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An irreducible polynomial functional basis of two-dimensional Eshelby tensors*

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Abstract The two-dimensional (2D) Eshelby tensors are discussed. Based upon the complex variable method, an integrity basis of ten isotropic invariants of the 2D Eshelby tensors is obtained. Since an integrity basis is always a polynomial functional basis, these ten isotropic invariants are further proven to form an irreducible polynomial functional basis of the 2D Eshelby tensors.

Key words Eshelby tensor, representation theorem, irreducible functional basis, isotropic invariant

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Nomenclature

$M^{(2)}$,	two-dimensional (2D) Eshelby tensor;
O_2,	orthogonal group in the 2D physical space;
R_2,	rotation group in the 2D physical space;
$Q(\theta)$,	rotation of angle θ in the 2D physical space;
$\widetilde{Q}$,	special reflection in the 2D physical space;
H^m,	mth-order irreducible tensor space in the 2D physical space;
e_i,	orthonormal base in the 2D physical space;
$\mathbb{R}$,	real number field;
$\mathbb{C}^n$,	complex number field with the dimension n;
$\mathrm{Re}(x)$,	real part of a complex number x;
$\otimes$,	tensor product.

1 Introduction

The Eshelby problem of linear elasticity exists in the infinite region Ω induced by releasing either the transformation strains or the eigenstrains in a subdomain ω, called an inclusion. To describe this objection precisely, we could express the strain field $\epsilon_{ij}(x)$ in a linear form as follows:

$$\epsilon_{ij}(x) = \Sigma^{\omega}_{ijkl}(x)\epsilon^0_{kl},$$

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where ϵ^0_{ij} is a second-order constant eigenstrain tensor, and the fourth-order tensor $\Sigma^{\omega}_{ijkl}(x)$ is the Eshelby tensor field, corresponding to an inclusion ω. Eshelby[1–2] proved that the Eshelby tensor field was uniform inside ω if the inclusion was elliptic and ellipsoidal in two-dimensional (2D) and three-dimensional (3D) elasticities, respectively. In such a condition, the Eshelby tensor field Σ^{ω} is called the Eshelby tensor, and this significant property is called the Eshelby uniformity. The Eshelby tensors and the Eshelby uniformity have wide applications in engineering and physical fields, e.g., elliptical and non-elliptical inclusions[3–5], matrix-inclusion composites[6–10], non-uniform Gaussian and exponential eigenstrains within ellipsoids[11], and have become the subject of some extensive studies[12–15].

Moreover, the tensor function representation theory is well established and of prime importance in both theoretical and applied mechanics[16]. It was introduced to describe the general consistent invariant forms of the nonlinear constitutive equations and to determine the number and the type of the involved scalar variables. In the latter half of the twentieth century, the representations in complete and irreducible forms of vectors, second-order symmetric tensors, and second-order skew-symmetric tensors for both isotropic and hemitropic invariants were thoroughly investigated by Zheng[16], Wang[17–19], Smith[20], and Boehler[21]. For higher order tensors, Smith and Bao[22] presented the minimal integrity bases for the third- and fourth-order symmetric and traceless tensors. In 2014, Olive and Auffray[23] presented an integrity basis with thirteen isotropic invariants of a third-order 3D symmetric tensor, and showed that the Olive-Auffray integrity basis was actually a minimal integrity basis. In 2017, Olive[24] and Olive et al.[25] gave a minimal integrity basis for the elasticity tensors with 297 invariants. Very recently, a number of new results appeared. Liu et al.[26] gave a minimal integrity basis and an irreducible functional basis for the isotropic invariants of the Hall tensors. Chen et al.[27] showed that any minimal integrity basis of a third-order 3D symmetric and traceless tensor was indeed an irreducible functional basis of that tensor. Chen et al.[28] presented an eleven invariant irreducible functional basis for a third-order 3D symmetric tensor. This eleven invariant irreducible functional basis is a proper subset of the Olive-Auffray minimal integrity basis of the tensor.

Even though the Eshelby tensor has wide applications in mechanics, its minimal integrity basis and irreducible functional basis have not been decided yet. Because the Eshelby tensor has a weaker symmetry than the elasticity tensor, i.e., $M^{(2)}_{ijkl} \neq M^{(2)}_{klij}$ in general, the Eshelby tensor owns more independent elements. Moreover, in the 3D physical space, as a conclusion previously introduced, the elasticity tensor has a minimal integrity basis with 297 invariants. For these two reasons, it may need a large number of invariants to form a minimal integrity basis for 3D Eshelby tensors. Consequently, in this article, we only study the invariants of Eshelby tensors in the 2D physical space.

