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The symmetry and loading-independency of multiple inclusions enclosing uniform stresses in an infinite elastic plane*

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Abstract The identification of multiple interacting inclusions with uniform internal stresses in an infinite elastic matrix subjected to a uniform remote loading is of fundamental importance in the mechanics and design of particulate composite materials. In anti-plane shear and plane deformations, certain sufficient conditions have been established in the literature which guarantee uniform internal stresses inside multiple interacting inclusions displaying various symmetries when the matrix is subjected to specific uniform remote loading. Correspondingly, sufficient conditions which allow for the design of multiple interacting inclusions independent of any specific form of (uniform) remote loading have also been established. In this paper, we demonstrate rigorously that, in all cases, these sufficient conditions are also necessary conditions and indeed allow for the identification of all possible collections of such inclusions.

Key words uniform stress, Eshelby's conjecture, multiple inclusion

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1 Introduction

Significant efforts[1–5] have been devoted to the design of multiple (interacting) inclusions, each of which achieves a uniform stress field in the presence of certain uniform remote loadings imposed on the surrounding elastic matrix (or equivalently uniform eigenstrains imposed on the inclusions themselves). Among these inclusions, those with geometric symmetry (for example, see Figs. 1–3 in Ref. [1], Fig. 2 in Ref. [2], Figs. 2, 3, and 7 in Ref. [3], Figs. 5–8 in Ref. [4], and Figs. 8–13 in Ref. [5]) and those whose shapes can be designed independently of any specific (uniform) external loadings (for example, see Figs. 2–4 in Ref. [3], Figs. 2(a), 2(c), 4(a), 7, and 8

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in Ref. [4], and Figs. 3, 6, 9, and 12 in Ref. [5]) are particularly attractive in terms of practicality in the design of advanced composite materials.

Simple conditions were established[4–5] for the design of collections of two symmetric inclusions (about a line), multiple rotationally symmetric inclusions (about a point), and multiple loading-independent inclusions with uniform stresses in an infinite elastic plane under anti-plane shear and plane deformations (see Subsections 4.2 and 4.3 of Ref. [4] and Subsections 4(b), 4(c), and 4(d) of Ref. [5] for more details). However, the question as to whether these conditions allow for the identification of all possible collections of such inclusions (those displaying certain types of symmetry as well as those designed independent of loading (uniform)) with uniform internal stress distributions in an infinite elastic plane subjected to the corresponding (anti-plane or plane) deformations remains unanswered. In this paper, we address this question by examining the necessity of the aforementioned (sufficient) conditions.

2 Analysis

Refer to the x_1x_2-coordinate system and consider n elastic inclusions, each of which is bounded by a smooth closed curve L_i (i=1, 2, $\cdots$, n), in an elastic infinite matrix subjected to uniform remote anti-plane shear or in-plane loadings. We denote by S and S_i (i=1, 2, $\cdots$, n) the (open) domains occupied by the matrix and the inclusions, respectively. In Refs. [4] and [5], it is shown that the geometry of the curves L_i must satisfy the following system of integral equations in the complex z-plane (here, $z = x_1+\mathrm{i}x_2$, in which we have chosen to represent the imaginary unit by the letter i to avoid confusion with the subscript i):

$$A_i z + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{n} B_j \oint_{L_j} \frac{\bar{t}}{t-z}\mathrm{d}t = -C_i, \quad \forall z \in S_i \ (i = 1, 2, \cdots, n), \tag{1}$$

in order for each of the inclusions to achieve a certain prescribed uniform stress field under a given uniform remote loading. In Eq. (1), the known constants A_i and B_j (i, j=1, 2, $\cdots$, n) are associated with the uniform remote loading, the uniform stress field inside each inclusion and the material constants of the matrix and each inclusion (we mention that, specifically, $B_j \neq 0$), while the unknown constants C_i (i=1, 2, $\cdots$, n) can be determined from Eq. (1) as part of the solution, although they play no role in the determination of the geometry of the curves L_i.

