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Mathematical model of micropolar fluid in two-phase immiscible fluid flow through porous channel*

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Abstract This paper is concerned with the flow of two immiscible fluids through a porous horizontal channel. The fluid in the upper region is the micropolar fluid/the Eringen fluid, and the fluid in the lower region is the Newtonian viscous fluid. The flow is driven by a constant pressure gradient. The presence of micropolar fluids introduces additional rotational parameters. Also, the porous material considered in both regions has two different permeabilities. A direct method is used to obtain the analytical solution of the concerned problem. In the present problem, the effects of the couple stress, the micropolarity parameter, the viscosity ratio, and the permeability on the velocity profile and the microrotational velocity are discussed. It is found that all the physical parameters play an important role in controlling the translational velocity profile and the microrotational velocity. In addition, numerical values of the different flow parameters are computed. The effects of the different flow parameters on the flow rate and the wall shear stress are also discussed graphically.

Key words micropolar fluid, immiscible fluid, porous medium, couple stress, micropolarity parameter

Chinese Library Classification O357.3
2010 Mathematics Subject Classification 76E05, 76S05, 76T99

Nomenclature

K_i,	non-dimensional permeability of the porous medium;
p_i,	non-dimensional pressure;
$\overline{x}, \overline{y}$,	coordinates along the channel;
$\overline{u}_i, \overline{v}_i$,	velocities in the x-direction;
Re_i,	Reynolds number;
c,	micropolarity parameter;
$c_0, c_{\mathrm{a}}, c_{\mathrm{d}}$,	coefficients of angular viscosities;
s,	couple stress;
ρ_i,	density of the fluid;
$\overline{\mu_i}$,	viscosity of the fluid;
$\overline{\omega_i}$,	microrotational velocity of the fluid;
$\overline{\mu}_{\mathrm{r}}$,	dynamic microrotation viscosity of the micropolar fluid;
λ,	viscosity ratio.

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1 Introduction

The study of fluid mechanics has shown vast applications of fluid flow problems in various fields such as industries, biological processes, engineering fields. The widespread applications of fluid flow problems are motivating the researchers to work on the fluid flow problem with different aspects. Most of industrial fluids or fluids with applications in industries, biological processes etc. are non-Newtonian. The study of immiscible fluids passing through channels (horizontal or vertical) is emerging as a challenging work for the researchers. Likely, flows of immiscible fluids through a porous channel have many applications such as the petroleum industry, the extraction of crude oil through soils, flows through pipes and tubes in environmental and ecological applications, and blood flows through arteries. The micropolar fluid is a class of fluids which are described by intrinsic motion of its particles. Eringen[1–3] introduced the fluids with the substructure. The micropolar fluid is a subclass of polar fluids which possess the number of governing equations. Classical fluids possess only three degrees of translational motion, but Eringen fluids possess six degrees of freedom, three translational degrees and three rotational degrees. Due to independent rotational and translational motions of fluid elements, the stress and moment stress tensors are no more symmetric. It was found experimentally by Hoyt and Fabula[4] that the presence of such additives in these fluids results in reduced skin friction. Ariman et al.[5] reviewed the mechanics of microcontinuum hypothesis. Lukaszewicz[6] described the theory and applications of micropolar fluids thoroughly. Examples of micropolar fluids can be found in the above mentioned references, and some of them are the human blood, the animal blood, liquid crystals, polymer suspensions, colloidal particles and so on. It can be seen in the work of Philip and Chandra[7] and Ellahi et al.[8] that the problem of blood flows through the stenosed arteries is considered to be a problem of micropolar fluid flow. They solved the problem for medical purposes. Chen et al.[9] gave an explanation on the theory of micropolar fluids and did numerical simulations of the involved governing equations. Ariman and Cakmak[10] discussed Couette, Poiseuille, and rotational flows of micropolar fluids. They found the deviation of these basic flows for micropolar fluids compared with that of the flow for classical Newtonian fluids. The combination of non-Newtonian fluids and porous media is emerging as a challenge for the practical purposes. Sochi[11] discussed the theory of non-Newtonian fluids through porous media with various applications. Brinkman[12] gave an extended law of Darcy's equation which explains the flow through porous media with high permeability.

Nowadays, the two-phase flow in a porous channel is presenting a challenge and motivating the researchers and engineers. This model is useful in extraction of oil from a porous pipe with the help of some water, human joints liquids, chemical reactors, gas liquid filtration, dust collector, sewage treatment, heat pipes, etc.[13]. Berman[14] was the first to study the steady and laminar fluid flow through the channels with porous walls. Kapur and Shukla[15] did a remarkable work on the flow of number of immiscible fluids through parallel plates. Shail[16] gave a practical description of his study by considering steady and incompressible flows of immiscible fluids in presence of a magnetic field. He numerically found that, if the ratios of viscosities, magnetic numbers, and depths are chosen suitably, then there will be an increase in the flow rate by 30%. Vafai and Thiyagaraja[17] analyzed the flow and heat transfer in three interfacial regions of the porous media. They obtained both theoretical and numerical results for velocity profiles and temperature profiles. Srinivasan and Vafai[18] extended the work of Muskat and Wyckoff[19] by considering boundary and inertial effects with Darcy's law. Chamkha[20] studied MHD flows of the two-fluid flow problem through porous and non-porous vertical channels. Along with an analytical solution, he used an implicit finite-difference method to validate the results obtained with the analytical method. Awartani and Hamdan[21] proposed the problem of the two-phase model with fluid and solid matrices of porous media. They discussed Poiseuille, Couette, and Poiseuille-Couette flows and found that there is a decrease in the flow velocity compared with the flows in absence of the porous structure. Umavathi et al.[22] studied the unsteady and laminar flow of two immiscible fluids in a horizontal channel. They considered only one fluid to be electrically conducting and also discussed heat transfer in the channel. Umavathi et al.[23] solved the problem of unsteady flows of fluids in the porous region sandwiched between layers of viscous fluids.

In all the above discussed works, the two immiscible fluids taken in their model were Newtonian fluids with different viscosities. Some motivational works were also done for Eringen fluids through the channels[24–26]. Joneidi et al.[27] explained the work on the flow of micropolar fluids through the porous channel. They solved equations of flows with high mass transfer by using an optimal homotopy asymptotic method. An analytical study on the Poiseuille flow for two immiscible micropolar fluids flows between two horizontal and parallel plates is done by Murthy and Srinivas[28]. They concluded that micropolar fluids can be used as good lubricants with an increase in the micropolarity parameter. Murthy et al.[29] considered different models compared with the above discussed models. They considered immiscible micropolar fluids to flow between the zones bounded by porous channels, and the viscous fluids were allowed to pass through porous channels of large and low permeability, respectively. The flow of two immiscible Newtonian fluids through the channel filled with the porous medium is discussed by Ansari and Deo[30]. The problem of flow of the reactive fluid through the channel filled with a porous material was solved by Adesanya et al.[31] using the Adomian decomposition method. Srinivas and Murthy[32] discussed the flow of two immiscible micropolar fluids flowing between two porous beds. They observed the effect of entropy generation on the flow problem by using the first and the second law of thermodynamics. Kumar et al.[33] considered the flow of micropolar fluids and Newtonian fluids through the vertical channel. They assumed the flow to be free-convective. They concluded that the presence of micropolar fluid parameters has remarkable effects on the velocity profile.

