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Novel method for random vibration analysis of single-degree-of-freedom vibroimpact systems with bilateral barriers*

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Abstract The vibroimpact systems with bilateral barriers are often encountered in practice. However, the dynamics of the vibroimpact system with bilateral barriers is full of challenges. Few closed-form solutions were obtained. In this paper, we propose a novel method for random vibration analysis of single-degree-of-freedom (SDOF) vibroimpact systems with bilateral barriers under Gaussian white noise excitations. A periodic approximate transformation is employed to convert the equations of the motion to a continuous form. The probabilistic description of the system is subsequently defined through the corresponding Fokker-Planck-Kolmogorov (FPK) equation. The closed-form stationary probability density function (PDF) of the response is obtained by solving the reduced FPK equation and using the proposed iterative method of weighted residue together with the concepts of the circulatory probability flow and the potential probability flow. Finally, the versatility of the proposed approach is demonstrated by its application to two typical examples. Note that the solution obtained by using the proposed method can be used as the benchmark to examine the accuracy of approximate solutions obtained by other methods.

Key words bilateral barrier, vibroimpact system, weighted residue method, iterative, random vibration

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1 Introduction

Systems with the vibroimpact interaction are often encountered in practice, ranging from imple toys, such as a skipping stone on the water surface, to practical engineering systems, such as impact machines, centrifugal dewatering machines, heat exchanger tube fretting due to adjacent tubes interaction, and ship roll motion against one-sided barrier. The vibroimpact

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interaction makes the system strongly nonlinear and discontinuous that leads to various nonlinear phenomena, e.g., frequency doubling, grazing phenomena, and even chaotic response[1]. Therefore, it is of great importance to investigate the dynamic properties of this kind of system. In fact, the vibroimpact system has been studied extensively in the past decades. Excellent books or monographs[2–6], reviews[7–8], and huge regular papers are now available in this field.

As for the investigation of stochastic vibroimpact systems, several methods have been suggested. But most of these methods belong to the ones popularly applied in the smooth dynamical systems at first and then extended to stochastic vibroimpact systems. For example, the stochastic averaging method was in dominant position to tackle the random vibration problems[9–12]. In the last decades, this method has been widely adopted to the stochastic vibroimpact systems[13–22] under the assumptions of small energy loss, weakly random intensity, and light damping. The generalized cell mapping method could compute both the nonstationary and stationary response of stochastic systems[23–26]. Recently, the generalized cell mapping method has been elegantly suggested to calculate the stochastic response of a class of impact systems[27]. Monte Carlo simulation is one of the most general techniques to calculate the response of stochastic vibroimpact systems[28] except for the challenge in computational efficiency and convergence. The exponential-polynomial closure (EPC) method was first explored by Er[29] and later has been generalized to obtain the stationary probability density function (PDF) solutions of the vibroimpact systems with a unilateral barrier under Gaussian white noise excitation[30] or Poisson white noise excitation[31]. Dimentberg et al.[32] replaced the Zhuravlev transformation[33] with the Zhuravlev-Ivanov transformation, and examined the stationary PDF solution of stochastic mass-spring vibroimpact systems with high energy losses and one side impacts by using a path integration method. Kumar and his coworkers introduced the same transformation as that in Ref. [32], and studied a stochastically excited vibroimpact Duffing-van der Pol oscillator with unilateral rigid barrier[34] or bilateral barriers[35] with the aid of the finite element method. However, it should be noted that the solutions obtained by the aforementioned methods are almost limited to smooth approximations of the non-smooth response of the intrinsically discontinuous vibroimpact oscillators.

Very recently, Chen et al.[36] proposed a new procedure for the closed-form stationary PDF of the response of Gaussian white noise excited single-degree-of-freedom (SDOF) vibroimpact system with unilateral rigid barrier. The obtained solution is piecewise form which reflects the intrinsically discontinuous characteristic of vibroimpact oscillators. In this paper, we continue this work in the field of stochastic vibroimpact system, and aim to propose a new scheme for the closed-form solution of stationary response of SDOF vibroimpact system with bilateral barrier under random excitations. First, a piecewise differentiable periodic transformation is utilized to convert the equations of motion to a continuous form. Then, the piecewise solution of stationary PDF is then obtained by solving the reduced FPK equation and using the iterative method of weighted residue procedure. Finally, two examples are worked out to illustrate the proposed scheme. The validity of the obtained closed-form solution is confirmed by using the Monte Carlo simulation data of the original system.

The layout of the paper is as follows. Section 2 formulates the problem of stationary PDF solution of the SDOF vibroimpact systems with bilateral barriers under Gaussian white noise excitation. Section 3 introduces the iterative method of weighted residue procedure to construct the approximate solution of the reduced FPK equation associated with the converted systems of vibroimpact systems with bilateral barriers. In Section 4, two examples are studied to demonstrate the proposed scheme. The accuracy of the approximate PDF is examined by the Monte Carlo simulations. Section 5 concludes the paper.

2 Problem formulation

Consider an SDOF vibroimpact oscillator under Gaussian white noise excitations as

$$\ddot{X} + g_1(X, \dot{X})\dot{X} + g_2(X) = \sum_{i=1}^{l} h_i(X, \dot{X}) W_i(t), \quad |X| \leqslant \frac{\Delta}{2}, \tag{1}$$

where $X, \dot{X}$, and $\ddot{X}$ denote the system displacement, velocity, and acceleration, respectively. $g_1(X, \dot{X})\dot{X}$ is the linear or nonlinear damping, $g_2(X)$ represents restoring forces, $h_i(X, \dot{X})$ is a linear or nonlinear function of X and $\dot{X}$, and $W_i(t)$ denotes Gaussian white noises with zero mean and correlation matrix, i.e., $E[W_i(t)W_j(t+\tau)] = 2D_{ij}\delta(\tau)$ $(i, j = 1, 2, \cdots, l)$.

The impact condition for the inelastic instantaneous interaction is

$$\dot{X}_+ = -r\dot{X}_-, \quad X = \pm\frac{\Delta}{2}, \tag{2}$$

where r is a restitution factor ($0 < r \leqslant 1$). The subscripts "–" and "+" denote the values of the velocity before and after impact, respectively. $|X| = \frac{\Delta}{2}$ is the position of the rigid barrier.

First of all, we shall define a non-dimensional displacement variable as follows:

$$Y = \frac{X\pi}{\Delta}. \tag{3}$$

The equation of motion for Y reads

$$\ddot{Y} + g_1\Big(\frac{Y\Delta}{\pi}, \frac{\dot{Y}\Delta}{\pi}\Big)\dot{Y} + \frac{\pi}{\Delta} g_2\Big(\frac{Y\Delta}{\pi}\Big) = \frac{\pi}{\Delta}\sum_{i=1}^{l} h_i\Big(\frac{Y\Delta}{\pi}, \frac{\dot{Y}\Delta}{\pi}\Big) W_i(t), \quad |Y| \leqslant \frac{\pi}{2} \tag{4}$$

with the impact condition defined as

$$\dot{Y}_+ = -r\dot{Y}_-, \quad Y = \pm\frac{\pi}{2}. \tag{5}$$

Second, we shall introduce the piecewise smooth variable transformation[33] as follows:

$$\begin{cases} Y = \Pi(Z) + \lambda N(Z), \\ \dot{Y} = (M(Z) + \lambda\Pi(Z))\dot{Z}, \\ \ddot{Y} = (M(Z) + \lambda\Pi(Z))\ddot{Z} + \lambda M(Z)\dot{Z}^2, \end{cases} \tag{6}$$

where $\Pi(Z), N(Z)$, and $M(Z)$ are 2π periodic functions defined by, respectively,