The complex variable method is our fundamental tool for recovering the Eshelby tensor from a set of isotropic invariants, i.e., the integrity basis. It was established by Pierce[29], and further applied to plane elasticity by Vianello[30]. In particular, Olive et al.[25] gave a summary of this method from an algebraic viewpoint. We denote 2D Eshelby tensors as $M^{(2)}$. By the orthogonal irreducible decomposition[31], $M^{(2)}$ could be split into six parts, i.e., three scalars (λ, μ, and υ), two second-order irreducible (symmetric and traceless) tensors (D^1 and D^2), and one fourth-order irreducible tensor D. Therefore, we only need to study the actions on D^1, D^2, and D of the 2D orthogonal group O_2. It is a familiar conclusion in the invariant theory that, for any fixed positive integer m, the dimension of an mth-order 2D irreducible tensor space is two, which helps us to construct a one-to-one correspondence between an irreducible tensor and a complex number. Moreover, there exists an isomorphism between the action of O_2 on a second-order irreducible tensor and the action of the same group on the products of the complex planes. Based on these two elementary facts, complexifying the problem becomes a useful approach to study the action of O_2 on the second-order irreducible tensor space.

The structure of this paper is as follows. To make our statement as self-contained as possible, we first give some notations and briefly review some basic definitions in the representation theory for the tensor functions in Section 2. In Section 3, starting from the irreducible decomposition of $M^{(2)}$, we further review the complex variable method, and propose a set of ten polynomial isotropic invariants of $M^{(2)}$. In Section 4, we prove that these ten invariants are functionally irreducible. Consequently, we obtain an irreducible polynomial functional basis of $M^{(2)}$, which is the main goal of this paper.

2 Preliminaries

In this paper, we denote H^m as the mth-order irreducible tensor space in the 2D physical space. As a classical terminology, O_2 is the group of orthogonal transformations in the 2D physical space, and R_2 is the rotation subgroup of O_2. $e_1 := (1, 0)^{\mathrm{T}}$ and $e_2 := (0, 1)^{\mathrm{T}}$ are a pair of orthonormal bases in the 2D physical space. $\widetilde{Q}$ is the reflection transformation such that

$$\widetilde{Q}e_1 = e_1, \quad \widetilde{Q}e_2 = -e_2.$$

Obviously, O_2 is generated by R_2 and $\widetilde{Q}$.

Let T be an mth-order tensor represented by $T_{i_1 i_2 \cdots i_m}$ under some orthonormal coordinates. A scalar-valued polynomial function $f(T_{i_1 i_2 \cdots i_m})$ is called a polynomial isotropic invariant of T if the function value is independent of the selection of the coordinate system, i.e.,

$$f(T_{i_1 i_2 \cdots i_m}) = f(Q_{i_1 j_1} Q_{i_2 j_2} \cdots Q_{i_m j_m} T_{j_1 j_2 \cdots j_m}). \tag{1}$$

We could rewrite Eq. (1) in a short form as follows:

$$f(T) = f(Q * T),$$

where Q is an arbitrary orthogonal matrix. $*$ is called the O_2-action and defined as the right-side in Eq. (1). Moreover, a set of tensors

$$O_2 * T = \{g * T : g \in O_2\}$$

is called the O_2-orbit of T.

Once we have one polynomial invariant, we could easily construct an infinite number of polynomial invariants from it. For this reason, our main goal is to find a finite set of polynomial invariants separating the O_2-orbits. Therefore, we introduce the definitions of the integrity basis and the polynomial functional basis as follows.

Definition 1 *Let $\{f_1, f_2, \cdots, f_n\}$ be a finite set of polynomial isotropic invariants of T. If any polynomial isotropic invariant of T is polynomial of $f_1, f_2, \cdots, f_n$, we call $\{f_1, f_2, \cdots, f_n\}$ a set of integrity basis of T. In addition, an integrity basis is minimal if no proper subset of it is an integrity basis.*

Similarly, we give the definition for the polynomial functional basis (hereinafter called the functional basis).

Definition 2 *Let $\{f_1, f_2, \cdots, f_n\}$ be a finite set of polynomial isotropic invariants of T. If any isotropic invariant of T is expressible by a function of $\{f_1, f_2, \cdots, f_n\}$, we call $\{f_1, f_2, \cdots, f_n\}$ a set of functional basis of T. In addition, a functional basis is minimal (or irreducible) if no proper subset of it is a functional basis.*

It is known in the invariant theory that, the algebra of invariant polynomials on the finite-dimensional representation V of O_2 is finitely generated[32]. In other words, it claims the existence of a set of integrity basis of $M^{(2)}$. Since the integrity bases are also functional bases (but not vice versa), both the integrity bases and the functional bases could separate orbits.