The above works motivate us to discuss the present problem. The model discussed in this paper originates from the very well-known application of the fluid flow through the porous channel, which is the filtration of the ground water passing through various porous materials such as sands, soils, and rocks. The oil recovery process in the petroleum industry involves flows of more than one phase through the oil reservoir, due to which the power required to pump oil from the reservoir gets reduced. Most of the industrial fluids or biological fluids cannot be explained by using the classical Newtonian law of hydrodynamics. These types of fluids include the micropolar fluid, which deals with the intrinsic motion of its particles. Many authors have studied blood flows by assuming the blood as a micropolar fluid. Therefore, the blood flow through the porous tissues can be treated as the flow through porous horizontal channels.

In the present work, we consider a model of two-fluid flows through the horizontal porous channel. The steady, incompressible, and fully-developed flow of micropolar and Newtonian fluids takes place through the porous channel. The flow is driven by the constant pressure gradient, and it is in absence of any body and couple forces. The solutions obtained are used to interpret the effects of the couple stress, the micropolarity parameter, the viscosity ratio, and the permeability on the velocity profile, the microrotational velocity, the flow rate, and the wall shear stress.

2 Mathematical formulation

Here, we consider the flow of two immiscible fluids through a horizontal porous channel with an impermeable rigid boundary. It is assumed that the non-Newtonian micropolar fluid is flowing in the upper layer, and the Newtonian fluid is flowing in the lower region. Let $\boldsymbol{u} = \boldsymbol{u}(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ be the velocity of the Newtonian fluid and $\boldsymbol{v} = \boldsymbol{v}(\overline{v}_1, \overline{v}_2, \overline{v}_3)$ be the velocity of the non-Newtonian micropolar fluid in the horizontal porous channel.

The governing equations for both regions are given as follows:

(i) For the Newtonian region,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u}, \tag{1}$$

$$\rho\Big(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big) = -\nabla p + \mu\nabla^2\boldsymbol{u} - \frac{\mu}{K}\boldsymbol{u} + \rho\boldsymbol{f}. \tag{2}$$

(ii) For the micropolar region,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{v}, \tag{3}$$

$$\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + (\lambda+\mu-\mu_{\mathrm{r}})\nabla\mathrm{div}\boldsymbol{v} + (\mu+\mu_{\mathrm{r}})\nabla^2\boldsymbol{v} + 2\mu_{\mathrm{r}}\nabla\times\omega - \frac{\mu}{K}\boldsymbol{v} + \rho\boldsymbol{f}, \tag{4}$$

$$\rho I\frac{\mathrm{D}\omega}{\mathrm{D}t} = 2\mu_{\mathrm{r}}(\nabla\times\boldsymbol{v} - 2\omega) + (c_0 + c_{\mathrm{d}} - c_{\mathrm{a}})\nabla\mathrm{div}\omega + (c_{\mathrm{a}} + c_{\mathrm{d}})\nabla^2\omega + \rho g, \tag{5}$$

where λ and μ are the usual viscosity coefficients such that $\mu \geqslant 0$ and $3\lambda+2\mu \geqslant 0$. μ_{r} is known as the dynamic microrotation viscosity, c_0, c_{a}, and c_{d} are the coefficients of angular viscosities, p is the pressure applied along the direction of the flow, ω is the microrotational velocity, ρ is the density of the fluid, $\boldsymbol{f}$ and g are the external forces, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are the velocities of Newtonian and micropolar fluids, respectively.

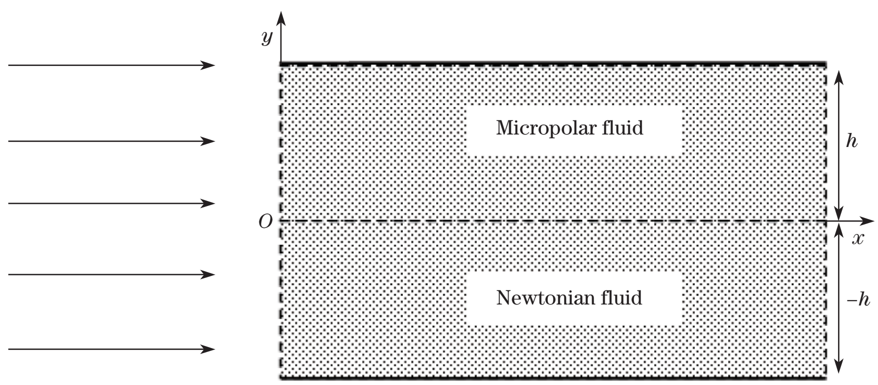

Fig. 1 Mathematical model of flow of immiscible fluids through porous channel

Let us consider that the immiscible fluids flows are steady, incompressible, and fully-developed, and the flow is driven by a constant pressure gradient. It is also assumed that there is no external forces and body couple forces. We also consider that the flow is in the x-direction only with microrotation along the z-direction. Let ρ_i, K_i, μ_i, and p_i $(i = 1, 2)$ be the density, permeability, viscosity, and pressure of Newtonian and non-Newtonian micropolar fluids, respectively.

Then, the governing equations (1)–(5) of motion will be reduced as follows:

(i) For the Newtonian region,

$$\frac{\partial\overline{u}_1}{\partial\overline{x}} = 0, \tag{6}$$

$$\overline{\mu}_1\frac{\partial^2\overline{u}_1}{\partial\overline{y}^2} - \frac{\overline{\mu}_1}{\overline{K}_1}\overline{u}_1 = \frac{\partial\overline{p}_1}{\partial\overline{x}}. \tag{7}$$

(ii) For the micropolar region,

$$\frac{\partial \overline{v}_1}{\partial \overline{x}} = 0, \tag{8}$$

$$(\overline{\mu}_2 + \overline{\mu}_{\rm r})\frac{\partial^2 \overline{v}_1}{\partial \overline{y}^2} + 2\overline{\mu}_{\rm r}\frac{\partial \overline{\omega}_3}{\partial \overline{y}} - \frac{\overline{\mu}_2}{\overline{K}_2}\overline{v}_1 = \frac{\partial \overline{p}_2}{\partial \overline{x}}, \tag{9}$$

$$2\overline{\mu}_{\rm r}\Big(\frac{\partial \overline{v}_1}{\partial \overline{y}} - 2\overline{\omega}_3\Big) + (\overline{c}_{\rm a} + \overline{c}_{\rm d})\frac{\partial^2 \overline{\omega}_3}{\partial \overline{y}^2} = 0, \tag{10}$$

where the symbol $-$ denotes the dimensional form of the flow quantity.

