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Free vibration of non-uniform axially functionally graded beams using the asymptotic development method*

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Abstract The asymptotic development method is applied to analyze the free vibration of non-uniform axially functionally graded (AFG) beams, of which the governing equations are differential equations with variable coefficients. By decomposing the variable flexural stiffness and mass per unit length into reference invariant and variant parts, the perturbation theory is introduced to obtain an approximate analytical formula of the natural frequencies of the non-uniform AFG beams with different boundary conditions. Furthermore, assuming polynomial distributions of Young's modulus and the mass density, the numerical results of the AFG beams with various taper ratios are obtained and compared with the published literature results. The discussion results illustrate that the proposed method yields an effective estimate of the first three order natural frequencies for the AFG tapered beams. However, the errors increase with the increase in the mode orders especially for the cases with variable heights. In brief, the asymptotic development method is verified to be simple and efficient to analytically study the free vibration of non-uniform AFG beams, and it could be used to analyze any tapered beams with an arbitrary varying cross width.

Key words axially functionally graded (AFG) beam, non-uniform, natural frequency, asymptotic development method

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1 Introduction

Functionally graded materials (FGMs), which are a kind of non-homogeneous composites with gradually varying material gradients from one surface to another, are made into variety of structures, such as beams[1–2], plates[3–4], and cylindrical shells[5–6], and are widely applied in aeronautical, nuclear, and mechanical engineering owing to various advantages over the classical composite laminates. Generally, the gradient variation may be orientated in the cross

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section or/and in the axial direction. For the former, there have been a large number of studies devoted to the free vibration of FGM beams[2,7–10]. For axially functionally graded (AFG) beams, precise analytical solutions are difficult to obtain because of the variable coefficients of the governing equation. In recent years, some numerical methods have been proposed to handle the free vibration of non-uniform structures, such as the power series method[11–12], the wavelet series method[13], the differential quadrature element method[14], and the high-order Chebyshev expansion method[15].

The asymptotic development method, which is a type of perturbation analysis method, is always used to solve nonlinear vibration equations. For example, Chen and Chen[16] and Yan et al.[17] investigated the nonlinear parametric vibration of axially accelerating viscoelastic strings via the asymptotic perturbation method. Ding et al.[18–19] used the method of multiple scales to obtain the approximate analytical solutions of the natural frequency of an axially moving viscoelastic beam. Chen[20] studied the finite deformations of a pre-stressed, hyperelastic, compressible plate using an asymptotic perturbation technique. Andrianov and Danishevs'kyy[21] developed an asymptotic approach to locate the periodic response of a beam with cubic nonlinearity. In addition, the asymptotic development method presented in the Poincaré-Lindstedt version[22] was introduced to obtain the accurate analytical approximations of the natural frequencies. Lenci et al.[23] employed the asymptotic development method and obtained approximate analytical expressions for the natural frequencies of non-uniform cables and beams. Tarnopolskaya et al.[24] presented the first comprehensive study of the mode transition phenomenon in the vibration of beams with arbitrarily varying curvature and cross section based on an asymptotic analysis.

For the AFG beams, many researchers have studied the free vibration by using numerical and analytical methods. For instance, Hein and Feklistova[13] used the Haar wavelet series to resolve the vibration problems of the AFG beams with various boundary conditions and varying cross sections. Kukla and Rychlewska[25] proposed a new approach to study the free vibration analysis of an AFG beam. The approach relies on replacing functions characterizing functionally graded beams with piecewise exponential functions. Huang and Li[26] transformed the governing equation of AFG beams to the Fredholm integral equation and solved the natural frequencies by requiring that the resulting Fredholm integral equation contains a non-trivial solution. Huang et al.[27] and Huang and Rong[28] proposed an exact analytical method to investigate the vibration behaviors of AFG beams with arbitrary axial gradients and non-uniform cross sections. Xie et al.[29] presented a spectral collocation approach based on integrated polynomials combined with the domain decomposition technique for the free vibration analyses of beams with axially variable cross sections, moduli of elasticity, and mass densities. Akgöz and Civalek[30] examined the free vibrations of AFG tapered Euler-Bernoulli micro-beams based on the Bernoulli-Euler beam and modified the couple stress theory. Zhao et al.[15] introduced a new approach based on the Chebyshev polynomial theory to investigate the free vibration of AFG beams with non-uniform cross sections. Fang and Zhou[31] used the Chebyshev-Ritz method to study the free vibration of the rotating tapered Timoshenko beams composed of axially FGMs. Shahba et al.[32–33] and Shahba and Rajasekaran[34] studied the free vibration of AFG beams using the finite element method and differential transform element method, respectively.