3 Orthogonal transformations and ten isotropic invariants of $M^{(2)}$

3.1 Orthogonal irreducible decomposition of $M^{(2)}$

Due to the minor index symmetry of $M^{(2)}$, we have

$$M_{ijkl}^{(2)} = M_{jikl}^{(2)} = M_{ijlk}^{(2)}.$$

The irreducible decomposition of $M^{(2)}$ takes the form[33]:

$$M_{ijkl}^{(2)} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{i\hat{k}}\delta_{j\hat{l}} + v(\delta_{i\hat{k}}\epsilon_{j\hat{l}} + \delta_{j\hat{k}}\epsilon_{i\hat{l}}) + \delta_{ij}D_{kl}^1 + \delta_{kl}D_{ij}^2 + D_{ijkl}, \tag{2}$$

where

$$\lambda = \frac{3}{8}M_{iikk}^{(2)} - \frac{1}{4}M_{ikik}^{(2)}, \quad \mu = \frac{1}{4}M_{ikik}^{(2)} - \frac{1}{8}M_{iikk}^{(2)}, \quad v = \frac{1}{4}\epsilon_{ij}M_{ikjk}^{(2)} \tag{3}$$

are three scalars,

$$D_{ij}^1 = \frac{1}{2}M_{kkij}^{(2)} - \frac{1}{4}M_{kkll}^{(2)}\delta_{ij}, \quad D_{ij}^2 = \frac{1}{2}M_{ijkk}^{(2)} - \frac{1}{4}M_{kkll}^{(2)}\delta_{ij} \tag{4}$$

are two second-order irreducible tensors (denoted as D^1 and D^2, respectively), and $D\,(= D_{ijkl})$ is a fourth-order irreducible tensor deduced by Eqs. (2), (3), and (4). Here, the symbol $\hat{}$ in the subscripts means calculating the average of the circulant symmetric monomials, e.g.,

$$\delta_{i\hat{k}}\delta_{j\hat{l}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

3.2 Orthonormal bases of H^2 and H^4 and orthogonal transformations

It is known that, for any fixed positive integer m, the dimension of H^m is two. Inspired by Vilanello[30], we could find two appropriate pairs of orthonormal bases in H^2 and H^4, respectively. In particular, we choose

$$E_1 = \frac{\sqrt{2}}{2}(e_1 \otimes e_1 - e_2 \otimes e_2), \quad E_2 = \frac{\sqrt{2}}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$$

as a pair of orthonormal bases in H^2 and

$$\begin{aligned} F_1 = \frac{\sqrt{8}}{8}(&e_1 \otimes e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \otimes e_2 - e_1 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_2 \otimes e_1 \otimes e_2 \\ &- e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_2 \otimes e_1 \otimes e_1), \\ F_2 = \frac{\sqrt{8}}{8}(&e_1 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \otimes e_1 \\ &- e_2 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_2 \otimes e_2 \otimes e_2) \end{aligned}$$

as a pair of orthonormal bases in H^4. Here, $\otimes$ stands for the tensor product. Similar to the representation of a vector in the 2D Cartesian coordinates, let θ_1, θ_2, and θ_3 be the angles between D^1 and E_1, D^2 and E_1, and D and F_1, respectively. Besides, some definitions are necessary:

$$\begin{aligned} &H_1 := |D^1|\cos\theta_1 = D^1 \cdot E_1, \quad H_2 := |D^1|\sin\theta_1 = D^1 \cdot E_2, \\ &L_1 := |D^2|\cos\theta_2 = D^2 \cdot E_1, \quad L_2 := |D^2|\sin\theta_2 = D^2 \cdot E_2, \\ &K_1 := |D|\cos\theta_3 = D \cdot F_1, \quad K_2 := |D|\sin\theta_3 = D \cdot F_2. \end{aligned}$$

With some calculations, a rotation $Q(\theta) \in R_2$ could satisfy the following identities:

$$\begin{cases} Q(\theta) * E_1 = \cos(2\theta)E_1 + \sin(2\theta)E_2, \\ Q(\theta) * E_2 = -\sin(2\theta)E_1 + \cos(2\theta)E_2, \\ Q(\theta) * F_1 = \cos(4\theta)F_1 + \sin(4\theta)F_2, \\ Q(\theta) * F_2 = -\sin(4\theta)F_1 + \cos(4\theta)F_2. \end{cases} \tag{5}$$

Moreover, for $\widetilde{Q}$, we have

$$\widetilde{Q} * E_1 = E_1, \quad \widetilde{Q} * E_2 = -E_2, \quad \widetilde{Q} * F_1 = F_1, \quad \widetilde{Q} * F_2 = -F_2. \tag{6}$$