3 Solution of problem

To find the solution of the above problem, let us define the following non-dimensional variables:

$$x = \frac{\overline{x}}{h}, \quad y = \frac{\overline{y}}{h}, \quad u_1 = \frac{\overline{u}_1}{U_0}, \quad v_1 = \frac{\overline{v}_1}{U_0}, \quad \omega_3 = \frac{\overline{\omega}_3 h}{U_0}, \quad K_i = \frac{\overline{K}_i}{h^2},$$
$$Re_i = \frac{\rho_i U_0 h}{\mu_i}, \quad c = \frac{\overline{\mu}_{\rm r}}{\overline{\mu}_2}, \quad s = \frac{2\overline{\mu}_{\rm r} h^2}{\overline{c}_{\rm a} + \overline{c}_{\rm d}}, \quad p_i = \frac{\overline{p}_i}{\rho_i U_0^2}\ (i = 1, 2).$$

Therefore, the governing equations (6)–(10) will become as follows.

(i) For the Newtonian region,

$$\frac{\partial u_1}{\partial x} = 0, \tag{11}$$

$$\frac{\partial^2 u_1}{\partial y^2} - \frac{1}{K_1}u_1 = Re_1\frac{\partial p_1}{\partial x}. \tag{12}$$

(ii) For the micropolar region,

$$\frac{\partial v_1}{\partial x} = 0, \tag{13}$$

$$\frac{\partial^2 v_1}{\partial y^2} + \frac{2c}{1+c}\frac{\partial \omega_3}{\partial y} - \frac{1}{(1+c)K_2}v_1 = \frac{Re_2}{1+c}\frac{\partial p_2}{\partial x}, \tag{14}$$

$$\frac{\partial^2 \omega_3}{\partial y^2} - s\frac{\partial v_1}{\partial y} - 2s\omega_3 = 0, \tag{15}$$

where c is the micropolarity parameter.

By solving Eq. (12), we have

$$u_1 = C_1{\rm e}^{Ay} + C_2{\rm e}^{-Ay} - \frac{Re_1 P}{A^2}, \tag{16}$$

$$v_1 = C_3{\rm e}^{\alpha y} + C_4{\rm e}^{-\alpha y} + C_5{\rm e}^{\beta y} + C_6{\rm e}^{-\beta y} - Re_2 K_2 P, \tag{17}$$

$$\omega_3 = -\eta(C_3{\rm e}^{\alpha y} - C_4{\rm e}^{-\alpha y}) - \xi(C_5{\rm e}^{\beta y} - C_6{\rm e}^{-\beta y}), \tag{18}$$

where $A^2 = \frac{1}{K_1}$, P is the constant pressure, because of which flows take place in both regions of the porous horizontal channel, C_1, C_2, C_3, C_4, C_5, and C_6 are the arbitrary constants, and

$$\alpha = \sqrt{\frac{(2sK_2+1) + \sqrt{(2sK_2-1)^2 - 8sK_2c}}{2K_2(1+c)}}, \quad \eta = \frac{\alpha^3 K_2(1+c) + \alpha(2scK_2 - 1)}{4scK_2},$$
$$\beta = \sqrt{\frac{(2sK_2+1) - \sqrt{(2sK_2-1)^2 - 8sK_2c}}{2K_2(1+c)}}, \quad \xi = \frac{\beta^3 K_2(1+c) + \beta(2scK_2 - 1)}{4scK_2}.$$

In order to find the solution of the boundary value problem, it is necessary to specify the suitable boundary conditions. The suitable boundary conditions in the non-dimensional form, which are physically and mathematically consistent, can be taken as follows.

No-slip boundary conditions at the walls of channel, i.e.,

$$v_1 = 0 \quad \text{at} \quad y = 1, \tag{19}$$

$$u_1 = 0 \quad \text{at} \quad y = -1, \tag{20}$$

$$\omega_3 = 0 \quad \text{at} \quad y = 1. \tag{21}$$

The constant cell rotational velocity[5], i.e.,

$$\frac{\mathrm{d}\omega_3}{\mathrm{d}y} = 0 \quad \text{at} \quad y = 0. \tag{22}$$

The continuity of velocity and stress at the interface of two immiscible fluids, i.e.,

$$u_1 = v_1 \quad \text{at} \quad y = 0, \tag{23}$$

$$\tau_1 = \tau_2 \quad \text{or} \quad (1+c)\frac{\mathrm{d}v_1}{\mathrm{d}y} + 2c\omega_3 = \lambda\frac{\mathrm{d}u_1}{\mathrm{d}y} \quad \text{at} \quad y = 0. \tag{24}$$

With the values of u_1, v_1, and ω_3 in Eqs. (19)–(24), we have

$$C_3\mathrm{e}^{\alpha} + C_4\mathrm{e}^{-\alpha} + C_5\mathrm{e}^{\beta} + C_6\mathrm{e}^{-\beta} - Re_2K_2P = 0, \tag{25}$$

$$C_1\mathrm{e}^{-A} + C_2\mathrm{e}^{A} - \frac{Re_1P}{A^2} = 0, \tag{26}$$

$$-\eta(C_3\mathrm{e}^{\alpha} - C_4\mathrm{e}^{-\alpha}) - \xi(C_5\mathrm{e}^{\beta} - C_6\mathrm{e}^{-\beta}) = 0, \tag{27}$$

$$\alpha\eta(C_3 + C_4) - \beta\xi(C_5 + C_6) = 0, \tag{28}$$

$$C_1 + C_2 - C_3 - C_4 - C_5 - C_6 - \frac{Re_1P}{A^2} + Re_2K_2P = 0, \tag{29}$$

$$\begin{aligned} &-A\lambda C_1 + A\lambda C_2 + (\alpha(c+1) - \eta c)C_3 + (\eta c - \alpha(c+1))C_4 \\ &+ (\beta(c+1) - c\xi)C_5 + (c\xi - \beta(c+1))C_6 = 0. \end{aligned} \tag{30}$$

By solving the above system of equations with the help of MATHEMATICA 10.3, we can obtain the arbitrary constants C_1, C_2, C_3, C_4, C_5, and C_6, whose values are cumbersome, which may lose the interest of the readers. Therefore, these constants are not presented here. Substituting these values of arbitrary constants into Eqs. (16)–(18), we can obtain the values of the flow velocity of Newtonian and micropolar fluids in their respective regions.

Other important fluid flow quantities which are discussed in this paper are the wall shear stress and the volume flow rate.