In the present work, the free vibration of non-uniform AFG beams is studied using the asymptotic development method. First, the governing differential equation for the free vibration of a non-uniform AFG beam is summarized and rewritten in a dimensionless equation based on the Euler-Bernoulli beam theory. By decomposing the variable coefficients into reference invariant and variant parts, the asymptotic development method is introduced to resolve the governing equation, and an approximate formula of the natural frequencies of the non-uniform AFG beams is obtained. Furthermore, assuming polynomial distributions of Young's modulus and mass density, the natural frequencies of the AFG beams with various taper ratios are analyzed considering different boundary configurations. The results are compared with the published literature results to validate the effectiveness of the asymptotic development method. Finally, the conclusions are presented.

2 Problem formulation

Herein, four types of AFG beams with various taper ratios are considered, shown in Fig. 1, where B_{L} and B_{R} are the widths of the left and right ends of the beams, respectively, H_{L} and H_{R} are the heights of the left and right ends of the beams, respectively, and L is the length of the beams. Here, it is assumed that the geometrical properties of the AFG beam vary linearly along the longitudinal direction. Thus, the cross-sectional area $A(x)$ and the moment of inertia $I(x)$ varying along the beam axis can be expressed as

$$A(x) = A_{\mathrm{L}}\left(1 - c_{\mathrm{b}}\frac{x}{L}\right)\left(1 - c_{\mathrm{h}}\frac{x}{L}\right), \quad I(x) = I_{\mathrm{L}}\left(1 - c_{\mathrm{b}}\frac{x}{L}\right)\left(1 - c_{\mathrm{h}}\frac{x}{L}\right)^3, \tag{1}$$

where $c_{\mathrm{b}} = 1 - \frac{B_{\mathrm{R}}}{B_{\mathrm{L}}}$ and $c_{\mathrm{h}} = 1 - \frac{H_{\mathrm{R}}}{H_{\mathrm{L}}}$ are the breadth and height taper ratios, respectively. A_{L} and I_{L} are the cross-sectional area and area moment of inertia at $x = 0$, respectively. It is noteworthy that if $c_{\mathrm{b}} = c_{\mathrm{h}} = 0$, the beam would be uniform; if $c_{\mathrm{h}} = 0, c_{\mathrm{b}} \neq 0$, the beam would be tapered with a constant height; if $c_{\mathrm{b}} = 0, c_{\mathrm{h}} \neq 0$, the beam would be tapered with a constant width; if $c_{\mathrm{b}} \neq 0, c_{\mathrm{h}} \neq 0$, the beam would be double tapered. These four cases correspond to Figs. 1(a)–1(d), respectively. Moreover, the material properties such as Young's modulus $E(x)$ and the mass density $\rho(x)$ along the beam axis are assumed as

$$E(x) = E_{\mathrm{L}}\Big(1 + \frac{x}{L}\Big), \quad \rho(x) = \rho_{\mathrm{L}}\Big(1 + \frac{x}{L} + \Big(\frac{x}{L}\Big)^2\Big), \tag{2}$$

where E_{L} and ρ_{L} are Young's modulus and the mass density at $x = 0$, respectively.