Therefore, each $Q(\theta) \in R_2$ acts on H^2 or H^4 as a rotation of 2θ or 4θ, respectively. Under the reflection transformation $\widetilde{Q}$, E_1 and F_1 are unchanged, while E_2 and F_2 are turned to the opposite ones. In view of this conclusion, a one-to-one mapping from an irreducible tensor in H^2 or H^4 to a complex number could be constructed. More precisely, D^1, D^2, and D are related to the complex numbers z_1, z_2, and z_3 in sequence. The relations could be described as follows:

$$z_1 = H_1 + \mathrm{i}H_2 = |D^1| \cdot \mathrm{e}^{\mathrm{i}\theta_1},$$

$$z_2 = L_1 + \mathrm{i}L_2 = |D^2| \cdot \mathrm{e}^{\mathrm{i}\theta_2},$$

$$z_3 = K_1 + \mathrm{i}K_2 = |D| \cdot \mathrm{e}^{\mathrm{i}\theta_3},$$

where i is the imaginary unit. In other words, we regard the component of the "horizontal" axis as the real part of z, and the component of the "vertical" axis as the imaginary part of z. Moreover, the action of $Q(\theta)$ or $\widetilde{Q}$ on the spaces H^2 and H^4 could be seen as an action on the complex plane $\mathbb{C}$. More precisely, according to Eqs. (5) and (6), we have

$$Q(\theta) * (z_1,\ z_2,\ z_3) = (z_1 \cdot \mathrm{e}^{\mathrm{i}(2\theta)},\ z_2 \cdot \mathrm{e}^{\mathrm{i}(2\theta)},\ z_3 \cdot \mathrm{e}^{\mathrm{i}(4\theta)}) \in \mathbb{C}^3, \tag{7}$$

$$\widetilde{Q} * (z_1,\ z_2,\ z_3) = (\bar{z}_1,\ \bar{z}_2,\ \bar{z}_3) \in \mathbb{C}^3 \tag{8}$$

for any $\theta \in [0,\ 2\pi)$.

3.3 Polynomial invariants of $M^{(2)}$

Now, we take our aim at the polynomial invariants of $M^{(2)}$. For any polynomial function p of $M^{(2)}$, we can rewrite $p(M^{(2)})$ as follows:

$$p(M^{(2)}) = \sum C_{abcdefgjk} \lambda^a \mu^b v^c z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k, \tag{9}$$

where a, b, c, d, e, f, g, j, and k are nine nonnegative integers. $C_{abcdefgjk} \in \mathbb{C}$ is the coefficient of each monomial. Since $p(M^{(2)})$ is a real-valued polynomial, we have

$$C_{abcdefgjk} = \overline{C}_{abcedgfkj}.$$

In addition, because O_2 is generated by R_2 and $\widetilde{Q}$, $p(M^{(2)})$ should be invariant under any rotation $Q(\theta)$ and the reflection $\widetilde{Q}$, which yields

$$p(M^{(2)}) = p(Q(\theta) * M^{(2)}) \quad \text{and} \quad p(M^{(2)}) = p(\widetilde{Q} * M^{(2)}), \tag{10}$$

where θ is an arbitrary angle. In a viewpoint of complex field, the action of $Q(\theta)$ takes the form

$$p(Q(\theta) * M^{(2)}) = \sum C_{abcdefgjk} \lambda^a \mu^b v^c z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k \mathrm{e}^{\mathrm{i}\theta(2(d-e)+2(f-g)+4(j-k))},$$

while the action of $\widetilde{Q}$ takes the form

$$p(\widetilde{Q} * M^{(2)}) = \sum C_{abcdefgjk}\lambda^a\mu^b v^c z_1^e \bar{z}_1^d z_2^g \bar{z}_2^f z_3^k \bar{z}_3^j.$$

Combining with Eq. (10), we conclude that the degrees of each monomial in Eq. (9) satisfy the Diophantine equation as follows:

$$d - e + f - g + 2(j - k) = 0, \tag{11}$$

and the coefficients are restricted to

$$C_{abcdefgjk} = C_{abcedgfkj}. \tag{12}$$

From Eqs. (10) and (12), we know that each coefficient $C_{abcdefgjk}$ is a real number. Furthermore, each monomial $C_{abcdefgjk}\lambda^a\mu^b v^c z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k$ should obey the Diophantine equation (11).

A solution of the Diophantine equation is irreducible if it is not the sum of two or more nonnegative and nontrivial solutions. The following proposition gives a maximal irreducible solution of Eq. (11). It can deduce that any nonnegative solution of Eq. (11) is a sum of these irreducible solutions. For convenience, we denote $w = (d, e, f, g, j, k)$ as a vector of six components.

Proposition 1 *Let*

$$\begin{aligned}
&w_1 = (1,1,0,0,0,0), \quad w_2 = (0,0,1,1,0,0), \quad w_3 = (0,0,0,0,1,1),\\
&w_4 = (2,0,0,0,0,1), \quad w_5 = (0,0,2,0,0,1), \quad w_6 = (1,0,0,1,0,0),\\
&w_7 = (1,0,1,0,0,1), \quad w_8 = (0,1,1,0,0,0), \quad w_9 = (0,1,0,1,1,0),\\
&w_{10} = (0,2,0,0,1,0), \quad w_{11} = (0,0,0,2,1,0).
\end{aligned}$$

Then, (i) $w_1, w_2, \cdots, w_{11}$ *are eleven irreducible solutions of Eq.* (11); (ii) *each non-negative solution of Eq.* (11) *is a sum of these irreducible solutions.*

Proof Property (i) can be easily verified. In order to prove Property 1 (ii), we denote $\Gamma = d + e + f + g + j + k$ and complete the proof by mathematical induction.