The wall shear stress on the upper wall is given as

$$\tau_{\mathrm{w}1} = (1+c)\frac{\mathrm{d}v_1}{\mathrm{d}y} + 2c\omega_3. \tag{31}$$

By substituting the values of v_1 and ω_3 into Eq. (31), we get

$$\begin{aligned} \tau_{\mathrm{w}1} =& (c+1)(\alpha(\mathrm{e}^{\alpha y}C_3 - \mathrm{e}^{-\alpha y}C_4) + \beta(\mathrm{e}^{\beta y}C_5 - \mathrm{e}^{-\beta y}C_6)) \\ &+ 2c((-\eta)(\mathrm{e}^{\alpha y}C_3 - \mathrm{e}^{-\alpha y}C_4) - \xi(\mathrm{e}^{\beta y}C_5 - \mathrm{e}^{-\beta y}C_6)). \end{aligned} \tag{32}$$

Similarly, we can evaluate the wall shear stress on the lower wall,

$$\tau_{\mathrm{w}2} = A\lambda(\mathrm{e}^{Ay}C_1 - \mathrm{e}^{-Ay}C_2), \tag{33}$$

where $\lambda = \frac{\mu_1}{\mu_2}$.

By using the values of arbitrary constants, i.e., C_1, C_2, C_3, C_4, C_5, and C_6, respectively, we can obtain the wall shear stress on the wall of the horizontal porous channel,

$$\begin{aligned}\tau_{\mathrm{w}1}|_{y=1} =&(c+1)P(\alpha\xi-\beta\eta)(Ak_2Re_2((\mathrm{e}^{\alpha}-1)(\mathrm{e}^{2\beta}+1)\beta\xi((\mathrm{e}^{\alpha}+1)(\mathrm{e}^{2A}-1)c\eta\\
&-(\mathrm{e}^{\alpha}-1)A(\mathrm{e}^{2A}+1)\lambda)+\alpha((\mathrm{e}^{2\alpha}+1)(\mathrm{e}^{\beta}-1)\eta(A(\mathrm{e}^{2A}+1)(\mathrm{e}^{\beta}-1)\lambda-(\mathrm{e}^{2A}-1)\\
&\cdot(\mathrm{e}^{\beta}+1)c\xi)+(\mathrm{e}^{2A}-1)\beta(c+1)((\mathrm{e}^{2\alpha}+1)(\mathrm{e}^{2\beta}-1)\eta-(\mathrm{e}^{2\alpha}-1)(\mathrm{e}^{2\beta}+1)\xi)))\\
&-2(\mathrm{e}^{A}-1)^2\lambda Re_1(\alpha\eta(-\mathrm{e}^{2\alpha+\beta})-\alpha\mathrm{e}^{\beta}\eta+\mathrm{e}^{\alpha}\beta\xi+\beta\xi\mathrm{e}^{\alpha+2\beta}))/(A(4\alpha^2(\mathrm{e}^{2A}-1)\\
&\cdot(c+1)\eta\xi\mathrm{e}^{\alpha+\beta}+\alpha((\mathrm{e}^{2A}-1)\beta(-c-1)((\mathrm{e}^{2\alpha}+1)(\mathrm{e}^{2\beta}+1)\eta^2-2(\mathrm{e}^{2\alpha}-1)\\
&\cdot(\mathrm{e}^{2\beta}-1)\eta\xi+(\mathrm{e}^{2\alpha}+1)(\mathrm{e}^{2\beta}+1)\xi^2)+\eta((\mathrm{e}^{2A}-1)(-c)\xi((\mathrm{e}^{2\alpha}-1)(\mathrm{e}^{2\beta}-1)\xi\\
&-\eta(-4\mathrm{e}^{\alpha+\beta}+\mathrm{e}^{2(\alpha+\beta)}+\mathrm{e}^{2\alpha}+\mathrm{e}^{2\beta}+1))-A(\mathrm{e}^{2A}+1)\lambda((\mathrm{e}^{2\alpha}+1)(\mathrm{e}^{2\beta}-1)\eta\\
&-(\mathrm{e}^{2\alpha}-1)(\mathrm{e}^{2\beta}+1)\xi)))+\beta\xi(\mathrm{e}^{2(\alpha+\beta)}(\eta-\xi)(A\lambda+c\eta)-(\eta-\xi)\mathrm{e}^{2(\alpha+A+\beta)}\\
&\cdot(c\eta-A\lambda)+4\eta\mathrm{e}^{\alpha+2A+\beta}(\beta+\beta c-c\xi)-\mathrm{e}^{2\alpha}(\eta+\xi)(A\lambda+c\eta)+\mathrm{e}^{2(\alpha+A)}(\eta+\xi)\\
&\cdot(c\eta-A\lambda)+\mathrm{e}^{2(A+\beta)}(\eta+\xi)(A\lambda+c\eta)-\mathrm{e}^{2\beta}(\eta+\xi)(c\eta-A\lambda)-\mathrm{e}^{2A}(\eta-\xi)\\
&\cdot(A\lambda+c\eta)+(\eta-\xi)(c\eta-A\lambda)-4\eta\mathrm{e}^{\alpha+\beta}(\beta+\beta c-c\xi)))),\end{aligned}\tag{34}$$