$$\begin{cases} \Pi(Z) = \begin{cases} Z, & \text{as} \quad -\dfrac{\pi}{2} \leqslant Z < \dfrac{\pi}{2}, \\ -Z + \pi, & \text{as} \quad \dfrac{\pi}{2} \leqslant Z \leqslant \dfrac{3\pi}{2}, \end{cases} \\ N(Z) = \begin{cases} \dfrac{Z^2}{2} - \dfrac{\pi^2}{8}, & \text{as} \quad -\dfrac{\pi}{2} \leqslant Z < \dfrac{\pi}{2}, \\ -\dfrac{(Z-\pi)^2}{2} + \dfrac{\pi^2}{8}, & \text{as} \quad \dfrac{\pi}{2} \leqslant Z \leqslant \dfrac{3\pi}{2}, \end{cases} \\ M(Z) = \begin{cases} 1, & \text{as} \quad -\dfrac{\pi}{2} \leqslant Z < \dfrac{\pi}{2}, \\ -1, & \text{as} \quad \dfrac{\pi}{2} \leqslant Z \leqslant \dfrac{3\pi}{2}, \end{cases} \end{cases} \tag{7}$$

and $\lambda = \frac{2}{\pi}(\frac{1-r}{1+r})$. Note that the transformation (6) has just a two-order approximation, and would be valid when $1 - r$ is supposed to be small. Thus, the cases studied herein are limited to the elastic impact or weakly inelastic impact.

Now, let us substitute these transformations in Eq. (6) into Eq. (4). We can derive the equation of motion as follows:

$$\ddot{Z} + \hat{g}_1(Z, \dot{Z}) + \hat{g}_2(Z) = \sum_{i=1}^{l} \hat{h}_i(Z, \dot{Z}) W_i(t), \tag{8}$$

where

$$\hat{g}_1(Z, \dot{Z}) = \frac{\lambda M(Z)\dot{Z}^2}{M(Z) + \lambda\Pi(Z)} + g_1\Big(\frac{\Delta}{\pi}(\Pi(Z) + \lambda N(Z)), \frac{\Delta}{\pi}(M(Z) + \lambda\Pi(Z))\dot{Z}\Big)\dot{Z}, \tag{9}$$

$$\hat{g}_2(Z) = \frac{\pi}{\Delta} \frac{g_2(\frac{\Delta}{\pi}(\Pi(Z) + \lambda N(Z)))}{M(Z) + \lambda\Pi(Z)}, \tag{10}$$

$$\hat{h}_i(Z, \dot{Z}) = \frac{\pi}{\Delta} \frac{h_i(\frac{\Delta}{\pi}(\Pi(Z) + \lambda N(Z)), \frac{\Delta}{\pi}(M(Z) + \lambda\Pi(Z))\dot{Z})}{M(Z) + \lambda\Pi(Z)}. \tag{11}$$

Then, writing Eq. (8) in the state space form yields

$$\begin{cases} \dfrac{\mathrm{d}Z_1}{\mathrm{d}t} = Z_2, \\ \dfrac{\mathrm{d}Z_2}{\mathrm{d}t} = -\hat{g}_1(Z_1, Z_2) - \hat{g}_2(Z_1) - \displaystyle\sum_{i=1}^{l} \hat{h}_i(Z_1, Z_2) W_i(t). \end{cases} \tag{12}$$

The stationary PDF, $p_{z_1 z_2} = p(z_1, z_2)$, of the system response is governed by the reduced FPK equation,

$$0 = -\frac{\partial}{\partial z_1}(p_{z_1 z_2} m_1) - \frac{\partial}{\partial z_2}(p_{z_1 z_2} m_2) + \frac{1}{2}\frac{\partial^2}{\partial z_2^2}(p_{z_1 z_2} b_{22}), \tag{13}$$

where the drift and diffusion terms m_1, m_2, and b_{22} are given by

$$\begin{cases} m_1 = z_2, \\ m_2 = -\hat{g}_1(z_1, z_2) - \hat{g}_2(z_1) + \displaystyle\sum_{i=1}^{l}\sum_{j=1}^{l} \Big(D_{ij} \frac{\partial \hat{h}_i(z_1, z_2)}{\partial z_2} \hat{h}_j(z_1, z_2)\Big), \\ b_{22} = \displaystyle\sum_{i=1}^{l}\sum_{j=1}^{l} \Big(2D_{ij} \hat{h}_i(z_1, z_2) \hat{h}_j(z_1, z_2)\Big). \end{cases} \tag{14}$$

According to the detail balance method, the reduced Fokker-Planck-Kolmogorov (FPK) equation (13) is solved under the following conditions:

$$-z_2 \frac{\partial p_{z_1 z_2}}{\partial z_1} + \hat{g}_2(z_1) \frac{\partial p_{z_1 z_2}}{\partial z_2} = 0, \tag{15}$$

$$-\frac{\partial}{\partial z_2}(p_{z_1 z_2}(m_2 + \hat{g}_2(z_1))) + \frac{1}{2}\frac{\partial^2}{\partial z_2^2}(p_{z_1 z_2} b_{22}) = 0. \tag{16}$$

Equations (15) and (16) imply the equilibrium of the circulatory probability flow and vanishing the potential probability flow, respectively. Note that $p_{z_1 z_2}$ and its first derivative with respect to z_2 vanish at the infinity boundary, Eq. (16) is reduced to

$$-p_{z_1 z_2}(m_2 + \hat{g}_2(z_1)) + \frac{1}{2}\frac{\partial}{\partial z_2}(p_{z_1 z_2} b_{22}) = 0. \tag{17}$$

An exact stationary solution to the reduced FPK equation (13) can be obtained from Eqs. (15) and (17) under the solvability condition. However, such a condition is often not met in practice. In this case, the reduced FPK equation (13) has no exact solution in general. In the next section, we shall introduce the iterative method of weighted residue procedure[37] to compute the approximate solution of the reduced FPK equation (13).

3 Iterative method of weighted residue procedure

The iterative method of weighted residue procedure may consist of three steps as follows: construct a trial solution of reduced FPK equation (13) on the basis of the concepts of circulatory probability flow and potential probability flow, determine the unknown parameters in the trial solution with application of weighted residue procedure, and impose the iterative method to improve the accuracy of the solution obtained with weighted residue procedure.

3.1 Trial solution

Assume that the approximate solution of the reduced FPK equation (13) is written as

$$\bar{p}_{z_1 z_2} = \bar{p}_{z_1 z_2}(z_1, z_2) = C_0 \exp(-\varphi(z_1, z_2)), \tag{18}$$

where C_0 is a normalization constant. $\varphi(z_1, z_2)$ denotes the probability potential, which can be divided into three parts according to Ref. [37]. That is,

$$\varphi(z_1, z_2) = \sum_{i,j \geqslant 0} \sum_{0<i+j \leqslant n} c_{ij} y_1^i y_2^j + k_1 \psi_1(z_1, z_2) + k_2 \psi_2(z_1, z_2), \tag{19}$$

in which c_{ij}, k_1, and k_2 are unknown parameters that needed to be determined. y_1 and y_2 are respectively given by the following equations:

$$\begin{cases} y_1 = \Pi(z_1) + \lambda N(z_1), \\ y_2 = (M(z_1) + \lambda \Pi(z_1)) z_2. \end{cases} \tag{20}$$

The term ψ_1 denotes the circulatory probability flow and is determined by Eq. (15), while ψ_2 denotes the circulatory probability flow and is determined by Eq. (17). One set of expressions for ψ_1 and ψ_2 is given below,

$$\psi_1 = \frac{(M(z_1) + \lambda \Pi(z_1))^2 z_2^2}{2} + \int_0^{\Pi(z_1)+\lambda N(z_1)} \frac{\pi}{\Delta} g_2\Big(\frac{s\Delta}{\pi}\Big) \mathrm{d}s, \tag{21}$$

$$\psi_2 = \int_0^{z_2} \frac{2}{b_{22}(z_1, z_2)} \Big(- m_2(z_1, z_2) - \hat{g}_2(z_1) + \frac{\partial b_{22}(z_1, z_2)}{\partial z_2} \Big) \mathrm{d}z_2. \tag{22}$$

