Appl. Math. Mech. -Engl. Ed., **39**(3), 353–364 (2018)

Applied Mathematics and Mechanics (English Edition)

https://doi.org/10.1007/s10483-018-2304-9

Synchronization in a fractional-order dynamic network with uncertain parameters using an adaptive control strategy*

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(Received Jun. 30, 2017 / Revised Aug. 8, 2017)

Abstract This paper studies synchronization of all nodes in a fractional-order complex dynamic network. An adaptive control strategy for synchronizing a dynamic network is proposed. Based on the Lyapunov stability theory, this paper shows that tracking errors of all nodes in a fractional-order complex network converge to zero. This simple yet practical scheme can be used in many networks such as small-world networks and scale-free networks. Unlike the existing methods which assume the coupling configuration among the nodes of the network with diffusivity, symmetry, balance, or irreducibility, in this case, these assumptions are unnecessary, and the proposed adaptive strategy is more feasible. Two examples are presented to illustrate effectiveness of the proposed method.

Key words fractional-order chaotic system, synchronization, complex dynamic network, adaptive control

Chinese Library Classification O415.5
2010 Mathematics Subject Classification 93C83, 93D05

1 Introduction

Fractional-order calculus is an old mathematical topic with more than 300 years history. Its applications, however, have been found in physics and engineering only in recent years. In fact, fractional-order chaotic systems have more complex dynamical behaviors than integer-order systems, and they have been widely used to ensure communication security[1]. Owing to the memory property of fractional-order differential equations, they have remarkable advantages in describing memory and hereditary properties of various materials and processes, such as anomalous diffusion phenomena and dynamic behaviors of financial systems[2–3]. Besides, fractional-order calculus serves as a valuable instrument in discussing signal processing and image filtering[4], viscoelastic materials[5–6], bioengineering[7], robotics[8], and mechanics[9]. These

∗ Citation: Du, L., Yang, Y., and Lei, Y. M. Synchronization in a fractional-order dynamic network with uncertain parameters using an adaptive control strategy. *Applied Mathematics and Mechanics* (*English Edition*), **39**(3), 353–364 (2018) https://doi.org/10.1007/s10483-018-2304-9

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Project supported by the National Natural Science Foundation of China (Nos. 11672231 and 11672233), the Natural Science Foundation of Shaanxi Province (No. 2016JM1010), the Fundamental Research Funds for the Central Universities (No. 3102017AX008), and the Seed Foundation of Innovation and Creation for Graduate Students at the Northwestern Polytechnical University of China (No. Z2017187)

further perfectly clarify the importance of consideration and analysis of dynamical systems with fractional-order models.

Recently, fractional-order calculus has been reported in a dynamic network which is a collection of nonlinear dynamical systems interacted by links with complex topological properties and can describe lots of complex circuits and systems in various fields of the real world. Examples include neural systems, electrical power grids, social network, and highway or subway systems, which display substantial nontrivial topological features with patterns of connection between their elements that are neither purely regular nor purely random. Such features include a heavy tail in the degree distribution, a small world, and a high clustering coefficient. In the directed networks, the reciprocity, triad significance profile, and other features are also included. By now, much effort has been devoted to studying the complex dynamical behavior of dynamic networks[10–14]. In Ref. [10], the authors designed an effective distributed adaptive strategy to tune the coupling weights of a network, which can be extended to the case where only a small fraction of coupling weights can be adjusted, but the coupling configuration is required to satisfy balance or symmetry. Pagani and Aiello[12] investigated the properties of different power grid infrastructures using complex network analysis techniques and methodologies. However, due to the limited theories of fractional-order calculus, it is still a challenging work to investigate the fractional-order dynamic network. Synchronization, as an effective method to study the emergent behavior and coordinated motion in fractional-order complex networks, has attracted increasing attention[15–18]. In Ref. [15], a new synchronous motion was obtained by tuning a coupling parameter to synchronization of a general fractional-order dynamic network. Asheghan et al.[16] proposed an open-plus-closed-loop scheme to synchronize complex networks with fractional-order dynamics. An adaptive method was considered in Ref. [17] to synchronize two fractional-order dynamic networks. Lately, Ma et al.[18] designed an adaptive control method to study the hybrid projective synchronization of two coupled general fractional-order complex networks with different sizes. However, the above studies just took into account synchronization between two complex dynamic networks, which cannot achieve the synchronization of all nodes in a complex network. In Refs. [19] and [20], some effective methods were introduced to make all nodes in a network synchronized. Chen et al.[19] proposed cluster synchronization in a fractional-order complex dynamic network, in which only the nodes in one community that have direct connections to the nodes in other communities need controllers, resulting in reduced control cost. In Ref. [20], nonlinear controllers were constructed to synchronize the weighted fractional-order complex dynamic networks, which can achieve the synchronization of complex networks with nonidentical nodes. In these works, however, the coupling configuration among the nodes of the network required diffusivity and irreducibility, and the controllers were designed using a solution to the node system $D^{\alpha}x = f(x)$. Moreover, most of the results mentioned above just dealt with the synchronization of complex networks without considering uncertain parameters, which play a significant role in physics and engineering. In Ref. [21], the authors investigated the robust stability and stabilization of fractional-order linear systems with nonlinear uncertain parameters, which appeared in the form of a combination of additive uncertainty and multiplicative uncertainty. The robust stability of fractional-order linear time-invariant interval systems has been considered in Ref. [22], in which the models with uncertain parameters showed that it has deterministic linear coupling relationship between fractional-order and other model parameters.