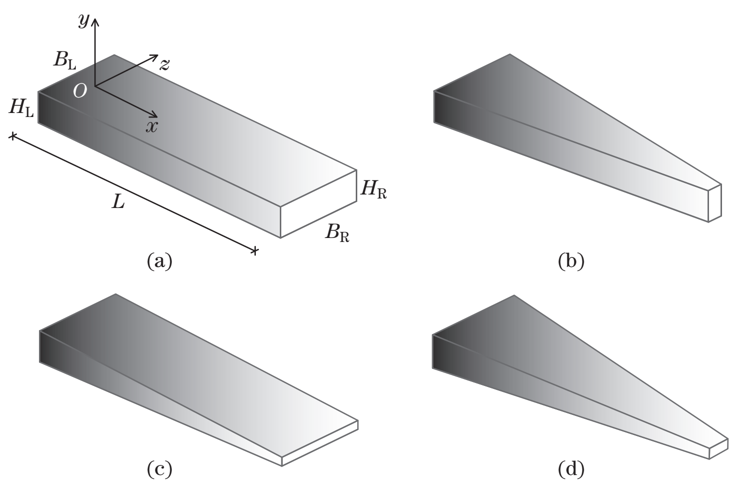

Fig. 1 Geometry and coordinate system of an AFG beam for different taper ratios, (a) Case 1: $c_{\mathrm{b}} = c_{\mathrm{h}} = 0$, (b) Case 2: $c_{\mathrm{h}} = 0$, $c_{\mathrm{b}} \neq 0$, (c) Case 3: $c_{\mathrm{b}} = 0$, $c_{\mathrm{h}} \neq 0$, and (d) Case 4: $c_{\mathrm{b}} \neq 0$, $c_{\mathrm{h}} \neq 0$

Based on the Euler-Bernoulli beam theory, the governing differential equation can be written as[35]

$$\frac{\partial^2}{\partial x^2}\Big(E(x)I(x)\frac{\partial^2 w(x,t)}{\partial x^2}\Big) + \rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad 0 \leqslant x \leqslant L, \tag{3}$$

where x is the axial coordinate, $w(x,t)$ is the transverse deflection at position x and time t, $E(x)I(x)$ is the flexural stiffness, which depends on both Young's modulus $E(x)$ and the area moment of inertia $I(x)$, and $\rho(x)A(x)$ is the mass of the beam per unit length, which depends on both the material mass density $\rho(x)$ and the cross-sectional area $A(x)$.

For the non-uniform AFG beams, the flexural stiffness $E(x)I(x)$ and the mass $\rho(x)A(x)$ both vary with the axial coordinate, which makes it difficult to resolve the differential equation with variable coefficients. To overcome this difficulty, we introduce a reference flexural stiffness EI_0 and a reference mass per unit length ρA_0, which will be derived and presented in Section 3. Let $E(x)I(x) = EI_0 + \overline{E(x)I(x)}$ and $\rho(x)A(x) = \rho A_0 + \overline{\rho(x)A(x)}$, where EI_0 and ρA_0 are the invariant parts and $\overline{E(x)I(x)}$ and $\overline{\rho(x)A(x)}$ are the variant parts of the bending stiffness and mass per unit length, respectively. Moreover, a non-dimensional equation is convenient for computational purposes. By introducing the non-dimensional space variable, defined by $\xi = x/L$, and the non-dimensional time, defined by $\tau = \frac{t}{L^2}\sqrt{\frac{EI_0}{\rho A_0}}$, Eq. (3) can be rewritten in the non-dimensional form as follows:

$$\frac{\partial^2}{\partial \xi^2}\Big((1+f_1(\xi))\,\frac{\partial^2 w(\xi,\tau)}{\partial \xi^2}\Big) + (1+f_2(\xi))\,\frac{\partial^2 w(\xi,\tau)}{\partial \tau^2} = 0, \quad 0 \leqslant \xi \leqslant 1, \tag{4}$$

where

$$f_1(\xi) = \frac{\overline{E(\xi)I(\xi)}}{EI_0}, \quad f_2(\xi) = \frac{\overline{\rho(\xi)A(\xi)}}{\rho A_0} \tag{5}$$

are the non-dimensional variant parts of the flexural stiffness and of the mass per unit length, respectively.