$$\begin{aligned}\tau_{\mathrm{w}2}|_{y=-1} =&-\mathrm{e}^{A}P\lambda((4(c+1)\mathrm{e}^{\alpha}(-1+\mathrm{e}^{\beta})^2\eta\xi\alpha^2+2(c\eta\xi((-1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\xi\\
&-(1+\mathrm{e}^{\alpha})^2(-1+\mathrm{e}^{\beta})^2\eta)-(c+1)\beta(-(1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{\beta})^2\eta^2+2(-1+\mathrm{e}^{2\alpha})\\
&\cdot(-1+\mathrm{e}^{2\beta})\xi\eta-(-1+\mathrm{e}^{\alpha})^2(1+\mathrm{e}^{2\beta})\xi^2))\alpha+2(-1+\mathrm{e}^{\alpha})\beta\eta\xi(c\mathrm{e}^{\alpha+2\beta}(\eta-\xi)\\
&+c(\xi-\eta)-c\mathrm{e}^{\alpha}(\eta+\xi)+c\mathrm{e}^{2\beta}(\eta+\xi)-2\mathrm{e}^{\beta}(c\beta+\beta-c\xi)+2\mathrm{e}^{\alpha+\beta}\\
&\cdot(c\beta+\beta-c\xi)))k_2Re_2A^2+(-4(c+1)\mathrm{e}^{-A+\alpha+\beta}(-1+\mathrm{e}^{A})^2\eta\xi\alpha^2+((c+1)\mathrm{e}^{-A}\beta\\
&\cdot((1+\mathrm{e}^{2\alpha})(1+\mathrm{e}^{2\beta})\eta^2-2(-1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\xi\eta+(1+\mathrm{e}^{2\alpha})(1+\mathrm{e}^{2\beta})\xi^2)\\
&\cdot(-1+\mathrm{e}^{A})^2+\eta(c\mathrm{e}^{-A}(-1+\mathrm{e}^{A})^2\xi((-1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\xi-(1+\mathrm{e}^{2\alpha}+\mathrm{e}^{2\beta}\\
&-4\mathrm{e}^{\alpha+\beta}+\mathrm{e}^{2(\alpha+\beta)})\eta)-A\mathrm{e}^{A}(-1+\mathrm{e}^{-2A})\lambda((1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\eta-(-1+\mathrm{e}^{2\alpha})\\
&\cdot(1+\mathrm{e}^{2\beta})\xi)))\alpha+\mathrm{e}^{-A}(-1+\mathrm{e}^{A})\beta\xi(\mathrm{e}^{A}(c\eta+A\lambda)(\eta-\xi)-\mathrm{e}^{2(\alpha+\beta)}(c\eta+A\lambda)\\
&\cdot(\eta-\xi)-(c\eta-A\lambda)(\eta-\xi)+\mathrm{e}^{A+2(\alpha+\beta)}(c\eta-A\lambda)(\eta-\xi)+\mathrm{e}^{2\alpha}(c\eta+A\lambda)\\
&\cdot(\eta+\xi)-\mathrm{e}^{A+2\beta}(c\eta+A\lambda)(\eta+\xi)-\mathrm{e}^{A+2\alpha}(c\eta-A\lambda)(\eta+\xi)+\mathrm{e}^{2\beta}(c\eta-A\lambda)\\
&\cdot(\eta+\xi)+4\mathrm{e}^{\alpha+\beta}\eta(c\beta+\beta-c\xi)-4\mathrm{e}^{A+\alpha+\beta}\eta(c\beta+\beta-c\xi)))Re_1)/(A(-4(c+1)\\
&\cdot\mathrm{e}^{\alpha+\beta}(-1+\mathrm{e}^{2A})\eta\xi\alpha^2+((c+1)(-1+\mathrm{e}^{2A})\beta((1+\mathrm{e}^{2\alpha})(1+\mathrm{e}^{2\beta})\eta^2-2(-1+\mathrm{e}^{2\alpha})\\
&\cdot(-1+\mathrm{e}^{2\beta})\xi\eta+(1+\mathrm{e}^{2\alpha})(1+\mathrm{e}^{2\beta})\xi^2)+\eta(c(-1+\mathrm{e}^{2A})\xi((-1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\xi\\
&-(1+\mathrm{e}^{2\alpha}+\mathrm{e}^{2\beta}-4\mathrm{e}^{\alpha+\beta}+\mathrm{e}^{2(\alpha+\beta)})\eta)+A(1+\mathrm{e}^{2A})\lambda((1+\mathrm{e}^{2\alpha})(-1+\mathrm{e}^{2\beta})\eta\\
&-(-1+\mathrm{e}^{2\alpha})(1+\mathrm{e}^{2\beta})\xi)))\alpha+\beta\xi((A\lambda-c\eta)(\eta-\xi)+\mathrm{e}^{2A}(c\eta+A\lambda)(\eta-\xi)\\
&-\mathrm{e}^{2(\alpha+\beta)}(c\eta+A\lambda)(\eta-\xi)+\mathrm{e}^{2(A+\alpha+\beta)}(c\eta-A\lambda)(\eta-\xi)+\mathrm{e}^{2\alpha}(c\eta+A\lambda)(\eta+\xi)\\
&-\mathrm{e}^{2(A+\beta)}(c\eta+A\lambda)(\eta+\xi)-\mathrm{e}^{2(A+\alpha)}(c\eta-A\lambda)(\eta+\xi)+\mathrm{e}^{2\beta}(c\eta-A\lambda)(\eta+\xi)\\
&+4\mathrm{e}^{\alpha+\beta}\eta(c\beta+\beta-c\xi)-4\mathrm{e}^{2A+\alpha+\beta}\eta(c\beta+\beta-c\xi)))).\end{aligned}\tag{35}$$

The total volume flow rate is defined as

$$Q = \int_0^1 v_1 \mathrm{d}y + \int_{-1}^0 u_1 \mathrm{d}y. \tag{36}$$

With Eq. (36), we get

$$\begin{aligned} Q = & \frac{PRe_1}{A^2} + \frac{(1-\mathrm{e}^{-A})C_1}{A} + \frac{(\mathrm{e}^{A}-1)C_2}{A} + \frac{(\mathrm{e}^{\alpha}-1)C_3}{\alpha} \\ & + \frac{(1-\mathrm{e}^{-\alpha})C_4}{\alpha} + \frac{(\mathrm{e}^{\beta}-1)C_5}{\beta} + \frac{(1-\mathrm{e}^{-\beta})C_6}{\beta} - K_2 PRe_2. \end{aligned} \tag{37}$$

By substituting the values of the arbitrary constants, i.e., C_1, C_2, C_3, C_4, C_5, and C_6, respectively, we can evaluate the expression for the volume flow rate.

4 Results and discussion

4.1 Effect of micropolarity parameter or cross viscosity on velocity profile and microrotation

From Figs. 2–3, it is interpreted that the translational velocity and the microrotational velocity decrease with an increase in the micropolarity parameter c. From Fig. 2, we conclude that the translational velocities in both regions (micropolar fluid region and Newtonian region) decrease towards the wall of the porous channel. The rate of decrease in the translational velocity is larger in the micropolar fluid region in comparison with that in the Newtonian region. The above effect is because of the increase in the cross viscosity or the micropolarity parameter and hence the more involvement of the particles into the rotation. Due to more involvement of particles, the momentum gets transferred into the rotation which results in the decreasing linear velocity. The abrupt decrease in the microrotational velocity near the wall of the micropolar fluid region is observed (see Fig. 3). The similar nature of variation in the translational and the microrotational velocities with the micropolarity parameter was also discussed by Murthy and Srinivas[28].

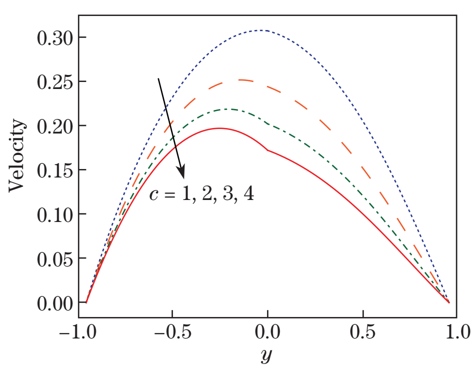

Fig. 2 Effect of micropolarity parameter c on velocity profile when $s = 5$, $K_1 = 1.5$, $K_2 = 2.5$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 1.2$, and $Re_2 = 1.6$

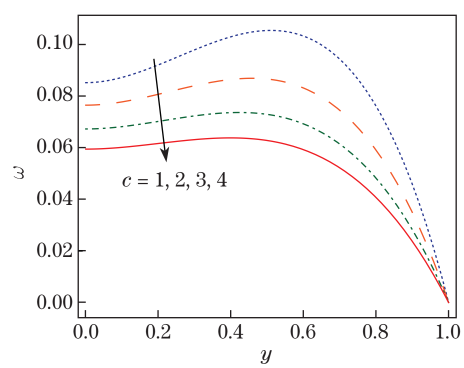

Fig. 3 Effect of micropolarity parameter c on microrotation ω when $s = 5$, $K_1 = 1.5$, $K_2 = 2.5$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 1.2$, and $Re_2 = 1.6$

4.2 Effect of couple stress on velocity profile and microrotation

The effects of the couple stress parameter on the velocity profile and microrotation are shown in Figs. 4 and 5. From these figures, it is observed that both translational and microrotational velocities increase with the increase in the couple stress parameter of the micropolar fluid.