When $\Gamma = 2,\ 3$, it is easy to testify that $w_1,\ w_2, \cdots,\ w_{11}$ form the whole feasible solutions of Eq. (11) in these two cases. To take a further step, we assume that if the sum of the six components of a solution is not more than Γ ($\geqslant 3$), this solution could be a sum of $w_1,\ w_2, \cdots,\ w_{11}$. Now, we consider a new feasible solution $w = (d,\ e,\ f,\ g,\ j,\ k)$ satisfying

$$d + e + f + g + j + k = \Gamma + 1, \quad d - e + f - g + 2(j - k) = 0.$$

We finish the proof within two cases.

Case 1 If $j = k$, we have $d - e + f - g = 0$.

In this case, if $d = e$, we have

$$f = g, \quad j + k = \Gamma + 1 \geqslant 4,$$

which implies $j,\ k \geqslant 1$. Therefore, $(d,\ e,\ f,\ g,\ j-1,\ k-1)$ with the sum $\Gamma - 1$ is also a solution. By the assumption, $(d,\ e,\ f,\ g,\ j-1,\ k-1)$ can be represented as the sum of $w_1,\ w_2, \cdots,\ w_{11}$. Combined with $(d,\ e,\ f,\ g,\ j,\ k) = (d,\ e,\ f,\ g,\ j-1,\ k-1) + w_3$, the conclusion is valid.

If $d \neq e$, without loss of generality, let $d > e$. Then, $g > f$. Similarly, we have $d,\ g \geqslant 1$ and $(d-1,\ e,\ f,\ g-1,\ j,\ k)$ can be represented as the sum of $w_1,\ w_2, \cdots,\ w_{11}$, indicating that the conclusion is valid.

Case 2 If $j \neq k$, we could assume $j > k$. Then, $e+g \geqslant 2+d+f \geqslant 2$. Since

$$(d,e,f,g,j,k) = (d,e-1,f,g-1,j-1,k) + w_9 := u_1 + w_9,$$

$$(d,e,f,g,j,k) = (d,e-2,f,g,j-1,k) + w_{10} := u_2 + w_{10},$$

$$(d,e,f,g,j,k) = (d,e,f,g-2,j-1,k) + w_{11} := u_3 + w_{11},$$

at least one of u_1, u_2, and u_3 is a non-negative solution when $e+g \geqslant 2$, which completes the proof.

Then, we relate each solution $(d,\ e,\ f,\ g,\ j,\ k)$ to a complex monomial $z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k$, so that the eleven solutions $w_1,\ w_2,\ \cdots,\ w_{11}$ could correspond to eleven complex monomials. Since the invariants are real-valued functions, we only need to consider the real parts of these complex monomials. Hence, we obtain seven different polynomial invariants $J_1,\ J_2,\ \cdots,\ J_7$ of $M^{(2)}$. Their relations to D^1, D^2, and D are presented concurrently as follows:

$$\begin{cases} w_1 \rightarrow J_1 := \mathrm{Re}(z_1\bar{z}_1) = |z_1|^2 = H_1^2 + H_2^2 = D_{ij}^1 \cdot D_{ij}^1, \\ w_2 \rightarrow J_2 := \mathrm{Re}(z_2\bar{z}_2) = |z_2|^2 = L_1^2 + L_2^2 = D_{ij}^2 \cdot D_{ij}^2, \\ w_3 \rightarrow J_3 := \mathrm{Re}(z_3\bar{z}_3) = |z_3|^2 = K_1^2 + K_2^2 = D_{ijkl} \cdot D_{ijkl}, \\ w_4,\ w_{10} \rightarrow J_4 := \mathrm{Re}(z_1^2\bar{z}_3) = (H_1^2 - H_2^2)K_1 + 2H_1H_2K_2 = D_{ij}^1 \cdot D_{ijkl} \cdot D_{kl}^1, \\ w_5,\ w_{11} \rightarrow J_5 := \mathrm{Re}(z_2^2\bar{z}_3) = (L_1^2 - L_2^2)K_1 + 2L_1L_2K_2 = D_{ij}^2 \cdot D_{ijkl} \cdot D_{kl}^2, \\ w_6,\ w_8 \rightarrow J_6 := \mathrm{Re}(z_1\bar{z}_2) = H_1L_1 + H_2L_2 = D_{ij}^1 \cdot D_{ij}^2, \\ w_7,\ w_9 \rightarrow J_7 := \mathrm{Re}(z_1z_2\bar{z}_3) = H_1K_1L_1 + H_1K_2L_2 - H_2K_1L_2 + H_2K_2L_1 \\ \qquad\qquad\qquad\quad = D_{ij}^1 \cdot D_{ijkl} \cdot D_{kl}^2. \end{cases} \tag{13}$$