To investigate the free vibration of the beam, we assume

$$w(\xi,\tau) = W(\xi)\sin(\omega\tau), \tag{6}$$

where $W(\xi)$ is the amplitude of the vibration, and ω is the circular frequency of the vibration. We obtain the following equation by substituting Eq. (6) into Eq. (4):

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big((1+f_1(\xi))\,\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2}\Big) - \omega^2\,(1+f_2(\xi))\,W = 0, \quad 0 \leqslant \xi \leqslant 1. \tag{7}$$

3 Determination of the natural frequency

In this section, the asymptotic development method is applied to obtain a perturbative solution around the reference case of a uniform and homogeneous beam. First, we introduce a small perturbation parameter ε[36]. Then, Eq. (5) is changed to

$$f_1(\xi) \to \varepsilon f_1(\xi), \quad f_2(\xi) \to \varepsilon f_2(\xi). \tag{8}$$

According to the Poincaré-Lindstedt method[22–23], we assume the expansion for ω and $W(\xi)$ as

$$\begin{cases} \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \\ W(\xi) = W_0(\xi) + \varepsilon W_1(\xi) + \varepsilon^2 W_2(\xi) + \cdots. \end{cases} \tag{9}$$

Substituting these expressions into the governing equation (7) and subsequently expanding the expressions into ε-series, the following equations are obtained by equating the coefficients of ε^0 and ε^1 to zero, yielding a sequence of problems for the unknowns ω_i and $W_i(\xi)$,

$$\frac{\mathrm{d}^4 W_0}{\mathrm{d}\xi^4} - \omega_0^2 W_0 = 0, \tag{10}$$

$$\frac{\mathrm{d}^4 W_1}{\mathrm{d}\xi^4} - \omega_0^2 W_1 + h_1(\xi) - 2\omega_1\omega_0 W_0 = 0, \tag{11}$$

where

$$h_1(\xi) = 2\frac{\mathrm{d}f_1(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}^3W_0}{\mathrm{d}\xi^3} + \frac{\mathrm{d}^2f_1(\xi)}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2} + \omega_0^2\left(f_1(\xi) - f_2(\xi)\right)W_0. \tag{12}$$

3.1 Zeroth-order solution

For Eq. (10), the following general solution can be obtained[35]:

$$W_0 = A\sin(k\xi) + B\cos(k\xi) + C\sinh(k\xi) + D\cosh(k\xi), \tag{13}$$

where

$$k = \sqrt{\omega_0}. \tag{14}$$

For simplicity, we consider clamped-free (C-F) beams. The corresponding boundary conditions are

$$W_0 = \frac{\mathrm{d}W_0}{\mathrm{d}\xi} = 0, \quad \xi = 0, \tag{15}$$

$$\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2} = \frac{\mathrm{d}^3W_0}{\mathrm{d}\xi^3} = 0, \quad \xi = 1. \tag{16}$$

Under these conditions, we obtain the following equations:

$$\begin{cases} A + C = 0, \\ B + D = 0, \\ \dfrac{C}{D} = \dfrac{\sin k - \sinh k}{\cos k + \cosh k}. \end{cases} \tag{17}$$

The frequency equation can also be expressed as

$$\cos k\cosh k + 1 = 0. \tag{18}$$

The spatial mode shape can be expressed as

$$W_0 = \cosh(k\xi) - \cos(k\xi) + \frac{C}{D}\left(\sinh(k\xi) - \sin(k\xi)\right). \tag{19}$$

3.2 First-order solution

In Eq. (11), both $h_1(\xi)$ and W_1 are linearly correlated with W_0. The solution to Eq. (11) exists if and only if the condition of solvability[37]

$$\int_0^1 (h_1(\xi) - 2\omega_1\omega_0W_0)W_0\mathrm{d}\xi = 0 \tag{20}$$

is satisfied. As a result,

$$\omega_1 = \frac{\int_0^1 h_1(\xi)W_0\mathrm{d}\xi}{2\omega_0\int_0^1 W_0^2\mathrm{d}\xi}. \tag{21}$$

Because $h_1(\xi)$ is linearly correlated with W_0, the former equations indicate that the arbitrary amplitude of W_0 does not impact ω_1. This finding yields the first-order correction of the natural frequency ω_0 corresponding to a uniform and homogeneous beam.