In this work, a fractional-order adaptive feedback controller is designed to achieve complete synchronization in a dynamic network with uncertain parameters, which is a general, simple, and rigorous feedback scheme without assuming the coupling configuration with diffusivity, symmetry, balance, or irreducibility. The rest of this paper is organized as follows. In Section 2, some preliminary definitions on fractional calculus are introduced. Section 3 provides a fractional-order control to achieve the complete synchronization in a fractional-order dynamic network with uncertain parameters. We show that tracking errors of all nodes in the fractional-order complex network converge to zero by means of the Lyapunov stability analysis. In Section 4, two examples are used to show validity and feasibility of the proposed scheme through numerical simulations. Finally, conclusions are drawn in Section 5.

2 Preliminaries

Fractional-order calculus is a classical mathematical idea, which is the generalization of integer-order differentiation and integration to arbitrary order. Now, several definitions of fractional-order derivative have been displayed. The Riemann-Liouville and Caputo definitions are two most commonly used. The initial conditions of the Caputo fractional derivative take the same form as the integer-order version, which have physical meanings and more applications in physics and engineering. Besides, the Caputo derivative of a constant is equal to zero. Therefore, in the rest of this study, the Caputo derivative will be adopted. Then, this section gives some basic definitions of fractional calculus.

Definition 1[23] *The Riemann-Liouville fractional integral operator of a continuous function $f(t)$ with order $\alpha > 0$ is given by*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^T (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2[23] *The Riemann-Liouville fractional derivative with order $\alpha > 0$ of a continuous function $f(t)$ is defined as follows*:

$$D^{\alpha} f(t) = D^m J^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^T \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function, and $m-1 < \alpha \leqslant m$ $(m \in \mathbb{N})$.

Definition 3[23] *The Caputo fractional derivative with order $\alpha > 0$ of a continuous function $f(t)$ is described as follows*:

$$D_{\mathrm{c}}^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^T \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function, and $m-1 < \alpha \leqslant m$ $(m \in \mathbb{N})$.

3 Synchronization principles

Consider a general fractional-order complex dynamic network consisting of N nodes, and each node is an n-dimensional fractional-order chaotic dynamical oscillator. The state equation of this fractional-order dynamic network can be described as

$$D_{\mathrm{c}}^{\alpha} x_i = f(x_i) + g(x_i) p_i + \sum_{j=1}^{N} c_{ij} H(x_j - x_i), \quad i = 1, 2, \cdots, N, \tag{4}$$

where $0 < \alpha \leqslant 1, x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ is a state vector of the ith node, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $g(x_i) \in \mathbb{R}^{n\times m}$ are the nonlinear continuous vector functions, $p_i = (p_{i1}, p_{i2}, \cdots, p_{im})^{\mathrm{T}} \in \mathbb{R}^m$ is the uncertain parameter vector of the ith node, $H \in \mathbb{R}^{n\times n}$ is an inner-coupling constant matrix, and $C = (c_{ij})_{N\times N}$ is the coupling configuration diffusive matrix, in which $c_{ij} > 0$ if there is a connection between the ith and jth nodes; otherwise, $c_{ij} = 0$ $(i,\ j = 1, 2, \cdots, N)$.