In addition, we denote

$$J_8 := \lambda,\ J_9 := \mu,\ J_{10} := v.$$

As a result of the above discussion, we finally obtain a set of ten polynomial isotropic invariants $\{J_1,\ J_2,\ \cdots,\ J_{10}\}$ of $M^{(2)}$. In the next section, we will prove that these ten isotropic invariants are both minimal integrity bases and irreducible function bases of $M^{(2)}$.

4 Minimal integrity bases and irreducible functional bases of $M^{(2)}$

Now, our aim is to prove that $J_1,\ J_2,\ \cdots,\ J_{10}$ are both minimal integrity bases and irreducible function bases of $M^{(2)}$. We first confirm that any isotropic polynomial invariant is a polynomial of $J_1,\ J_2,\ \cdots,\ J_{10}$, i.e., $J_1,\ J_2,\ \cdots,\ J_{10}$ are integrity bases. As we have mentioned, an integrity basis is always a functional basis. Therefore, $J_1,\ J_2,\ \cdots,\ J_{10}$ also form a set of function basis of $M^{(2)}$. Next, we claim that $J_1,\ J_2,\ \cdots,\ J_{10}$ are functionally irreducible. Consequently, they are proven to be a set of irreducible functional basis of $M^{(2)}$, which is the main goal of this paper.

First, we give the following proposition to show that $J_1,\ J_2,\ \cdots,\ J_{10}$ form a set of integrity basis of $M^{(2)}$.

Proposition 2 *Any isotropic polynomial invariant of $M^{(2)}$ is a polynomial of*

$$J_1,\ J_2,\ \cdots,\ J_{10}.$$

Proof In the beginning, we rewrite the forms of $J_1,\ J_2,\ \cdots,\ J_7$ in Eq. (13) by using

$$H := |D^1|, \quad L := |D^2|, \quad K := |D|$$

as three norms of $D^1,\ D^2,\ D$ and $\theta_1,\ \theta_2,\ \theta_3$ as three angles defined as in Subsection 3.2. More explicitly, we have

$$\begin{cases} J_1 := H_1^2 + H_2^2 = H^2, \quad J_2 := L_1^2 + L_2^2 = L^2, \quad J_3 := K_1^2 + K_2^2 = K^2, \\ J_4 := (H_1^2 - H_2^2)K_1 + 2H_1H_2K_2 = H^2K \cdot \cos(2\theta_1 - \theta_3), \\ J_5 := (L_1^2 - L_2^2)K_1 + 2L_1L_2K_2 = L^2K \cdot \cos(2\theta_2 - \theta_3), \\ J_6 := H_1L_1 + H_2L_2 = HL \cdot \cos(\theta_1 - \theta_2), \\ J_7 := H_1K_1L_1 + H_1K_2L_2 - H_2K_1L_2 + H_2K_2L_1 \\ \qquad = HKL \cdot \cos(\theta_1 + \theta_2 - \theta_3), \end{cases} \tag{14}$$

where $H,\ L,\ K,\ \theta_1,\ \theta_2$, and θ_3 are independent of each other. Moreover, we introduce six scalar-valued functions of $H,\ L,\ K,\ \theta_1,\ \theta_2$, and θ_3 as follows:

$$\begin{cases} J_{11} := H^2L^2K^2 \sin(2\theta_1 - \theta_3)\sin(2\theta_2 - \theta_3), \\ J_{12} := H^3LK \sin(2\theta_1 - \theta_3)\sin(\theta_1 - \theta_2), \\ J_{13} := H^3LK^2 \sin(2\theta_1 - \theta_3)\sin(\theta_1 + \theta_2 - \theta_3), \\ J_{14} := HL^3K \sin(2\theta_2 - \theta_3)\sin(\theta_1 - \theta_2), \\ J_{15} := HL^3K^2 \sin(2\theta_2 - \theta_3)\sin(\theta_1 + \theta_2 - \theta_3), \\ J_{16} := H^2L^2K \sin(\theta_1 - \theta_2)\sin(\theta_1 + \theta_2 - \theta_3). \end{cases} \tag{15}$$

By some calculations, we have

$$J_{11} = J_6^2 \cdot J_3 - J_7^2, \quad J_{12} = J_1 \cdot J_7 - J_4 \cdot J_6, \quad J_{13} = J_1 \cdot J_3 \cdot J_6 - J_4 \cdot J_7,$$

$$J_{14} = J_5 \cdot J_6 - J_2 \cdot J_7, \quad J_{15} = J_2 \cdot J_3 \cdot J_6 - J_5 \cdot J_7, \quad J_{16} = \frac{1}{2}(J_1 \cdot J_5 - J_2 \cdot J_4).$$

Thus, $J_{11},\ J_{12},\ \cdots,\ J_{16}$ are polynomials of $J_1,\ J_2,\ \cdots,\ J_7$. In view of this, they are also polynomial invariants of $M^{(2)}$, and we only need to testify that any polynomial invariant of $M^{(2)}$ is polynomial of $J_1,\ J_2,\ \cdots,\ J_{16}$.