Integrating by parts, we obtain

$$\int_0^1 h_1(\xi)W_0\mathrm{d}\xi = \Big(\frac{\mathrm{d}f_1}{\mathrm{d}\xi}\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}W_0 + f_1\frac{\mathrm{d}^3W_0}{\mathrm{d}\xi^3}W_0 - f_1\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\frac{\mathrm{d}W_0}{\mathrm{d}\xi}\Big)\Big|_0^1 + \int_0^1 \Big(f_1\Big(\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\Big)^2 - \omega_0^2 f_2 W_0^2\Big)\mathrm{d}\xi. \tag{22}$$

By choosing

$$EI_0 = \Big(\Big(\frac{\mathrm{d}\,(E(\xi)I(\xi))}{\mathrm{d}\xi}\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}W_0 + E(\xi)I(\xi)\frac{\mathrm{d}^3W_0}{\mathrm{d}\xi^3}W_0 - E(\xi)I(\xi)\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\frac{\mathrm{d}W_0}{\mathrm{d}\xi}\Big)\Big|_0^1 + \int_0^1 E(\xi)I(\xi)\Big(\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\Big)^2\mathrm{d}\xi\Big)\Big(\Big(\frac{\mathrm{d}^3W_0}{\mathrm{d}\xi^3}W_0 - \frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\frac{\mathrm{d}W_0}{\mathrm{d}\xi}\Big)\Big|_0^1 + \int_0^1\Big(\frac{\mathrm{d}^2W_0}{\mathrm{d}\xi^2}\Big)^2\mathrm{d}\xi\Big)^{-1}, \tag{23}$$

$$\rho A_0 = \frac{\int_0^1 \rho(\xi)A(\xi)W_0^2\mathrm{d}\xi}{\int_0^1 W_0^2\mathrm{d}\xi}, \tag{24}$$

we obtain $\omega_1 = 0$. These values are the properties of the equivalent uniform and homogeneous beam having the same frequency (at least up to the first order) as the given non-homogeneous beam with non-uniform cross section.

Finally, the nth natural circular frequency of the AFG beam can be derived as

$$\lambda_n = \frac{1}{L^2}\sqrt{\frac{EI_0}{\rho A_0}}\omega_0, \tag{25}$$

where ω_0 can be determined from the multi-roots (positive roots from small to large) of the frequency equation (18). All the indispensable expressions have been computed above. Equation (25) is an approximate formula of the natural frequencies of the non-uniform AFG beam.

For the other classical support situations, the formulae can be easily obtained by transforming Eqs. (15) and (16) to the corresponding boundary conditions. These formulae can be treated similarly as described above. Because of space limitations, we omit the details of the derivation herein. The final results are listed in Table 1.

Table 1 Frequency equations and mode shapes for various beams

Boundary condition	Frequency equation	Mode shape
Simply supported (S-S)	$\sin k = 0$	$W_0 = \sin(k\xi)$
Clamped-clamped (C-C)	$\cos k \cosh k - 1 = 0$	$W_0 = \cosh(k\xi) - \cos(k\xi) + \frac{\sin k + \sinh k}{\cos k - \cosh k} \cdot (\sinh(k\xi) - \sin(k\xi))$

4 Results and discussion

In this section, the numerical solutions are obtained and discussed. Based on the analytical equation above, the first three order non-dimensional natural frequencies ($\Omega_n = \lambda_n L^2 \sqrt{\rho_{\mathrm{L}} A_{\mathrm{L}}/(E_{\mathrm{L}} I_{\mathrm{L}})}$) of the four cases of non-uniform AFG beams with different boundary configurations are obtained. The results are listed in Tables 2–7, respectively, and are also compared with those by Shahba et al.[32].