The rate of increase in the translational velocity with respect to the couple stress parameter is lower compared with that of the microrotational velocity. The translational velocity is almost the same for different values of the couple stress parameter near the wall of the porous horizontal channel (see Fig. 4). From Fig. 5, it is also observed that the microrotational velocity increases and then decreases towards the wall of the porous horizontal channel for larger values of the couple stress parameter. Although, for lower values of the couple stress parameter, the microrotational velocity always decreases towards the wall of the porous horizontal channel.

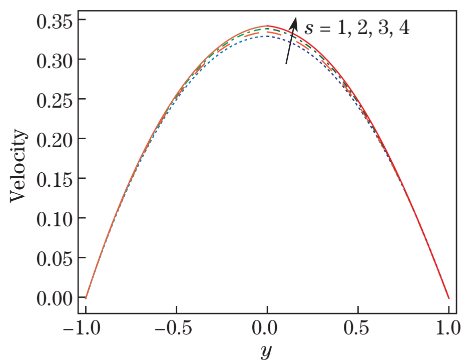

Fig. 4 Effect of couple stress s on velocity profile when $c = 0.6$, $K_1 = 1.5$, $K_2 = 2.5$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 1.2$, and $Re_2 = 1.6$

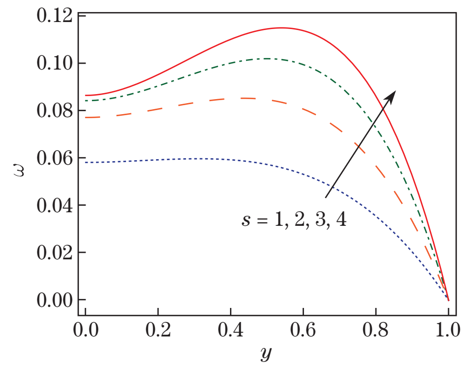

Fig. 5 Effect of couple stress s on microrotation when $c = 0.6$, $K_1 = 1.5$, $K_2 = 2.5$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 1.2$, and $Re_2 = 1.6$

For $s = 1$, the microrotational velocity (see Fig. 5) of the micropolar fluid has a very low value compared with those for $s = 2, 3, 4$.

4.3 Effect of viscosity ratio on velocity profile and microrotation

The effects of viscosity ratio on the velocity and microrotation are shown in Figs. 6–7, which show that the translational velocity and the microrotational velocity both increase with the increase in the viscosity ratio. The variations in the translational velocity and the microrotational velocity are observed to be maximum near the micropolar-Newtonian fluid interface and minimum near the wall of the porous channel. The microrotational velocity slightly increases and then decreases rapidly towards the wall of the porous channel.

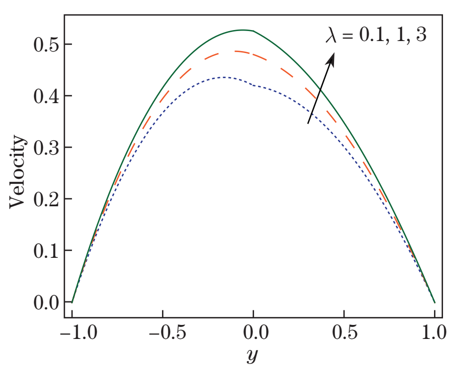

Fig. 6 Effect of viscosity ratio λ on velocity when $s = 2$, $c = 0.6$, $K_1 = 1.5$, $K_2 = 1.7$, $P = -0.7$, $Re_1 = 2$, and $Re_2 = 3$

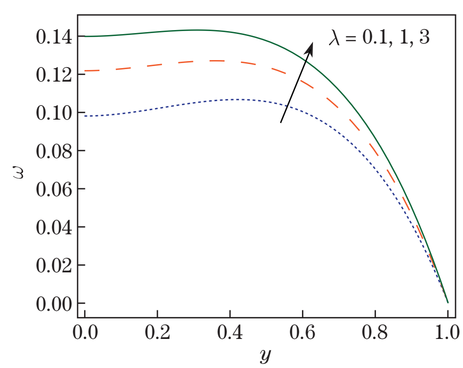

Fig. 7 Effect of viscosity ratio λ on microrotation when $s = 2$, $c = 0.6$, $K_1 = 1.5$, $K_2 = 1.7$, $P = -0.7$, $Re_1 = 2$, and $Re_2 = 3$

4.4 Effect of permeability on velocity profile and microrotation

The effects of permeability for both regions of the porous horizontal channel on the velocity profile and microrotation are discussed in Figs. 8 and 9. From Fig. 8, we conclude that the translational velocity is minimum throughout the regions of the porous channel, when the permeabilities of both regions are equal, and it is maximum near the micropolar-Newtonian fluid interface, when the permeability of the region I is smaller than that of the region II, i.e., $K_1 < K_2$. However, the microrotational velocity is maximum when the permeability of the region I is larger than that of the region II, i.e., $K_1 > K_2$, and it is minimum when the permeability of the region I is smaller than that of the region II, i.e., $K_1 < K_2$ (see Fig. 9).

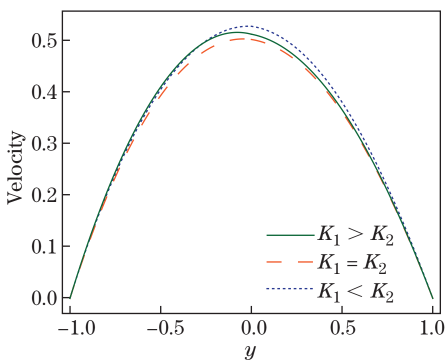

Fig. 8 Effect of permeabilities K_1 and K_2 on velocity when $s = 2$, $c = 0.6$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 2$, and $Re_2 = 3$

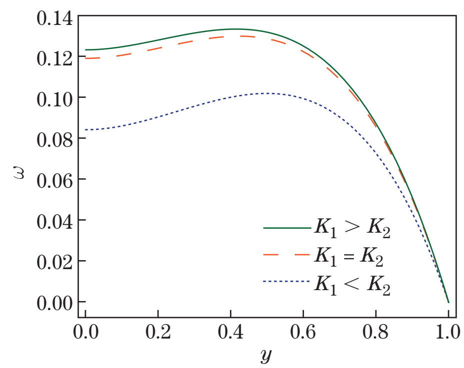

Fig. 9 Effect of permeabilities K_1 and K_2 on microrotation when $s = 2$, $c = 0.6$, $\lambda = 0.8$, $P = -0.7$, $Re_1 = 2$, and $Re_2 = 3$

4.5 Effect of couple stress and micropolarity parameter on flow rate

The influence of the couple stress and the micropolarity parameter on the flow rate is presented in Fig. 10. From this figure, we conclude that the flow rate increases with the increase in the couple stress and decreases with the increase in the micropolarity parameter. The nature of variation in the flow rate is almost the same for different values of the micropolarity parameter.