Recall that each non-zero monomial

$$C_{abcdefgjk}\lambda^a\mu^b v^c z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k$$

should satisfy $C_{abcdefgjk} = C_{abcedgfkj} \in \mathbb{R}$ and the Diophantine equation (11). Then, the remaining work is to prove that any sum of two conjugated monomials

$$W := C_{abcdefgjk}\ \{\ \lambda^a\mu^b v^c z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k + \lambda^a\mu^b v^c z_1^e \bar{z}_1^d z_2^g \bar{z}_2^f z_3^k \bar{z}_3^j\ \}$$

with the degrees satisfying that Eq. (11) is a polynomial of $J_1,\ J_2,\ \cdots,\ J_{16}$. Omitting the scalars, we denote

$$\begin{aligned} \widehat{W} :&= z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k + z_1^e \bar{z}_1^d z_2^g \bar{z}_2^f z_3^k \bar{z}_3^j \\ &= 2\mathrm{Re}\{z_1^d \bar{z}_1^e z_2^f \bar{z}_2^g z_3^j \bar{z}_3^k\} \\ &= 2H^{d+e}L^{f+g}K^{j+k} \cdot \cos((d-e)\theta_1 + (f-g)\theta_2 + 2(j-k)\theta_3), \end{aligned}$$

and further define eight angles $\beta_1, \beta_2, \cdots, \beta_8$ as follows:

$$\beta_1 = 2\theta_1 - \theta_3, \quad \beta_2 = 2\theta_2 - \theta_3, \quad \beta_3 = 2\theta_1 - \theta_2, \quad \beta_4 = \theta_1 + \theta_2 - \theta_3,$$

$$\beta_5 = -\theta_1 + \theta_2, \quad \beta_6 = -\theta_1 - \theta_2 + \theta_3, \quad \beta_7 = -2\theta_1 + \theta_3, \quad \beta_8 = -2\theta_2 + \theta_3.$$

Similar to the proof of Proposition 1, if

$$d - e + f - g + 2(j - k) = 0,$$

the linear combination $(d-e)\theta_1 + (f-g)\theta_2 + 2(j-k)\theta_3$ of $\theta_1,\ \theta_2$, and θ_3 would also be a linear combination of $\beta_1, \beta_2, \cdots, \beta_8$, i.e.,

$$(d-e)\theta_1 + (f-g)\theta_2 + 2(j-k)\theta_3 = \alpha_1\beta_1 + \cdots + \alpha_8\beta_8,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_8$ are all natural numbers.

With some simple calculations, $\cos(\alpha_1\beta_1 + \cdots + \alpha_8\beta_8)$ is a polynomial of

$$\cos\beta_1, \cos\beta_2, \cdots, \cos\beta_8 \quad \text{and} \quad \sin\beta_i \cdot \sin\beta_j \ \ (i, j \in \{1, 2, \cdots, 8\}).$$

Considering the forms of $J_1, J_2, \cdots, J_{16}$ in Eqs. (14) and (15), we claim that each sum of two conjugated monomials W is a polynomial of $J_1, J_2, \cdots, J_{16}$. Thus, we finish the proof.

To take a further step, we need to prove that $J_1, J_2, \cdots, J_{10}$ are functionally irreducible (then also polynomially irreducible). The method to prove the functionally irreducibility of invariants was first introduced by Pennisi and Trovato[34], with which we can prove the following proposition.

Proposition 3 $J_1, J_2, \cdots, J_{10}$ *are functionally irreducible.*

Proof Due to the orthogonal irreducible decompositions (2), it is clear that the three scalars $J_8,\ J_9,\ J_{10}$ are functionally irreducible. Therefore, we only need to consider about $J_1, J_2, \cdots, J_7$. Our goal is to change one of J_s $(s = 1, 2, \cdots, 7)$ while keep the other six invariants unchanged.

Case 1 When $s = 1$, let $L = K = 0$, which leads to $J_2 = J_3 = \cdots = J_7 = 0$. However, J_1 will change when H changes, so that J_1 cannot be a function of the others.