In addition, the approximate stationary PDF $\bar{p}_{z_1 z_2}$ of the system should be subject to the existence conditions,

$$\begin{cases} \exp(-\varphi(z_1, z_2)) = \text{finite}, \quad \text{as} \quad z_1 = -\dfrac{\pi}{2} \text{ and } \dfrac{3\pi}{2}, \\ \exp(-\varphi(z_1, z_2)) = 0, \quad \text{as} \quad |z_2| \to \infty, \end{cases} \tag{23}$$

and the integrability condition at the origin,

$$\varphi(z_1, z_2) \propto A^{\alpha}, \quad \alpha > -1, \quad \text{if} \quad \forall \theta \in [0, 2\pi) \quad \text{as} \quad A \to 0, \tag{24}$$

where $z_1 = A\cos\theta$ and $z_2 = A\sin\theta$.

Numerical examples studied in the next section will verify that the trial solution to Eq. (18) is very adaptable since the terms ψ_1 and ψ_2 already describe the main features of the system.

3.2 Weighted residue procedure

By substituting $\overline{p}_{z_1 z_2}$ for $p_{z_1 z_2}$ in Eq. (13), one can obtain a residual error, $\vartheta(z_1, z_2, c_{ij}, k_1, k_2)\overline{p}_{z_1 z_2}$, where

$$\begin{aligned} \vartheta(z_1, z_2, c_{ij}, k_1, k_2) = & -m_1 \frac{\partial \varphi}{\partial z_1} - \frac{\partial m_2}{\partial z_2} + m_2 \frac{\partial \varphi}{\partial z_1} \\ & + \frac{b_{22}}{2} \Big(\Big(\frac{\partial \varphi}{\partial z_2} \Big)^2 - \frac{\partial^2 \varphi}{\partial z_2^2} \Big) - \frac{\partial \varphi}{\partial z_2} \frac{\partial b_{22}}{\partial z_2} + \frac{1}{2} \frac{\partial^2 b_{22}}{\partial z_2^2}. \end{aligned} \tag{25}$$

By making use of the weighted residue procedure, $\vartheta(z_1, z_2, c_{ij},\ k_1, k_2)$ equals zero by a weighted average of a chosen set of weighting functions $M_l(z_1, z_2)\,(l = 1, 2, \cdots, n+2)$,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M_l(z_1, z_2)\vartheta(z_1, z_2, c_{ij}, k_1, k_2)\mathrm{d}z_1\mathrm{d}z_2 = 0. \tag{26}$$

Note that the key to the weighted residue procedure depends highly on the choice of the weighting function. Let us select a set of weighting functions in the following form:

$$\begin{cases} M_k(z_1, z_2) = p_m(z_1, z_2)y_1^i y_2^j, \quad k = 1, 2, \cdots, n, \\ M_{n+1}(z_1, z_2) = p_m(z_1, z_2)\psi_1(z_1, z_2), \\ M_{n+2}(z_1, z_2) = p_m(z_1, z_2)\psi_2(z_1, z_2), \end{cases} \tag{27}$$

where $p_m(z_1, z_2)$ can be the Gaussian PDF obtained from the system (4) by using equivalent linearization method, or can be directly constructed as follows:

$$p_m(z_1, z_2) = \exp(-\psi_1(z_1, z_2) - \psi_2(z_1, z_2)). \tag{28}$$

By inserting the weighting function (27) into Eq. (26), a set of nonlinear algebraic equations can be obtained and then solved by using any numerical solution under the conditions (23) and (24).

3.3 Iterative method

It should be emphasized that the accuracy of results obtained with the weighted residue procedure may not be sufficient. In Refs. [38]–[39], an iterative method is employed to improve the accuracy of the solution. In such procedure, the approximate PDF $\overline{p}_{z_1z_2}$ is obtained after applying the weighted residue procedure in the previous subsection, and let $\overline{p}^{(1)} = \overline{p}_{z_1z_2}$. Then, updating with p_m in Eq. (27) by $\overline{p}^{(k)}$ for all $k > 0$ and using the weighted residue procedure again yield the next approximate PDF denoted as $\overline{p}^{(k+1)}$. The iterative method is stopped as the convergence criterion with a preset tolerance ε_0,

$$\varepsilon = \sqrt{\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{-\infty}^{+\infty}\left(\overline{p}^{(k)} - \overline{p}^{(k+1)}\right)^2\mathrm{d}z_1\mathrm{d}z_2} \leqslant \varepsilon_0, \tag{29}$$

or

$$\varepsilon = \sqrt{\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{-\infty}^{+\infty}\left(\vartheta\overline{p}^{(k)}\right)^2\mathrm{d}z_1\mathrm{d}z_2} \leqslant \varepsilon_0, \tag{30}$$

is satisfied.

Numerical computations have indicated that the convergence of the iterative method is quite fast. However, the iterative method also may converge to an insufficiently accurate solution in some cases, particularly for strongly nonlinear systems. In this regard, Chen and Sun[39] suggested a progressively way to computer the stationary PDFs of strongly nonlinear systems. By choosing a converged solution of a weakly nonlinear system as p_m, and then slowly increasing the nonlinear parameter when applying the iterative method of weighted residue procedure, the stationary PDFs of strongly nonlinear systems can be captured effectively. Here, the converged solution of the elastic impact case is chosen as p_m to search the solution of the weakly inelastic impact case progressively.

Once $\overline{p}_{z_1z_2}$ is obtained with the above procedure, the PDF $p_{Y\dot{Y}}$ of system (4) can be calculated from

$$\begin{aligned} p_{Y\dot{Y}} = p_{Y\dot{Y}}(y, \dot{y}) = \Big(&\overline{p}_{z_1z_2}\Big(f_1(y), \frac{\dot{y}}{1+\lambda f_1(y)}\Big)\left|\frac{\mathrm{d}f_1(y)}{\mathrm{d}y}\frac{1}{1+\lambda f_1(y)}\right|\Big) \\ &+ \overline{p}_{z_1z_2}\Big(f_2(y), \frac{\dot{y}}{-1-\lambda f_2(y)+\lambda\pi}\Big)\left|\frac{\mathrm{d}f_2(y)}{\mathrm{d}y}\frac{1}{-1-\lambda f_2(y)+\lambda\pi}\right|, \end{aligned} \tag{31}$$

where $f_1(y)$ is determined by

$$f_1(y)=\begin{cases} y, \quad \text{as} \quad \lambda=0, \\ \sqrt{\dfrac{2y}{\lambda}+\dfrac{\pi^2}{4}+\dfrac{1}{\lambda^2}}-\dfrac{1}{\lambda}, \quad \text{as} \quad \lambda\neq 0, \end{cases} \tag{32}$$

and $f_2(y)$ is determined by

$$f_2(y)=\begin{cases} \pi-y, \quad \text{as} \quad \lambda=0, \\ \sqrt{-\dfrac{2y}{\lambda}+\dfrac{\pi^2}{4}+\dfrac{1}{\lambda^2}}+\pi-\dfrac{1}{\lambda}, \quad \text{as} \quad \lambda\neq 0. \end{cases} \tag{33}$$

The marginal PDFs, p_Y and $p_{\dot{Y}}$, can be obtained from Eq. (31) as follows:

$$\begin{cases} p_Y=p_Y(y)=\displaystyle\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} p_{Y\dot{Y}}(y,\dot{y})\mathrm{d}\dot{y}, \\ p_{\dot{Y}}=p_{\dot{Y}}(\dot{y})=\displaystyle\int_{-\infty}^{+\infty} p_{Y\dot{Y}}(y,\dot{y})\mathrm{d}y. \end{cases} \tag{34}$$

4 Examples

In this section, we will study two typical examples to examine the effectiveness and accuracy of the proposed procedure.