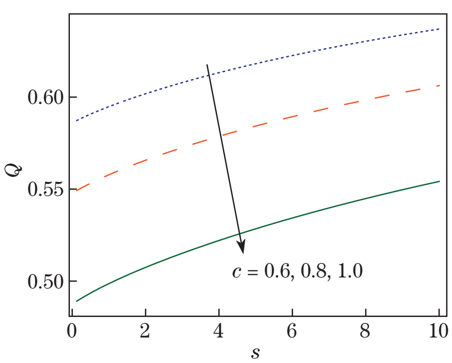

Fig. 10 Effect of couple stress and micropolarity parameter on flow rate when $K_1 = 1.5$, $K_2 = 1.7$, $\lambda = 0.4$, $P = -0.7$, $Re_1 = 2.1$, and $Re_2 = 2$

4.6 Effect of viscosity ratio and permeability on flow rate

The variation of the flow rate with the viscosity ratio and permeability is shown in Fig. 11, which shows that the flow rate increases with the increase in the viscosity ratio. From Fig. 11, it is also observed that the permeability of both regions of the horizontal porous channel plays an important role in controlling the flow rate. The flow rate is maximum when the permeability of the porous region I is larger than that of the porous region II, and it is minimum when the permeability of the porous region I is equal to that of the porous region II.

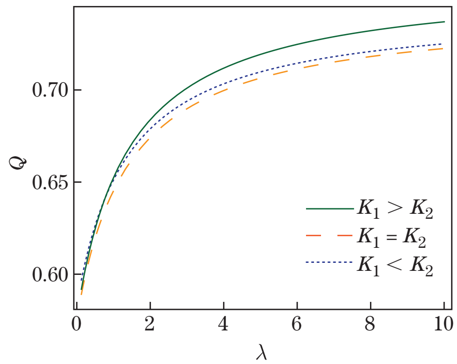

Fig. 11 Effect of permeability and viscosity ratio on flow rate when $s = 2$, $c = 0.6$, $\lambda = 0.4$, $P = -0.7$, $Re_1 = 2.1$, and $Re_2 = 2$

4.7 Effect of couple stress, micropolarity parameter, viscosity ratio, and permeability on wall shear stress of both regions

Numerical results of the wall shear stress are shown in Tables 1, 2, and 3, respectively.

It is observed that the wall shear stress for the micropolar fluid on the upper wall decreases with the increase in the couple stress, although the wall shear stress for the Newtonian fluid on the lower wall increases with the increase in the couple stress (see Table 1), whether $K_1 < K_2$, $K_1 > K_2$, or $K_1 = K_2$.

Table 1 Values of wall shear stress when $c = 1.0$, $\lambda = 0.4$, $P = -0.7$, $Re_1 = 2.1$, and $Re_2 = 2$

s	$K_1 < K_2$		$K_1 > K_2$		$K_1 = K_2$	
	τ_{w1}	τ_{w2}	τ_{w1}	τ_{w2}	τ_{w1}	τ_{w2}
2.00	1.251 5	0.417 0	1.240 9	0.419 6	1.238 1	0.415 4
5.00	1.251 0	0.424 3	1.204 4	0.426 8	1.201 8	0.422 5
7.00	1.206 8	0.427 8	1.195 6	0.430 3	1.193 0	0.425 9
9.00	1.202 1	0.430 7	1.190 7	0.433 2	1.188 1	0.428 7
11.00	1.199 3	0.433 3	1.187 6	0.435 6	1.185 0	0.431 2

Table 2 shows that the wall shear stress for the micropolar fluid on the upper wall increases with the increase in the micropolarity parameter, and the wall shear stress for the Newtonian fluid on the lower wall decreases with the increase in the micropolarity parameter. It means that the particle rotation has a noticeable effect on the wall shear stress in the three cases of permeability, i.e., $K_1 < K_2$, $K_1 > K_2$, and $K_1 = K_2$.

Table 2 Values of wall shear stress when $s = 3.0$, $\lambda = 0.4$, $P = -0.7$, $Re_1 = 2.1$, and $Re_2 = 2$

c	$K_1 < K_2$		$K_1 > K_2$		$K_1 = K_2$	
	τ_{w1}	τ_{w2}	τ_{w1}	τ_{w2}	τ_{w1}	τ_{w2}
0.10	1.191 5	0.473 2	1.176 2	0.475 3	1.172 5	0.469 8
0.50	1.217 9	0.445 5	1.205 1	0.447 9	1.202 0	0.443 0
0.80	1.228 7	0.429 2	1.217 3	0.431 6	1.214 0	0.427 2
1.20	1.237 0	0.411 7	1.227 1	0.414 2	1.224 0	0.410 1
1.50	1.240 4	0.400 9	1.251 5	0.403 5	1.229 0	0.399 7

The effect of the viscosity ratio on the wall shear stress is discussed in Table 3. It shows that the wall shear stress on the upper and lower walls of the porous channel increases with the increase in the viscosity ratio for all the cases of permeability.

From the above discussion, we also conclude that the wall shear stress on the upper wall of the porous channel is larger than that on the lower wall, when the permeability of both regions of porous channel is different.

Table 3 Values of wall shear stress when $s = 3.0$, $c = 1.0$, $P = -0.7$, $Re_1 = 2.1$, and $Re_2 = 2$

λ	$K_1 < K_2$		$K_1 > K_2$		$K_1 = K_2$	
	$\tau_{\mathrm{w}1}$	$\tau_{\mathrm{w}2}$	$\tau_{\mathrm{w}1}$	$\tau_{\mathrm{w}2}$	$\tau_{\mathrm{w}1}$	$\tau_{\mathrm{w}2}$
0.20	1.209 0	0.206 5	1.196 5	0.207 4	1.195 0	0.205 5
0.40	1.233 0	0.419 9	1.223 0	0.422 4	1.220 0	0.418 2
0.60	1.254 0	0.638 5	1.244 6	0.643 1	1.240 8	0.636 0
0.80	1.271 0	0.860 9	1.262 9	0.867 8	1.258 1	0.857 2
1.20	1.298 0	0.314 2	1.292 1	1.326 7	1.285 5	0.310 0

5 Conclusions

The problem of fully-developed and steady flows of two immiscible fluids (micropolar-Newtonian) through the horizontal porous channel is solved with the direct method. During the analysis, it is noticed that the value of the translational velocity in the upper region of the horizontal porous channel is small compared with that in the lower region of the horizontal porous channel. This result can be used in practical applications. It is observed that the permeability of both regions of horizontal porous channels can be used to control the velocity profile, flow rate, and wall shear stress. It is also found that the viscosity ratio and couple stress promote the velocity profile, and the micropolarity parameter suppresses the velocity.