Table 2 shows the first three order natural frequencies of the AFG beam, as shown in Fig. 1(a), which is uniform but non-homogeneous. We can firmly conclude that the analytical results obtained from the asymptotic development method are in good agreement with those in Ref. [32].

Table 2 Non-dimensional natural frequencies of the AFG uniform beam (Case 1) with different boundary conditions

Boundary condition	Method	First mode	Second mode	Third mode
C-F	Present	2.439	18.437	54.339
	Ref. [32]	2.426	18.604	55.180
S-S	Present	9.053	35.834	80.470
	Ref. [32]	9.029	36.372	81.732
C-C	Present	20.585	56.251	109.869
	Ref. [32]	20.472	56.549	110.947

Tables 3 and 4 show the values of first three order dimensionless natural frequencies for an AFG tapered beam for the cases of varying only the width and only the height, respectively. One can conclude that the present method is accurate and effective for the AFG tapered beam of constant height, while some fractional errors occur for the AFG tapered beam with variable height, and the larger the height taper ratio c_{h}, the larger is the error.

Table 3 Non-dimensional natural frequencies of the AFG tapered beam with a constant height (Case 2) and different boundary conditions

c_{b}	Method	C-F			S-S			C-C		
		First mode	Second mode	Third mode	First mode	Second mode	Third mode	First mode	Second mode	Third mode
0.2	Present	2.613	18.887	54.951	9.068	35.957	80.772	20.457	56.196	110.003
	Ref. [32]	2.605	19.004	55.534	9.060	36.342	81.685	20.415	56.472	110.862
0.4	Present	2.854	19.483	55.753	9.088	36.117	81.165	20.294	56.124	110.177
	Ref. [32]	2.851	19.530	56.023	9.087	36.315	81.645	20.288	56.298	110.671
0.6	Present	3.214	20.311	56.853	9.113	36.332	81.697	20.079	56.028	110.411
	Ref. [32]	3.214	20.296	56.800	9.099	36.297	81.624	20.019	55.921	110.250
0.8	Present	3.832	21.542	58.453	9.147	36.638	82.456	19.783	55.892	110.743
	Ref. [32]	3.831	21.676	58.435	9.069	36.277	81.639	19.385	54.971	109.142

Table 4 Non-dimensional natural frequencies of the AFG tapered beam with a constant width (Case 3) and different boundary conditions

c_{b}	Method	C-F			S-S			C-C		
		First mode	Second mode	Third mode	First mode	Second mode	Third mode	First mode	Second mode	Third mode
0.2	Present	2.505 4	17.259 6	49.498 2	8.141 6	32.188 8	72.268 0	18.242 0	50.185 1	98.299 2
	Ref. [32]	2.505 1	17.380 2	50.049 1	8.134 1	32.523 6	73.113 8	18.217 0	50.480 1	99.173 4
0.4	Present	2.629 3	16.299 5	45.351 9	7.279 3	28.971 7	65.120 3	16.302 7	45.060 0	88.434 5
	Ref. [32]	2.615 5	16.070 5	44.618 1	7.153 1	28.474 7	63.994 2	15.828 2	44.024 6	86.627 2
0.6	Present	2.853 5	15.669 7	42.235 8	6.487 2	26.369 4	59.485 0	14.915 2	41.250 2	80.974 7
	Ref. [32]	2.783 5	14.650 8	38.744 6	6.035 7	24.110 1	54.092 1	13.229 3	36.965 3	72.874 0
0.8	Present	3.288 9	15.566 2	40.655 4	5.796 6	24.637 1	55.973 4	14.223 3	39.182 3	76.769 0
	Ref. [32]	3.087 1	13.114 2	32.130 9	4.652 0	19.131 4	42.695 4	10.223 5	28.749 2	56.810 9

For Case 4 (see Fig. 1(d)), the AFG beam is non-uniform with the double tapered variation at width and longitude. The natural frequencies of three types of boundary conditions, including clamped-free, simply supported, and clamped-clamped, are tabulated in Tables 5–7, respectively. It can be seen that the lower order frequencies agree with those in Ref. [32], but some errors exist when the height taper ratio c_h increases, especially for the higher order modes.