Case 2 When $s = 2$, let $H = K = 0$, which leads to $J_1 = J_3 = \cdots = J_7 = 0$. However, J_2 will change when L changes, so that J_2 cannot be a function of the others.

Case 3 When $s = 3$, let $H = L = 0$, which leads to $J_1 = J_2 = J_4 = \cdots = J_7 = 0$. However, J_3 will change when K changes, so that J_3 cannot be a function of the others.

Case 4 When $s = 4$, let $L = 0$ and K and H be two fixed and non-zero numbers, which leads to

$$J_2 = J_5 = J_6 = J_7 = 0, \quad J_1 = H^2, \quad J_3 = K^2.$$

However, $J_4 = H^2K \cdot \cos(2\theta_1 - \theta_3)$ will change when $2\theta_1 - \theta_3$ changes, so that J_4 cannot be a function of the others.

Case 5 When $s = 5$, let $H = 0$ and K and L be two fixed and non-zero numbers, which leads to

$$J_1 = J_4 = J_6 = J_7 = 0, \quad J_2 = L^2, \quad J_3 = K^2.$$

However, $J_5 = L^2K \cdot \cos(2\theta_2 - \theta_3)$ will change when $2\theta_2 - \theta_3$ changes, so that J_5 cannot be a function of the others.

Case 6 When $s = 6$, let $K = 0$ and H and L be two fixed and non-zero numbers, which leads to

$$J_3 = J_4 = J_5 = J_7 = 0, \quad J_1 = H^2, \quad J_2 = L^2.$$

However, $J_6 = HL \cdot \cos(\theta_1 - \theta_2)$ will change when $\theta_1 - \theta_2$ changes, so that J_6 cannot be a function of the others.

Case 7 When $s = 7$, let K, H, and L be three fixed and non-zero numbers, which leads to

$$J_1 = H^2, \quad J_2 = L^2, \quad J_3 = K^2.$$

Now, let

$$\theta_1 = \frac{\pi}{2}, \quad \theta_2 = 0, \quad \theta_3 = \frac{3\pi}{4}.$$

Then, we have

$$J_4 = \frac{\sqrt{2}}{2}H^2K, \quad J_5 = -\frac{\sqrt{2}}{2}L^2K, \quad J_6 = 0, \quad J_7 = \frac{\sqrt{2}}{2}HKL.$$

However, when $\theta_1 = \frac{3\pi}{2}$, $\theta_2 = 0$, and $\theta_3 = \frac{11\pi}{4}$, we have

$$J_4 = \frac{\sqrt{2}}{2}H^2K, \quad J_5 = -\frac{\sqrt{2}}{2}L^2K, \quad J_6 = 0, \quad J_7 = -\frac{\sqrt{2}}{2}HKL.$$

Only the value of J_7 changes, so that J_7 cannot be a function of the others.

In conclusion, $J_1, J_2, \cdots, J_{10}$ are functionally irreducible.

As a result of Propositions 2 and 3, finally, we have the following theorem.

Theorem 1 *Define*

$$J_1 := D^1_{ij} \cdot D^1_{ij}, \quad J_2 := D^2_{ij} \cdot D^2_{ij}, \quad J_3 := D_{ijkl} \cdot D_{ijkl},$$
$$J_4 := D^1_{ij} \cdot D_{ijkl} \cdot D^1_{kl}, \quad J_5 := D^2_{ij} \cdot D_{ijkl} \cdot D^2_{kl}, \quad J_6 := D^1_{ij} \cdot D^2_{ij},$$
$$J_7 := D^1_{ij} \cdot D_{ijkl} \cdot D^2_{kl}, \quad J_8 := \lambda, \quad J_9 := \mu, \quad J_{10} := v,$$

where λ, μ, v, D^1, D^2, *and* D *are three scalars, two 2nd-order irreducible tensors, and one 4th-order irreducible tensor in the orthogonal irreducible decomposition* (2), *respectively. Then,* $J_1, J_2, \cdots, J_{10}$ *are both a set of minimal integrity basis and a set of irreducible polynomial functional basis of* $M^{(2)}$.

5 Conclusions

We study the isotropic invariants of 2D Eshelby tensors $M^{(2)}$. The complex variable method is our fundamental tool, which helps to construct a one-to-one mapping from an irreducible tensor to a complex number. With this method, we obtain a set of integrity basis of ten isotropic invariants $\{J_1, J_2, \cdots, J_{10}\}$ of $M^{(2)}$, and then further prove that they are also a set of irreducible functional basis of $M^{(2)}$, as in the previous sections. The contributions of this article are as follows:

(i) $\{J_1, J_2, \cdots, J_{10}\}$ is a minimal integrity basis of 2D Eshelby tensors $M^{(2)}$.

(ii) $\{J_1, J_2, \cdots, J_{10}\}$ is also an irreducible polynomial functional basis of 2D Eshelby tensors $M^{(2)}$.

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