In Subsection 4.2 of Ref. [4] and Subsections 4(c) and 4(d) of Ref. [5], certain conditions were given which allowed for the construction of two symmetric inclusions (about a line) and multiple rotationally symmetric inclusions (about a point) with internal uniform stresses, while in Subsection 4.3 of Ref. [4] and Subsection 4(b) of Ref. [5], particular conditions were proposed for identifying multiple inclusions whose internal stress distributions are always uniform irrespective of the specific uniform loading applied remotely. In fact, from the mathematical point of view, all of these conditions are based on a simple proposition that the following two systems of equations (resulting from Eq. (1)):

$$A_i^{(1)} z + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{n} B_j^{(1)} \oint_{L_j} \frac{\bar{t}}{t-z}\mathrm{d}t = -C_i^{(1)}, \quad \forall z \in S_i \ (i = 1, 2, \cdots, n), \tag{2}$$

$$A_i^{(2)} z + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{n} B_j^{(2)} \oint_{L_j} \frac{\bar{t}}{t-z}\mathrm{d}t = -C_i^{(2)}, \quad \forall z \in S_i \ (i = 1, 2, \cdots, n) \tag{3}$$

have the same solution for each of the curves L_i when

$$\frac{A_i^{(1)}}{B_i^{(1)}} = \frac{A_i^{(2)}}{B_i^{(2)}}, \quad \frac{B_j^{(1)}}{B_i^{(1)}} = \frac{B_j^{(2)}}{B_i^{(2)}}, \ i, j = 1, 2, \cdots, n. \tag{4}$$

Apparently, Eq. (4) is sufficient for Eqs. (2) and (3) to admit the same solution, but whether it is also necessary remains unclear. Only when the necessity of Eq. (4) is established, can the conditions proposed in Refs. [4] and [5] lead to all possible collections of symmetric, rotationally symmetric, and loading-independent inclusions with uniform stresses. We show the necessity of Eq. (4) in what follows.

We rewrite Eqs. (2) and (3) for a certain p $(1 \leqslant p \leqslant n)$ as

$$\frac{1}{2\pi \mathrm{i}} \oint_{L_p} \frac{\bar{t}}{t-z} \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{B_j^{(1)}}{B_p^{(1)}} \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t = -\frac{A_p^{(1)}}{B_p^{(1)}} z - \frac{C_p^{(1)}}{B_p^{(1)}}, \quad \forall z \in S_p, \tag{5}$$

$$\frac{1}{2\pi \mathrm{i}} \oint_{L_p} \frac{\bar{t}}{t-z} \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{B_j^{(2)}}{B_p^{(2)}} \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t = -\frac{A_p^{(2)}}{B_p^{(2)}} z - \frac{C_p^{(2)}}{B_p^{(2)}}, \quad \forall z \in S_p. \tag{6}$$

Subtracting Eq. (6) from Eq. (5) yields

$$\underbrace{\frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \Big(\frac{B_j^{(1)}}{B_p^{(1)}} - \frac{B_j^{(2)}}{B_p^{(2)}} \Big) \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t}_{f(z)} = \underbrace{\Big(\frac{A_p^{(2)}}{B_p^{(2)}} - \frac{A_p^{(1)}}{B_p^{(1)}} \Big) z + \frac{C_p^{(2)}}{B_p^{(2)}} - \frac{C_p^{(1)}}{B_p^{(1)}}}_{g(z)}, \quad \forall z \in S_p, \tag{7}$$

in which $f(z)$ and $g(z)$ denote the functions on the left- and right-hand sides, respectively. Here, it is worth noting that $f(z)$ and $g(z)$ are analytic not only in S_p but also in the entire $S_p \cup L_p \cup S$ (since $f(z)$ does not include the contour integral over the curve L_p). In complex analysis, the identity theorem states that if two functions analytic in a connected domain D coincide in a certain (connected) subdomain of D, then they must coincide in the entire domain D. Consequently, it follows from Eq. (7) that

$$f(z) = g(z), \quad \forall z \in S_p \cup L_p \cup S. \tag{8}$$

Further noting that $\lim\limits_{|z| \to \infty} f(z) = 0$, it is required in $g(z)$ that

$$\frac{A_p^{(1)}}{B_p^{(1)}} - \frac{A_p^{(2)}}{B_p^{(2)}} = \frac{C_p^{(1)}}{B_p^{(1)}} - \frac{C_p^{(2)}}{B_p^{(2)}} = 0, \tag{9}$$

which means that Eq. (8) becomes

$$f(z) = g(z) = 0, \quad \forall z \in S_p \cup L_p \cup S. \tag{10}$$

To this point, we have shown (from Eq. (9)) that the first part of Eq. (4) is necessary for Eqs. (2) and (3) to achieve the same solution.