Example 1 As the first example, the van der Pol oscillator with two-sided barriers subject to external and parametrical Gaussian white noise excitations is studied. The governing equation is given below,

$$\ddot{X}-\beta_0\dot{X}+\beta_1 X^2\dot{X}+\omega^2 X=W_1(t)+XW_2(t), \tag{35}$$

where β_0,β_1, and ω are positive, and $W_i(t)$ denotes the independent Gaussian white noise excitation with correlations function $2D_i(\tau)$. The impact condition is defined by Eq. (2).

Let $Y=\frac{X\pi}{\Delta}$. Then, Eq. (35) can be converted as follows:

$$\ddot{Y}-\beta_0\dot{Y}+\beta_1\frac{\Delta^2}{\pi^2}Y^2\dot{Y}+\omega^2 Y=\frac{\pi}{\Delta}W_1(t)+YW_2(t), \tag{36}$$

with the impact condition (5). With the application of the transformation in Eq. (6), a modified equation can be derived as follows:

$$\begin{aligned} &\ddot{Z}+\frac{\lambda M(Z)\dot{Z}^2}{M(Z)+\lambda\Pi(Z)}-\beta_0\dot{Z}+\beta_1\frac{\Delta^2}{\pi^2}(\Pi(Z)+\lambda N(Z))^3\dot{Z}+\omega^2\frac{\Pi(Z)+\lambda N(Z)}{M(Z)+\lambda\Pi(Z)} \\ &=\frac{\pi}{\Delta}\frac{W_1(t)}{M(Z)+\lambda\Pi(Z)}+\frac{\Pi(Z)+\lambda N(Z)}{M(Z)+\lambda\Pi(Z)}W_2(t). \end{aligned} \tag{37}$$

The reduced FPK equation governing the stationary PDF of the system (37) is the form as Eq. (13) with the following drift and diffusion coefficients:

$$\begin{cases} m_1=z_2, \\ m_2=-\dfrac{\lambda M(z_1)z_2^2}{M(z_1)+\lambda\Pi(z_1)}+\beta_0 z_2-\beta_1\dfrac{\Delta^2}{\pi^2}(\Pi(z_1)+\lambda N(z_1))^3 z_2-\omega^2\dfrac{\Pi(z_1)+\lambda N(z_1)}{M(z_1)+\lambda\Pi(z_1)}, \\ b_{22}=\dfrac{2D_1\pi^2}{\Delta^2(M(z_1)+\lambda\Pi(z_1))^2}+2D_2\Big(\dfrac{\Pi(Z)+\lambda N(Z)}{M(Z)+\lambda\Pi(Z)}\Big)^2. \end{cases} \tag{38}$$

$\psi_1(z_1, z_2)$ and $\psi_2(z_1, z_2)$ are written as follows:

$$\psi_1(z_1, z_2) = \frac{(M(z_1) + \lambda\Pi(z_1))^2 z_2^2}{2} + \frac{\omega^2(\Pi(z_1) + \lambda N(z_1))^2}{2}, \tag{39}$$

$$\psi_2(z_1, z_2) = \frac{2}{b_{22}}\Big(- \beta_0 \frac{z_2^2}{2} + \frac{1}{3}\frac{\lambda M(z_1) z_2^3}{M(z_1) + \lambda\Pi(z_1)} + \frac{\beta_1 \Delta^2}{\pi^2}\frac{(\Pi(z_1) + \lambda N(z_1))^3 z_2^2}{2}\Big). \tag{40}$$

Construct p_m for the case of $r = 1$ as

$$p_m = C_0 \exp(-\psi_1(z_1, z_2) - \psi_2(z_1, z_2)). \tag{41}$$

Let $\beta_0 = 0.25$, $\beta_1 = 1.0$, $\omega = 1.0$, $D_1 = D_2 = 0.25$, $\Delta = 4.0$, and $\varepsilon_0 = 10^{-3}$, unless otherwise mentioned. By using the iterative method of weighted residue procedure after 4 iterations, a piecewise closed-form solution of $r = 1.0$ and order $n = 4$ is obtained as follows:

$$\overline{p}_{z_1 z_2} = \begin{cases} \overline{p}_1(z_1, z_2), & \text{as} \quad -\dfrac{\pi}{2} \leqslant z_1 < \dfrac{\pi}{2}, \\ \overline{p}_2(z_1, z_2), & \text{as} \quad \dfrac{\pi}{2} \leqslant z_1 \leqslant \dfrac{3\pi}{2}, \end{cases} \tag{42}$$

where $\overline{p}_1(z_1, z_2)$ and $\overline{p}_2(z_1, z_2)$ are given by

$$\begin{cases} \overline{p}_1(z_1, z_2) = C_0 \exp\Big(- 0.133\,574\,42 \times z_2^4 + 0.470\,003\,83 \times z_1 z_2^3 \\ \qquad - 0.981\,340\,09 \times z_1^2 z_2^2 + 0.335\,359\,38 \times z_1^3 z_2 \\ \qquad - 0.524\,602\,97 \times z_1^4 - 0.049\,158\,02 \times z_2^2 \\ \qquad - 0.916\,977\,15 \times z_1 z_2 + 1.028\,157\,11 \times z_1^2 \\ \qquad + \dfrac{0.019\,473\,33 \times z_2^2(1.621\,138\,9 \times z_1^2 - 0.25)}{0.25 \times z_1^2 + 0.154\,212\,57}\Big), \\ p_2(z_1, z_2) = C_0 \exp\Big(- 0.133\,574\,42 \times z_2^4 + 0.470\,003\,83 \times (\pi - z_1) z_2^3 \\ \qquad - 0.981\,340\,09 \times (\pi - z_1)^2 z_2^2 + 0.335\,359\,38 \times (\pi - z_1)^3 z_2 \\ \qquad - 0.524\,602\,97 \times (\pi - z_1)^4 - 0.049\,158\,02 \times z_2^2 \\ \qquad - 0.916\,977\,15 \times (\pi - z_1) z_2 + 1.028\,157\,11 \times z_1^2 \\ \qquad + \dfrac{0.019\,473\,33 \times z_2^2((\pi - z_1)^2 - 0.25)}{0.25 \times (\pi - z_1)^2 + 0.154\,212\,57}\Big). \end{cases} \tag{43}$$

The approximate PDF solution of $r = 0.99$ and order $n = 4$ is obtained with the approximate PDF solution of $r = 1.0$ in Eq. (42) as p_m after 4 iterations. The detailed expression of the closed-form solution is given by Eq. (A1). In Figs. 1 and 2, the stationary PDFs of the cases $r = 1.0$ and $r = 0.99$ with $\Delta = 4.0$, respectively, of the system (37) are plotted. From Figs. 1 and 2, a fancy limited cycle can be observed, and the analytical solutions are in good agreement with those obtained with Monte Carlo simulation of a sample size of 4×10^7. It should be pointed out that the numerical studying of this example belongs to the strongly nonlinear van der Pol oscillator. The conventional method of stochastic averaging is not suitable.