Table 5 Non-dimensional natural frequencies of the AFG double tapered beam (Case 4) with the C-F boundary condition

Mode	c_h	Method	c_b			
			0.2	0.4	0.6	0.8
1	0.2	Present	2.687 3	2.938 0	3.311 3	3.945 5
		Ref. [32]	2.686 3	2.933 6	3.299 3	3.921 9
	0.4	Present	2.822 6	3.087 7	3.479 6	4.137 7
		Ref. [32]	2.798 7	3.048 6	3.418 1	4.047 1
	0.6	Present	3.064 0	3.350 6	3.770 0	4.462 5
		Ref. [32]	2.969 9	3.223 7	3.598 5	4.235 5
	0.8	Present	3.527 1	3.847 5	4.308 1	5.045 8
		Ref. [32]	3.279 4	3.540 1	3.923 2	4.569 5
2	0.2	Present	17.722 5	18.328 9	19.159 8	20.372 5
		Ref. [32]	17.750 1	18.237 9	18.950 1	20.243 2
	0.4	Present	16.782 2	17.406 1	18.245 8	19.441 8
		Ref. [32]	16.409 2	16.857 1	17.513 9	18.716 4
	0.6	Present	16.177 1	16.821 4	17.668 7	18.838 0
		Ref. [32]	14.956 7	15.362 7	15.961 6	17.069 4
	0.8	Present	16.094 7	16.749 3	17.583 6	18.687 7
		Ref. [32]	13.385 0	13.746 6	14.284 8	15.295 5
3	0.2	Present	50.219 4	51.153 4	52.411 4	54.199 5
		Ref. [32]	50.393 4	50.864 5	51.602 9	53.133 2
	0.4	Present	46.197 0	47.273 4	48.692 5	50.652 0
		Ref. [32]	44.950 4	45.400 3	46.095 7	47.512 9
	0.6	Present	43.204 2	44.411 7	45.961 3	48.026 9
		Ref. [32]	39.060 5	39.484 4	40.130 4	41.423 6
	0.8	Present	41.706 5	42.981 7	44.563 6	46.582 8
		Ref. [32]	32.422 9	32.812 3	33.398 6	34.552 1

Table 6 Non-dimensional natural frequencies of the AFG double tapered beam (Case 4) with the S-S boundary condition

Mode	c_{h}	Method	c_{b}			
			0.2	0.4	0.6	0.8
1	0.2	Present	8.168 2	8.201 8	8.245 6	8.305 1
		Ref. [32]	8.146 2	8.149 8	8.133 6	8.064 6
	0.4	Present	7.317 2	7.364 7	7.426 2	7.508 9
		Ref. [32]	7.145 5	7.125 4	7.079 4	6.970 3
	0.6	Present	6.535 7	6.596 0	6.673 2	6.775 4
		Ref. [32]	6.008 2	5.963 8	5.886 8	5.735 1
	0.8	Present	5.853 7	5.924 0	6.012 8	6.128 3
		Ref. [32]	4.604 6	4.535 5	4.426 4	4.228 3
2	0.2	Present	32.413 3	32.700 7	33.081 9	33.611 8
		Ref. [32]	32.512 3	32.507 9	32.516 4	32.532 6
	0.4	Present	29.297 1	29.707 6	30.241 9	30.966 5
		Ref. [32]	28.482 2	28.500 3	28.537 0	28.592 8
	0.6	Present	26.783 4	27.296 5	27.949 3	28.809 1
		Ref. [32]	24.137 1	24.179 1	24.246 9	24.349 7
	0.8	Present	25.103 2	25.668 3	26.368 3	27.259 0
		Ref. [32]	19.180 3	19.250 9	19.359 0	19.530 0
3	0.2	Present	72.817 9	73.523 7	74.462 5	75.773 2
		Ref. [32]	73.095 9	73.090 3	73.111 6	73.185 5
	0.4	Present	65.915 8	66.920 2	68.229 1	70.006 9
		Ref. [32]	64.005 4	64.035 0	64.100 7	64.237 4
	0.6	Present	60.492 2	61.739 2	63.324 3	65.408 9
		Ref. [32]	54.133 0	54.199 2	54.312 6	54.520 7
	0.8	Present	57.096 9	58.454 7	60.130 3	62.253 0
		Ref. [32]	42.767 7	42.874 2	43.043 6	43.345 1

