, A. J., and HUPPERT, H. E. Polydisperse particle-driven gravity currents. *Journal of Fluid Mechanics*, **472**, 333–371 (2002)

[22] BENNACER, L., AHFIR, N. D., BOUANANI, A., ALEM, A., and WANG, H. Suspended particles transport and deposition in saturated granular porous medium: particle size effects. *Transport in Porous Media*, **100**, 377–392 (2013)

[23] TRZASKUS, K., ELSHOF, M., KEMPERMAN, A., and NIJMEIJER, K. Understanding the role of nanoparticle size and polydispersity in fouling development during dead-end microfiltration. *Journal of Membrane Science*, **516**, 152–161 (2016)

[24] KUZMINA, L. I., OSIPOV, Y. V., and ZHEGLOVA, Y. G. Analytical model for deep bed filtration with multiple mechanisms of particle capture. *International Journal of Non-Linear Mechanics*, **105**, 242–248 (2018)

[25] ZHANG, H., MALGARESI, G. V. C. P., and BEDRIKOVETSKY, P. Exact solutions for suspension-colloidal transport with multiple capture mechanisms. *International Journal of Non-Linear Mechanics*, **105**, 27–42 (2018)

[26] POLYANIN, A. and ZAITSEV, V. *Handbook of Nonlinear Partial Differential Equations*, Chapman and Hall/CRC Press, Boca Raton (2012)

[27] VYAZMINA, E. A., BEDRIKOVETSKII, P. G., and POLYANIN, A. D. New classes of exact solutions to nonlinear sets of equations in the theory of filtration and convective mass transfer. *Theoretical Foundations of Chemical Engineering*, **41**, 556–564 (2007)

[28] ALVAREZ, A. C., HIME, G., MARCHESIN, D., and BEDRIKOVETSKY, P. G. The inverse problem of determining the filtration function and permeability reduction in flow of water with particles in porous media. *Transport in Porous Media*, **70**, 43–62 (2007)

[29] ALVAREZ, A. C., HIME, G., SILVA, J. D., and MARCHESIN, D. Analytic regularization of an inverse filtration problem in porous media. *Inverse Problems*, **29**, 025006 (2013)

[30] CHALK, P., GOODING, N., HUTTEN, S., YOU, Z., and BEDRIKOVETSKY, P. Pore size distribution from challenge coreflood testing by colloidal flow. *Chemical Engineering Research and Design*, **90**, 63–77 (2012)

[31] KUZMINA, L. I., NAZAIKINSKII, V. E., and OSIPOV, Y. V. On a deep bed filtration problem with finite blocking time. *Russian Journal of Mathematical Physics*, **26**, 130–134 (2019)

[32] KUZMINA, L. I. and OSIPOV, Y. V. Deep bed filtration asymptotics at the filter inlet. *Procedia Engineering*, **153**, 366–370 (2016)

[33] ANDREUCCI, D. and TEDEEV, A. F. Asymptotic behavior for the filtration equation in domains with noncompact boundary. *Communications in Partial Differential Equations*, **42**, 347–365 (2017)

[34] KUZMINA, L. I., OSIPOV, Y. V., and GALAGUZ, Y. P. A model of two-velocity particles transport in a porous medium. *International Journal of Non-Linear Mechanics*, **93**, 1–6 (2017)

[35] MALGARESI, G., COLLINS, B., ALVARO, P., and BEDRIKOVETSKY, P. Explaining non-monotonic retention profiles during flow of size-distributed colloids. *Chemical Engineering Journal*, **375**, 121984 (2019)

[36] POLYANIN, A. and DILMAN, V. V. *Methods of Modeling Equations and Analogies in Chemical Engineering*, CRC Press/Begell House, Boca Raton (1994)

[37] TANG, Y. D., JIN, T., and FLESCH, R. C. C. Effect of mass transfer and diffusion of nanofluid on the thermal ablation of malignant cells during magnetic hyperthermia. *Applied Mathematical Modelling*, **83**, 122–135 (2020)

[38] GERBER, G., WEITZ, D. A., and COUSSOT, P. Propagation and adsorption of nanoparticles in porous medium as traveling waves. *Physical Review Research*, **2**, 033074 (2020)

[39] CHALK, P., GOODING, N., HUTTEN, S., YOU, Z., and BEDRIKOVETSKY, P. Pore size distribution from challenge coreflood testing by colloidal flow. *Chemical Engineering Research and Design*, **90**, 63–77 (2012)

[40] GALAGUZ, Y. P., KUZMINA, L. I., and OSIPOV, Y. V. Problem of deep bed filtration in a porous medium with the initial deposit. *Fluid Dynamics*, **54**, 85–97 (2019)

[41] YOU, Z., BEDRIKOVETSKY, P., BADALYAN, A., and HAND, M. Particle mobilization in porous media: temperature effects on competing electrostatic and drag forces. *Geophysical Research Letters*, **42**, 2852–2860 (2015)

[42] BEDRIKOVETSKY, P., ZEINIJAHROMI, A., SIQUEIRA, F. D., FURTADO, C., and DE SOUZA, A. L. S. Particle detachment under velocity alternation during suspension transport in porous media. *Transport in Porous Media*, **91**, 173–197 (2012)

[43] YOU, Z., BADALYAN, A., YANG, Y., and BEDRIKOVETSKY, P. Formation damage challenges in geothermal reservoirs: laboratory and mathematical modeling. *Formation Damage During Improved Oil Recovery: Fundamentals and Applications*, Gulf Professional Publishing, Oxford (2018)

[44] DEL VIGO, A., ZUBELZU, S., and JUANA, L. Numerical routine for soil water dynamics from trickle irrigation. *Applied Mathematical Modelling*, **83**, 371–385 (2020)

[45] BORAZJANI, S., ROBERTS, A. J., and BEDRIKOVETSKY, P. Splitting in systems of PDEs for two-phase multicomponent flow in porous media. *Applied Mathematical Letters*, **53**, 25–32 (2015)

[46] BORAZJANI, S. and BEDRIKOVETSKY, P. Exact solutions for two-phase colloidal-suspension transport in porous media. *Applied Mathematical Modelling*, **44**, 296–320 (2017)