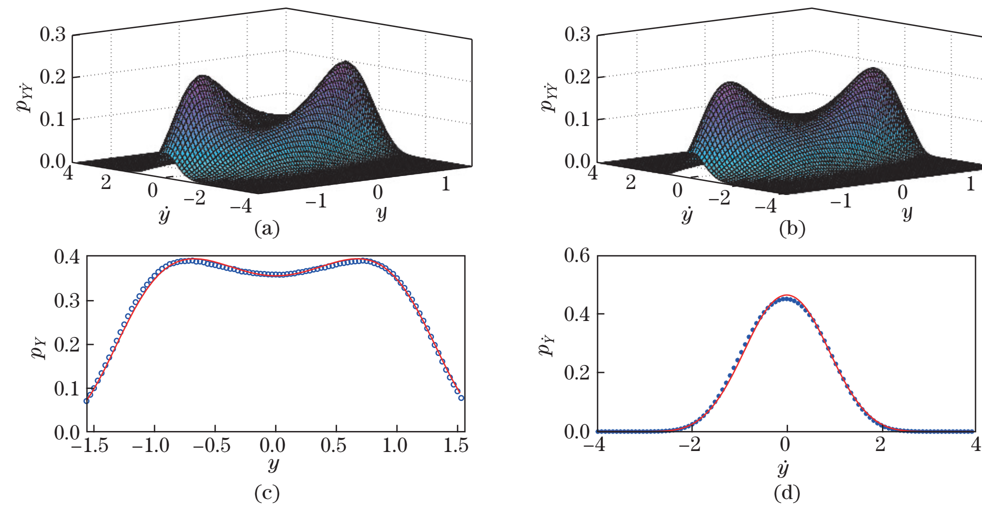

Fig. 1 Stationary PDFs of system (36) in the case of $r = 1.0$ and $\Delta = 4$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are $\beta_0 = 0.25, \beta_1 = 1.0, \omega = 1.0$, and $D_1 = D_2$ (color online)

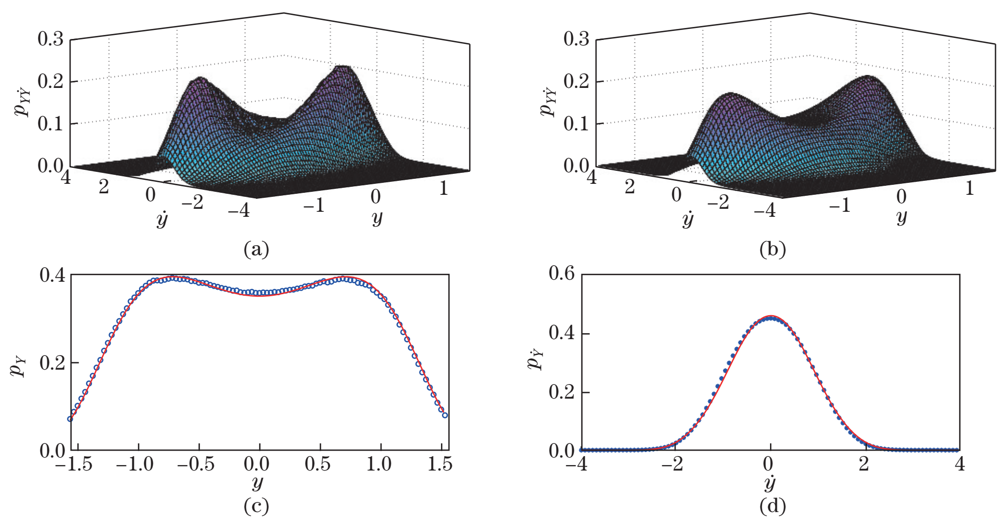

Fig. 2 Stationary PDFs of system (36) in the case of $r = 0.99$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are the same as those in Fig. 1 (color online)

Example 2 As the second example, the Duffing system with two-sided barriers subject to external Gaussian white noise excitations is considered. The equation of the motion is given below,

$$\ddot{X} + \beta_0\dot{X} + \omega^2 X + \alpha X^3 = W_1(t), \tag{44}$$

where β_0 and ω are positive constants, and $W_1(t)$ denotes the Gaussian white noise excitation with correlation function $2D_1\delta(\tau)$. The impact condition is defined by Eq. (2).

With application of non-dimensional transformation in Eq. (3), the equation of motion for Y reads

$$\ddot{Y} + \beta_0 \dot{Y} + \omega^2 Y + \alpha \frac{\Delta^2}{\pi^2} Y^3 = \frac{\pi}{\Delta} W_1(t) \tag{45}$$

with the impact condition in Eq. (5). By substituting the piecewise smooth variable transformation in Eq. (6) into Eq. (46), the equation of motion for Z is expressed by

$$\begin{aligned} &\ddot{Z} + \frac{\lambda M(Z)\dot{Z}^2}{M(Z)+\lambda\Pi(Z)} + \beta_0 \dot{Z} + \omega^2 \frac{\Pi(Z)+\lambda N(Z)}{M(Z)+\lambda\Pi(Z)} + \alpha \frac{\Delta^2}{\pi^2} \frac{(\Pi(Z)+\lambda N(Z))^3}{M(Z)+\lambda\Pi(Z)} \\ &= \frac{\pi}{\Delta} \frac{W_1(t)}{M(Z)+\lambda\Pi(Z)}. \end{aligned} \tag{46}$$

The reduced FPK equation associated with Eq. (46) is the same as Eq. (13) with the following drift and diffusion coefficients:

$$\begin{cases} m_1 = z_2, \\ m_2 = -\dfrac{\lambda M(z_1) z_2^2}{M(z_1)+\lambda\Pi(z_1)} - \beta_0 z_2 \\ \qquad -\omega^2 \dfrac{\Pi(z_1)+\lambda N(z_1)}{M(z_1)+\lambda\Pi(z_1)} - \alpha \dfrac{\Delta^2}{\pi^2} \dfrac{(\Pi(z_1)+\lambda N(z_1))^3}{M(z_1)+\lambda\Pi(z_1)}, \\ b_{22} = \dfrac{1}{(M(z_1)+\lambda\Pi(z_1))^2} \dfrac{2D_1\pi^2}{\Delta^2}. \end{cases} \tag{47}$$

Accordingly, $\psi_1(z_1, z_2)$ and $\psi_2(z_1, z_2)$ can be obtained as follows:

$$\begin{aligned} \psi_1(z_1, z_2) = &\frac{(M(z_1)+\lambda\Pi(z_1))^2 z_2^2}{2} + \frac{\omega^2}{2}(\Pi(z_1)+\lambda N(z_1))^2 \\ &+ \frac{\alpha\Delta^2}{4\pi^2}(\Pi(z_1)+\lambda N(z_1))^4, \end{aligned} \tag{48}$$

$$\psi_2(z_1, z_2) = \frac{2}{b_{22}} \Big(\frac{1}{3} \frac{\lambda M(z_1) z_2^3}{M(z_1)+\lambda\Pi(z_1)} + \beta_0 \frac{z_2^2}{2} \Big). \tag{49}$$

p_m is supposed as

$$p_m = C_0 \exp(-\psi_1(z_1, z_2) - \psi_2(z_1, z_2)). \tag{50}$$

Now, we first examine the sensibility of r to the accuracy of the proposed method. Let $\beta_0 = 0.1$, $\omega = 1.0$, $\alpha = 2.0$, $D_1 = 0.1$, $\Delta = 2.0$, and $\varepsilon_0 = 10^{-3}$, unless otherwise stated. By using the iterative method of weighted residue procedure after 2 iterations, a piecewise closed-form solution of $r = 1.0$ and order $n = 4$ can be obtained as

$$\overline{p}_{z_1 z_2} = \begin{cases} \overline{p}_1(z_1, z_2), & \text{as} \quad -\dfrac{\pi}{2} \leqslant z_1 < \dfrac{\pi}{2}, \\ \overline{p}_2(z_1, z_2), & \text{as} \quad \dfrac{\pi}{2} \leqslant z_1 \leqslant \dfrac{3\pi}{2}, \end{cases} \tag{51}$$

where $\overline{p}_1(z_1,z_2)$ and $\overline{p}_2(z_1,z_2)$ are given by the following equations, respectively:

$$\begin{cases} \overline{p}_1(z_1,z_2) = C_0\exp(-0.202\,642\,36\times z_1^2 - 1.31\times 10^{-11}\times z_1z_2 - 0.202\,642\,36\times z_2^2 \\ \qquad - 0.082\,127\,86\times z_1^4 + 5.41\times 10^{-13}\times z_1^3z_2 + 4.05\times 10^{-12}\times z_1^2z_2^2 \\ \qquad + 1.79\times 10^{-13}\times z_1z_2^3 - 4.20\times 10^{-12}\times z_2^4), \\ \overline{p}_2(z_1,z_2) = C_0\exp(-0.202\,642\,36\times(\pi-z_1)^2 - 1.31\times 10^{-11}\times(\pi-z_1)z_2 \\ \qquad - 0.202\,642\,36\times z_2^2 - 0.082\,127\,858\times(\pi-z_1)^4 \\ \qquad + 5.41\times 10^{-13}\times(\pi-z_1)^3z_2 + 4.05\times 10^{-12}\times(\pi-z_1)^2z_2^2 \\ \qquad + 1.79\times 10^{-13}\times(\pi-z_1)z_2^3 + 4.20\times 10^{-12}\times z_2^4), \end{cases} \tag{52}$$

The solution of $p_{Y\dot{Y}}$ can be obtained from Eq. (51) with the application of relation in Eq. (31). The solution of $p_{Y\dot{Y}}$ should be the exact stationary PDF of system (46) by neglecting the terms with coefficients of the order 10^{-11}. As far as we know, such a solution is obtained first time in the literature. It can be used as a benchmark problem in nonlinear random vibration. In Fig. 3, numerical results of $p_{Y\dot{Y}}$, and the marginal PDFs, p_Y and $p_{\dot{Y}}$, of system (45) computed by the proposed procedure and the Monte Carlo simulation are shown. Obviously, the analytical results agree well with the Monte Carlo simulation data.

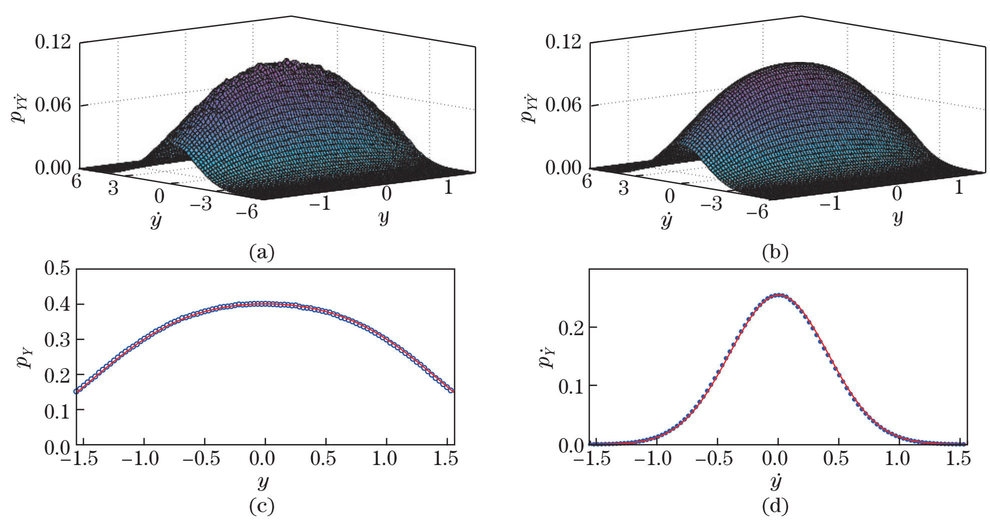

Fig. 3 Stationary PDFs of system (45) in the case of $r = 1.0$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are $\beta_0 = 0.1, \omega = 1.0$, $\alpha = 2.0, D_1 = 0.1, \Delta = 2.0$, and $\varepsilon_0 = 10^{-3}$ (color online)

Now let the elastic impact's PDF solution to Eq. (51) be p_m. The solution with $r = 0.99$ and order $n = 4$ is obtained after 2 iterations. The piecewise result is similar as the form in Eq. (51) with $\overline{p}_1(z_1,z_2)$ and $\overline{p}_2(z_1,z_2)$ are proposed in Eq. (A4). In Fig. 4, the stationary PDFs of the case $r = 0.99$ and $\Delta = 2$ are depicted. It can be observed that the analytical solutions obtained with the proposed technique are very close to the Monte Carlo simulation data of a sample size of 4×10^7.

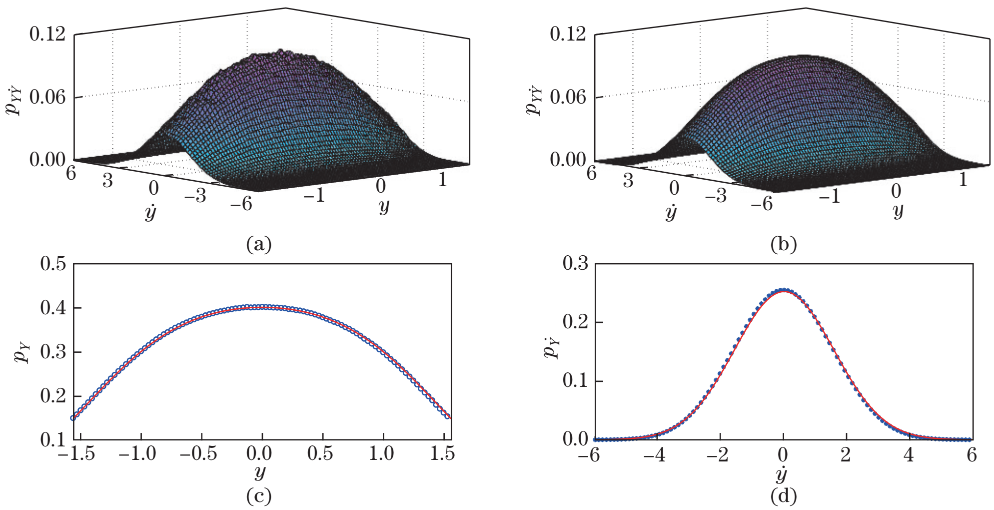

Fig. 4 Stationary PDFs of system (45) in the case of $r = 0.99$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are the same as those in Fig. 3 (color online)

Next, we examine the effect of barrier to the practicability of the proposed technique. The solution of $\Delta = 4$ for the case of $r = 0.99$ and order $n = 4$ is obtained with the approximate PDF solution in Eq. (A4) as p_m after 2 iterations. The expression of solution is given in Eq. (A5). Figure 5 presents comparison of analytical and Monte Carlo simulation results of a sample size of 4×10^7 for $p_{Y\dot{Y}}, p_Y$, and $p_{\dot{Y}}$ in the case of $r = 0.99$ and $\Delta = 4$. Similar good agreement can be observed again. From Fig. 5, a very small probability can be found in the collision boundary by comparison with the case of $\Delta = 2$. Thus, the stochastic P-bifurcation occurs as the change in Δ.