5 Conclusions

In this paper, the free vibration of non-uniform AFG beams with various classical boundary conditions is investigated by applying the asymptotic development method. Most significantly, the perturbation theory is used to analyze the differential equations with variable coefficients. First, the governing equation of the free vibrations of the non-uniform AFG beam is presented as a dimensionless equation. As a primary step, the variable coefficients, the flexural stiffness, and the mass of the beam per unit length of the differential equation are decomposed into invariant and variant parts. Using the asymptotic development method, an approximate analytical formula of the natural frequency is derived, which is expressed as a novel equivalent homogeneous and uniform beam model. Assuming polynomial distributions of Young's modulus and mass density, the first three order natural frequencies of the AFG beam with various taper ratios and three different boundary conditions are analyzed and discussed. The comparison between the present method and the published literature indicates good agreement for the first three order natural frequencies of the non-uniform AFG beams except high order frequencies for the cases

Table 7 Non-dimensional natural frequencies of the AFG double tapered beam (Case 4) with the C-C boundary condition

Mode	c_h		c_b			
			0.2	0.4	0.6	0.8
1	0.2	Present	18.277 9	18.323 1	18.381 8	18.461 2
		Ref. [32]	18.199 6	18.128 6	17.943 7	17.456 6
	0.4	Present	16.497 5	16.739 6	17.048 4	17.456 3
		Ref. [32]	15.849 8	15.835 0	15.736 7	15.402 5
	0.6	Present	15.251 2	15.662 2	16.177 1	16.842 3
		Ref. [32]	13.289 6	13.331 9	13.323 8	13.152 9
	0.8	Present	14.666 2	15.200 4	15.858 7	16.692 5
		Ref. [32]	10.322 9	10.425 5	10.516 8	10.533 9
2	0.2	Present	50.443 0	50.771 3	51.203 5	51.798 1
		Ref. [32]	50.456 5	50.359 9	50.101 7	49.372 8
	0.4	Present	45.625 7	46.334 6	47.249 5	48.476 3
		Ref. [32]	44.055 3	44.037 0	43.902 7	43.406 6
	0.6	Present	42.089 0	43.121 4	44.424 5	46.123 4
		Ref. [32]	37.050 9	37.113 7	37.110 4	36.867 8
	0.8	Present	40.215 1	41.461 4	42.997 5	44.942 0
		Ref. [32]	28.891 2	29.040 9	29.184 2	29.240 2
3	0.2	Present	98.299 2	99.746 6	100.821 9	102.313 0
		Ref. [32]	99.147 4	99.041 4	98.754 3	97.904 6
	0.4	Present	88.434 5	90.980 6	92.820 0	95.302 3
		Ref. [32]	86.660 8	86.641 4	86.493 2	85.917 6
	0.6	Present	80.974 7	84.459 8	86.896 7	90.085 5
		Ref. [32]	72.968 1	73.038 2	73.037 5	72.761 5
	0.8	Present	76.769 0	80.842 6	83.586 7	87.057 6
		Ref. [32]	56.967 4	57.134 1	57.299 1	57.378 7

of variable height. Fractional errors exist and increase with the increase in the mode orders especially for the cases with variable heights. To conclude, the asymptotic development method is verified to be simple and efficient to analytically study the free vibration of non-uniform AFG beams, and it can be used to analyze any tapered beams with arbitrary varying cross width.

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