References

[1] ERINGEN, A. C. *Microcontinuum Field Theories* II: *Fluent Media*, Springer, New York (2001)

[2] ERINGEN, A. C. Simple microfluids. *International Journal of Engineering Science*, **2**(2), 205–217 (1964)

[3] ERINGEN, A. C. Theory of micropolar fluids. *Journal of Mathematics and Mechanics*, **16**, 1–18 (1966)

[4] HOYT, J. W. and FABULA, A. G. *The Effect of Additives on Fluid Friction*, Defense Technical Information Center, Dayton (1964)

[5] ARIMAN, T., TURK, M., and SYLVESTER, N. D. Microcontinuum fluid mechanics —— a review. *International Journal of Engineering Science*, **11**, 905–930 (1973)

[6] LUKASZEWICZ, G. *Micropolar Fluids*: *Theory and Applications*, Springer Science and Business Media, New York (2004)

[7] PHILIP, D. and CHANDRA, P. Flow of Eringen fluid (simple microfluid) through an artery with mild stenosis. *International Journal of Engineering Science*, **34**(1), 87–99 (1996)

[8] ELLAHI, R., RAHMAN, S. U., GULZAR, M. M., NADEEM, S., and VAFAI, K. G. A mathematical study of non-Newtonian micropolar fluid in arterial blood flow through composite stenosis. *Applied Mathematics and Information Sciences*, **8**(4), 1567–1573 (2014)

[9] CHEN, J., LIANG, C., and LEE, J. D. Theory and simulation of micropolar fluid dynamics. *Proceedings of the Institution of Mechanical Engineers*, *Part N*: *Journal of Nanomaterials*, *Nanoengineering and Nanosystems*, **224**, 31–39 (2010)

[10] ARIMAN, T. and CAKMAK, A. S. Some basic viscous flows in micropolar fluids. *Rheologica Acta*, **7**(3), 236–242 (1968)

[11] SOCHI, T. Non-Newtonian flow in porous media. *Polymer*, **51**(22), 5007–5023 (2010)

[12] BRINKMAN, H. C. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Applied Scientific Research*, **2**(1), 155–161 (1951)

[13] ISHII, M. and HIBIKI, T. *Thermo-Fluid Dynamics of Two-Phase Flow*, Springer Science and Business Media, New York (2010)

[14] BERMAN, A. S. Laminar flow in channels with porous walls. *Journal of Applied Physics*, **24**(9), 1232–1235 (1953)

[15] KAPUR, J. N. and SHUKLA, J. B. The flow of incompressible immiscible fluids between two plates. *Applied Scientific Research*, **13**, 55–60 (1962)

[16] SHAIL, R. On laminar two-phase flows in magnetohydrodynamics. *International Journal of Engineering Science*, **11**(10), 1103–1108 (1973)

[17] VAFAI, K. and THIYAGARAJA, R. Analysis of flow and heat transfer at the interface region of a porous medium. *International Journal of Heat and Mass Transfer*, **30**(7), 1391–1405 (1987)

[18] SRINIVASAN, V. and VAFAI, K. Analysis of linear encroachment in two-immiscible fluid systems in a porous medium. *Journal of Fluids Engineering*, **116**(4), 1–5 (1994)

[19] MUSKAT, M. and WYCKOFF, R. D. *Flow of Homogeneous Fluids Through Porous Media*, McGraw-Hill Book Company, New York (1937)

[20] CHAMKHA, A. J. Flow of two-immiscible fluids in porous and nonporous channels. *Journal of Fluids Engineering*, **122**(1), 117–124 (2000)

[21] AWARTANI, M. M. and HAMDAN, M. H. Fully developed flow through a porous channel bounded by flat plates. *Applied Mathematics and Computation*, **169**(2), 749–757 (2005)

[22] UMAVATHI, J. C., CHAMKHA, A. J., MATEEN, A., and AL-MUDHAF, A. Oscillatory Hartmann two-fluid flow and heat transfer in a horizontal channel. *International Journal of Applied Mechanics and Engineering*, **11**(1), 155–178 (2006)

[23] UMAVATHI, J. C., LIU, I. C., PRATHAP-KUMAR, J., and SHAIK-MEERA, D. Unsteady flow and heat transfer of porous media sandwiched between viscous fluids. *Applied Mathematics and Mechanics* (*English Edition*), **31**(12), 1497–1516 (2010) https://doi.org/10.1007/s10483-010-1379-6

[24] DELHOMMELLE, J. and EVANS, D. J. Poiseuille flow of a micropolar fluid. *Molecular Physics*, **100**(17), 2857–2865 (2002)

[25] SRINIVASACHARYA, D. and SHIFERAW, M. Flow of micropolar fluid between parallel plates with Soret and Dufour effects. *Arabian Journal of Science and Engineering*, **39**(6), 5085–5093 (2014)

[26] SHEIKHOLESLAMI, M., HATAMI, M., and GANJI, D. D. Micropolar fluid flow and heat transfer in a permeable channel using analytical method. *Journal of Molecular Liquids*, **194**, 30–36 (2014)

[27] JONEIDI, A. A., GANJI, D. D., and BABAELAHI, M. Micropolar flow in a porous channel with high mass transfer. *International Communications in Heat and Mass Transfer*, **36**(10), 1082–1088 (2009)

[28] MURTHY, J. V. R. and SRINIVAS, J. Second law analysis for Poiseuille flow of immiscible micropolar fluids in a channel. *International Journal of Heat and Mass Transfer*, **65**, 254–264 (2013)

[29] MURTHY, J. V. R., SRINIVAS, J., and SAI, K. S. Flow of immiscible micropolar fluids between two porous beds. *Journal of Porous Media*, **17**(14), 287–300 (2014)

[30] ANSARI, I. A. and DEO, S. Effect of magnetic field on the two immiscible viscous fluids flow in a channel filled with porous medium. *National Academy Science Letters*, **40**(3), 211–214 (2017)

[31] ADESANYA, S. O., FALADE, J. A., UKAEGBU, J. C., and ADEKEYE, K. S. Mathematical analysis of a reactive viscous flow through a channel filled with a porous medium. *Journal of Mathematics*, **2016**, 1–8 (2016)

[32] SRINIVAS, J. and MURTHY, J. V. R. Second law analysis of the flow of two immiscible micropolar fluids between two porous beds. *Journal of Engineering thermophysics*, **25**(1), 126–142 (2016)

[33] KUMAR, J. P., UMAVATHI, J. C., CHAMKHA, A. J., and POP, I. Fully-developed free-convective flow of micropolar and viscous fluids in a vertical channel. *Applied Mathematical Modelling*, **34**(5), 1175–1186 (2010)