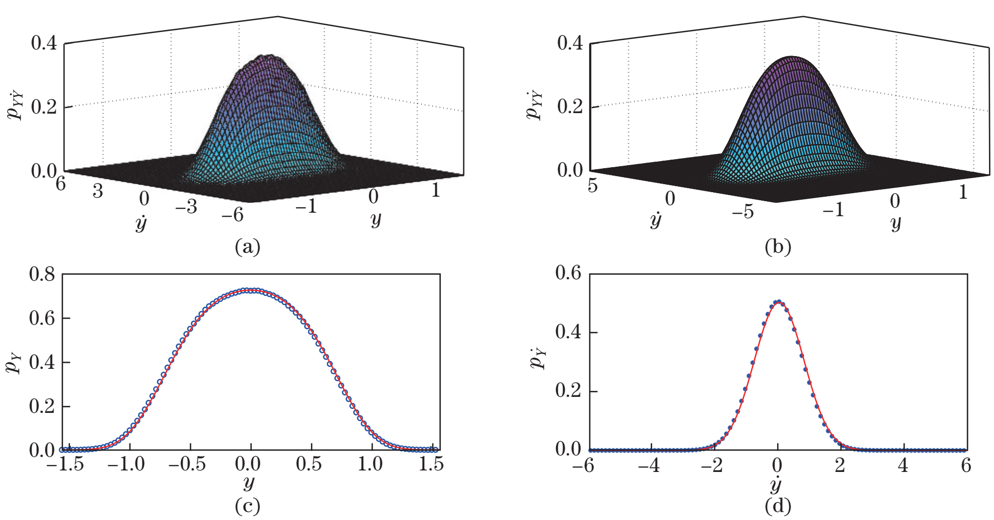

Fig. 5 Stationary PDFs of system (45) in the case of $r = 0.99$ and $\Delta = 4$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are the same as those in Fig. 3 (color online)

Finally, we examine the effect of excitation intensity D_1 to the accuracy of the proposed technique. Here, the solutions of the cases $D_1 = 0.5$ and 1.0 under $\Delta = 4$ and $r = 0.99$ are obtained with the approximate PDF solution in Eq. (A5) as p_m after 3 iterations, respectively. For the sake of brevity and the limited space, the expressions of the solutions for $D_1 = 0.5$ and 1.0 are omitted here. Figures 6 and 7 show the results $p_{Y\dot{Y}}$, p_Y, and $p_{\dot{Y}}$, given by the proposed method (order $n = 4$) and Monte Carlo simulation results of a sample size of 4×10^7, respectively. As expected, the related comparisons have demonstrated an excellent agreement between the present solution and Monte Carlo simulation.

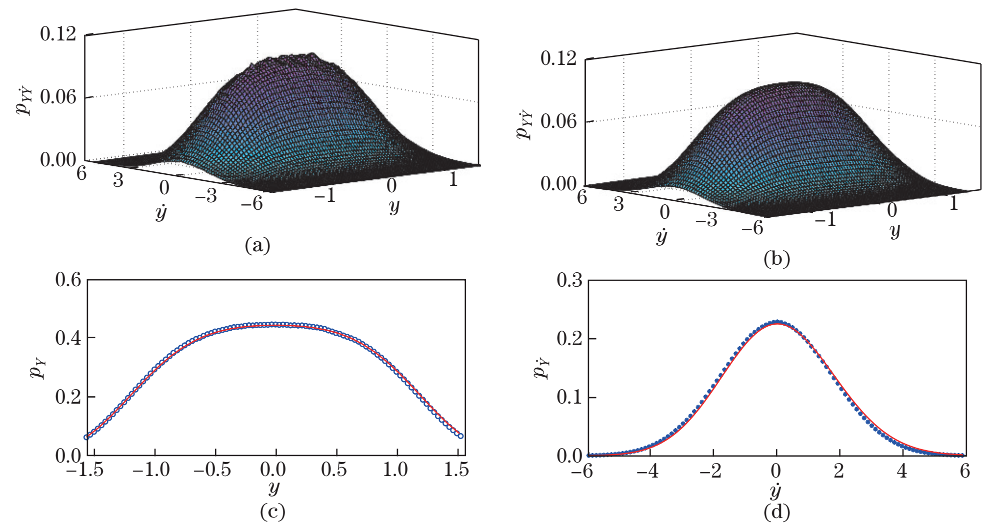

Fig. 6 Stationary PDFs of system (45) in the case of $r = 0.99$ and $D_1 = 0.5$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are the same as those in Fig. 3 (color online)

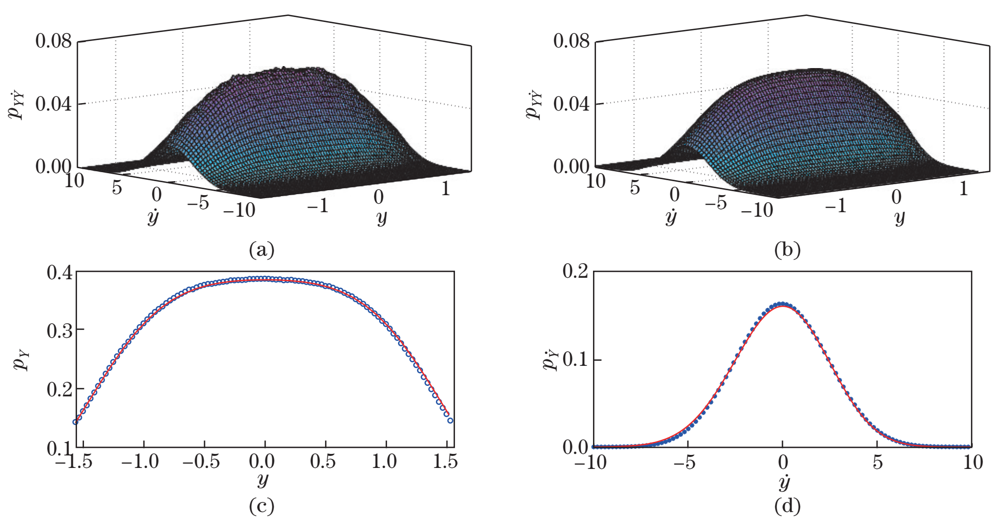

Fig. 7 Stationary PDFs of system (45) in the case of $r = 0.99$ and $D_1 = 1.0$. (a) denotes the joint PDFs of $p_{Y\dot{Y}}$ obtained with the Monte Carlo simulation; (b) represents the joint PDFs of $p_{Y\dot{Y}}$ obtained with the proposed scheme; (c) and (d) denote the marginal PDFs of p_Y and $p_{\dot{Y}}$, respectively. Solid line is the analytical result, and symbols are the Monte Carlo simulation data. The other parameters are the same as those in Fig. 3 (color online)

5 Conclusions

In this paper, a new method has been proposed to predict the stationary PDFs solution of the SDOF vibroimpact systems with bilateral barriers under a Gaussian white noise excitation. The method involves converting the non-smooth vibroimpact system with bilateral barriers to such a system without barriers through piecewise differentiable periodic transformation, constructing the approximate solution of the corresponding reduced FPK equation with the help of the concepts of the circulatory probability flow and the potential probability flow, and determining the unknown parameters in the trial solution by using the iterative method of weighted residue procedure. The proposed scheme has been applied to two typical examples of van der Pol oscillator and Duffing oscillator. It has been shown that the proposed scheme could yield a high precision in all results compared to the Monte Carlo simulation data. Besides, an interesting nonlinear characteristic of limit cycle together with two tentacles in the strongly nonlinear van der Pol oscillator has been fully captured and firstly expressed in the closed-form. As for the Duffing oscillator, the analytical closed-form stationary PDF solution in the case of elastic-impact has been obtained for the first time in the literature. The stochastic P-bifurcation has also been observed as some parameters change. Furthermore, all solutions obtained in this paper can be utilized as benchmark for the studies of vibroimpact systems with bilateral barriers.