We proceed to consider the necessity of the second part of Eq. (4). For a certain q $(q \neq p)$, we have from Eqs. (2) and (3) that

$$\frac{1}{2\pi \mathrm{i}} \oint_{L_p} \frac{\bar{t}}{t-z} \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{B_j^{(1)}}{B_p^{(1)}} \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t = -\frac{A_q^{(1)}}{B_p^{(1)}} z - \frac{C_q^{(1)}}{B_p^{(1)}}, \quad \forall z \in S_q, \tag{11}$$

$$\frac{1}{2\pi \mathrm{i}} \oint_{L_p} \frac{\bar{t}}{t-z} \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{B_j^{(2)}}{B_p^{(2)}} \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t = -\frac{A_q^{(2)}}{B_p^{(2)}} z - \frac{C_q^{(2)}}{B_p^{(2)}}, \quad \forall z \in S_q. \tag{12}$$

Subtracting Eq. (12) from Eq. (11) results in

$$\underbrace{\frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq p}}^{n} \Big(\frac{B_j^{(1)}}{B_p^{(1)}} - \frac{B_j^{(2)}}{B_p^{(2)}} \Big) \oint_{L_j} \frac{\bar{t}}{t-z} \mathrm{d}t}_{f(z)} = \Big(\frac{A_q^{(2)}}{B_p^{(2)}} - \frac{A_q^{(1)}}{B_p^{(1)}} \Big) z + \frac{C_q^{(2)}}{B_p^{(2)}} - \frac{C_q^{(1)}}{B_p^{(1)}}, \quad \forall z \in S_q, \tag{13}$$

where $f(z)$ denotes the same expression as that in Eq. (7). Using the Plemelj formulae, we describe the jump of $f(z)$ across the curve L_q as

$$f(t^+) - f(t^-) = \Big(\frac{B_q^{(1)}}{B_p^{(1)}} - \frac{B_q^{(2)}}{B_p^{(2)}}\Big)\bar{t}, \quad \forall t \in L_q, \tag{14}$$

where '+' and '−' identify the limits of $f(z)$ as z tends towards a certain value t on L_q from the interior and exterior of S_q, respectively. In particular, it follows from Eqs. (10) and (13) that

$$f(t^-) = 0, \quad f(t^+) = \Big(\frac{A_q^{(2)}}{B_p^{(2)}} - \frac{A_q^{(1)}}{B_p^{(1)}}\Big)t + \frac{C_q^{(2)}}{B_p^{(2)}} - \frac{C_q^{(1)}}{B_p^{(1)}}, \quad \forall t \in L_q. \tag{15}$$

Combining Eqs. (14) and (15) results in

$$\Big(\frac{B_q^{(1)}}{B_p^{(1)}} - \frac{B_q^{(2)}}{B_p^{(2)}}\Big)\bar{t} = \Big(\frac{A_q^{(2)}}{B_p^{(2)}} - \frac{A_q^{(1)}}{B_p^{(1)}}\Big)t + \frac{C_q^{(2)}}{B_p^{(2)}} - \frac{C_q^{(1)}}{B_p^{(1)}}, \quad \forall t \in L_q. \tag{16}$$

Since L_q is a closed curve and cannot be a straight line, one must require that in order to satisfy Eq. (16),

$$\frac{B_q^{(1)}}{B_p^{(1)}} - \frac{B_q^{(2)}}{B_p^{(2)}} = \frac{A_q^{(2)}}{B_p^{(2)}} - \frac{A_q^{(1)}}{B_p^{(1)}} = \frac{C_q^{(2)}}{B_p^{(2)}} - \frac{C_q^{(1)}}{B_p^{(1)}} = 0, \tag{17}$$

which completes the proof for the necessity of the second part of Eq. (4).

3 Concluding remarks

We have shown from Eqs. (9) and (17) that Eq. (4) is necessary for Eqs. (2) and (3) to yield the same solution for the geometry of each of the curves L_i. Consequently, we confirm that by using the conditions presented in Subsections 4.2 and 4.3 of Ref. [4] and Subsections 4(b), 4(c), and 4(d) of Ref. [5], one can indeed identify (completely) all possible groups of symmetric, rotationally symmetric, and loading-independent inclusions which achieve uniform internal stress distributions in an infinite elastic plane subjected to plane or anti-plane shear deformations.

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