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Appendix A

Example 1 The stationary PDF of van der Pol oscillator with bilateral barriers under $r = 0.99$ of order $n = 4$ is given below,

$$\overline{p}_{z_1 z_2} = \begin{cases} \overline{p}_1(z_1, z_2), & \text{as} \quad -\dfrac{\pi}{2} \leqslant z_1 < \dfrac{\pi}{2}, \\ \overline{p}_2(z_1, z_2), & \text{as} \quad \dfrac{\pi}{2} \leqslant z_1 \leqslant \dfrac{3\pi}{2}, \end{cases} \tag{A1}$$

where $\overline{p}_1(z_1, z_2)$ and $\overline{p}_2(z_1, z_2)$ are of the forms as follows:

$$\begin{cases} \overline{p}_1(z_1,z_2)=C_0\exp\Big(0.479\,899\,79\times s_1(z_1)w_1(z_1,z_2)^3-0.984\,912\,97\times s_1(z_1)^2w_1(z_1,z_2)^2\\ \qquad +0.353\,334\,95\times s_1(z_1)^3w_1(z_1,z_2)-0.569\,640\,76\times s_1(z_1)^4\\ \qquad +0.093\,163\,63\times w_1(z_1,z_2)^2-0.951\,304\,66\times s_1(z_1)w_1(z_1,z_2)\\ \qquad +1.150\,689\,89\times s_1(z_1)^2-0.140\,828\,18\times w_1(z_1,z_2)^4\\ \qquad +0.019\,705\,81\dfrac{w_1(z_1,z_2)^2(1.621\,138\,9\times s_1(z_1)^2-0.25)+0.002\,132\,68w_1(z_1,z_2)z_2^2}{0.308\,425\,13+0.5\times s_1(z_1)^2}\Big),\\ \overline{p}_2(z_1,z_2)=C_0\exp\Big(0.479\,899\,79\times s_2(z_1)w_2(z_1,z_2)^3-0.984\,912\,97\times s_2(z_1)^2w_2(z_1,z_2)^2\\ \qquad +0.353\,334\,95\times s_2(z_1)^3w_2(z_1,z_2)-0.569\,640\,76\times s_2(z_1)^4\\ \qquad +0.093\,163\,63\times w_2(z_1,z_2)^2-0.951\,304\,66\times s_2(z_1)w_2(z_1,z_2)\\ \qquad +1.150\,689\,89\times s_2(z_1)^2-0.140\,828\,18\times w_2(z_1,z_2)^4\\ \qquad +0.019\,705\,81\dfrac{w_2(z_1,z_2)^2(1.621\,138\,9\times s_2(z_1)^2-0.25)+0.002\,132\,68w_2(z_1,z_2)z_2^2}{0.308\,425\,13+0.5\times s_2(z_1)^2}\Big), \end{cases} \tag{A2}$$

where $s_1(z_1)$, $w_1(z_1,z_2)$, $s_2(z_1)$, and $w_2(z_1,z_2)$, respectively, are determined by

$$\begin{cases} s_1(z_1)=z_1+0.001\,599\,55\times z_1^2-0.003\,946\,72,\\ w_1(z_1,z_2)=(0.003\,199\,09\times z_1+1)z_2,\\ s_2(z_1)=-z_1-0.001\,599\,55\times(z_1-\pi)^2+\pi,\\ w_2(z_1,z_2)=-(0.989\,949\,74+0.003\,199\,09\times z_1)z_2. \end{cases} \tag{A3}$$

Example 2 $\overline{p}_1(z_1,z_2)$ and $\overline{p}_2(z_1,z_2)$ for the case $r=0.99$ of order $n=4$ are given by following equations, respectively,

$$\begin{cases} \overline{p}_1(z_1,z_2)=C_0\exp(-9.31\times10^{-7}\times w_1(z_1,z_2)^4+7.71\times10^{-8}\times s_1(z_1)w_1(z_1,z_2)^3\\ \qquad +3.34\times10^{-6}\times s_1(z_1)^2w_1(z_1,z_2)^2-1.85\times10^{-8}\times s_1(z_1)^3w_1(z_1,z_2)\\ \qquad -0.082\,131\,06\times w_1(z_1,z_2)^4-0.202\,638\,34\times w_1(z_1,z_2)^2\\ \qquad +0.98\times10^{-6}\times s_1(z_1)w_1(z_1,z_2)-0.202\,682\,11\times s_1(z_1)^2\\ \qquad +(0.11\times10^{-6}\times z_1+0.000\,035\,37)\times z_2^3),\\ \overline{p}_2(z_1,z_2)=C_0\exp(-9.31\times10^{-7}\times w_2(z_1,z_2)^4+7.71\times10^{-8}\times s_2(z_1)w_2(z_1,z_2)^3\\ \qquad +3.34\times10^{-6}\times s_2(z_1)^2w_2(z_1,z_2)^2-1.85\times10^{-8}\times s_2(z_1)^3w_2(z_1,z_2)\\ \qquad -0.082\,131\,06\times w_2(z_1,z_2)^4-0.202\,638\,34\times w_2(z_1,z_2)^2\\ \qquad +0.98\times10^{-6}\times s_2(z_1)w_2(z_1,z_2)-0.202\,682\,11\times s_2(z_1)^2\\ \qquad +(0.11\times10^{-6}\times z_1+0.000\,035\,37)\times z_2^3), \end{cases} \tag{A4}$$

in which $s_1(z_1)$, $w_1(z_1,z_2)$, $s_2(z_1)$, and $w_2(z_1,z_2)$, respectively, are given by Eq. (A3). The solution of $\Delta=4$ for the case of $r=0.99$ and order $n=4$ is given by

$$\overline{p}_{z_1z_2}=\begin{cases} \overline{p}_1(z_1,z_2), & \text{as}\quad -\dfrac{\pi}{2}\leqslant z_1<\dfrac{\pi}{2},\\ \overline{p}_2(z_1,z_2), & \text{as}\quad \dfrac{\pi}{2}\leqslant z_1\leqslant\dfrac{3\pi}{2}, \end{cases} \tag{A5}$$

where $\overline{p}_1(z_1, z_2)$ and $\overline{p}_2(z_1, z_2)$ are of the forms as follows:

$$\begin{cases} \overline{p}_1(z_1, z_2) = C_0 \exp(-0.000\,016\,57 \times w_1(z_1, z_2)^4 \\ \qquad\qquad + 0.000\,023\,71 \times s_1(z_1) w_1(z_1, z_2)^3 + 0.000\,132\,87 \times s_1(z_1)^2 w_1(z_1, z_2)^2 \\ \qquad\qquad - 0.000\,013\,51 \times s_1(z_1)^3 w_1(z_1, z_2) - 1.314\,142\,33 \times s_1(z_1)^4 \\ \qquad\qquad - 0.810\,523\,88 \times w_1(z_1, z_2)^2 - 0.000\,015\,11 \times s_1(z_1) w_1(z_1, z_2) \\ \qquad\qquad - 0.810\,444\,86 \times s_1(z_1)^2 + (6.718 \times 10^{-6} \times z_1 + 0.001\,044\,80) \times z_2^3), \\ \overline{p}_2(z_1, z_2) = C_0 \exp(-0.000\,016\,57 \times w_2(z_1, z_2)^4 \\ \qquad\qquad + 0.000\,023\,71 \times s_2(z_1) w_2(z_1, z_2)^3 + 0.000\,132\,87 \times s_2(z_1)^2 w_2(z_1, z_2)^2 \\ \qquad\qquad - 0.000\,013\,51 \times s_2(z_1)^3 w_2(z_1, z_2) - 1.314\,142\,33 \times s_2(z_1)^4 \\ \qquad\qquad - 0.810\,523\,88 \times w_2(z_1, z_2)^2 - 0.000\,015\,11 \times s_2(z_1) w_2(z_1, z_2) \\ \qquad\qquad - 0.810\,444\,86 \times s_2(z_1)^2 - (6.72 \times 10^{-6} \times z_1 + 0.001\,023\,69) \times z_2^3), \end{cases} \tag{A6}$$

in which $s_1(z_1)$, $w_1(z_1, z_2)$, $s_2(z_1)$, and $w_2(z_1, z_2)$, respectively, are given by Eq. (A3).