Remark 1 The matrix $C = (c_{ij})_{N\times N}$ reflects the complex topological properties, which play an important role in the behavior of the network. In the literature, the coupling configuration diffusive matrix C was strictly limited. In Refs. [10], [19], and [20], the following properties[16] were required:

Irreducibility There is no isolated cluster of the nodes.

Diffusivity The coupling configuration diffusive matrix C satisfies $\sum_{i=1}^{N} c_{ij} = 0$ $(j = 1, 2, \cdots, N)$.

Balance $\sum_{i=1}^{N} c_{ij} = \sum_{j=1}^{N} c_{ij}$ is required.

Symmetry The matrix C satisfies $c_{ij} = c_{ji}$ $(j = 1, 2, \cdots, N)$.

In this paper, we present that it is not essential to gratify the properties just mentioned above.

Assumption 1 *For any* $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^{\mathrm{T}}$, and $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})^{\mathrm{T}} \in \mathbb{R}^n$, *the nonlinear function* $f(\cdot, t)$ *satisfies the Lipschitz conditions*,

$$(x_i - y_i)^{\mathrm{T}}(f(x_i, t) - f(y_i, t)) \leqslant (x_i - y_i)^{\mathrm{T}} l_i (x_i - y_i), \quad i = 1, 2, \cdots, N, \tag{5}$$

where $l_i > 0$ *is a Lipschitz constant.*

Remark 2 The Lipschitz condition ensures existence and uniqueness of the nonlinear function $f(\cdot, t)$. Most of the well-known fractional-order chaotic systems meet the condition mentioned above, such as the fractional-order Chen system, the fractional-order Lorenz system, and the fractional-order hyperchaotic Chen system.

Lemma 1[24] *Consider a fractional-order system*

$$D_{\mathrm{c}}^{\alpha} Y(t) = f(Y(t)), \tag{6}$$

where $\alpha \in (0, 1]$, *and* $Y(t) \in \mathbb{R}^n$. *If the following inequality*

$$Y^{\mathrm{T}}(t) f(Y(t)) \leqslant 0 \tag{7}$$

is satisfied, the origin of the system (6) *is stable. If*

$$Y^{\mathrm{T}}(t) f(Y(t)) < 0, \quad \forall Y \neq 0, \tag{8}$$

the origin of the system (6) *is asymptotically stable.*

To synchronize all nodes in a fractional-order dynamic network, we add the controller $u_i \in \mathbb{R}^n$ to each node of the network,

$$D_{\mathrm{c}}^{\alpha} x_i = f(x_i) + g(x_i) p_i + \sum_{j=1}^{N} c_{ij} H(x_j - x_i) + u_i, \quad i = 1, 2, \cdots, N. \tag{9}$$

Let

$$\overline{X} = \frac{1}{N} \sum_{j=1}^{N} x_j.$$

Then, one has

$$\begin{aligned} D_{\mathrm{c}}^{\alpha} \overline{X}(t) &= G(x_1, x_2, \cdots, x_N) \\ &= \frac{1}{N} \sum_{j=1}^{N} \Big(f(x_j) + g(x_j) p_j + \sum_{i=1}^{N} c_{ij} H(x_i - x_j) + u_j \Big). \end{aligned} \tag{10}$$

Now, we define the error signal as $e_i = x_i - \overline{X}$ $(i = 1, 2, \cdots, N)$ for synchronization of complex dynamic network. Then, we can obtain the following error dynamic network:

$$\begin{aligned} D_{\mathrm{c}}^{\alpha} e_i &= D_{\mathrm{c}}^{\alpha} x_i - D_{\mathrm{c}}^{\alpha} \overline{X} \\ &= f(x_i) + g(x_i) p_i + \sum_{j=1}^{N} c_{ij} H(x_j - x_i) + u_i - G(x_1, x_2, \cdots, x_N). \end{aligned} \tag{11}$$

Construct the following adaptive feedback controller to realize synchronization in a dynamic network:

$$u_i = -K_i e_i - g(x_i) q_i, \quad i = 1, 2, \cdots, N, \tag{12}$$

where $K_i = \mathrm{diag}(k_{i1}, k_{i2}, \cdots, k_{in}) \in \mathbb{R}^{n\times n}$ and $q_i \in \mathbb{R}^m$ are usually called adaptive feedback strengths, which are adapted as the following adaptive law:

$$\begin{cases} D_{\mathrm{c}}^{\alpha} k_{il} = \varepsilon_i e_i^{\mathrm{T}} e_i, \ \ l = 1, 2, \cdots, n, \ \ \varepsilon_i > 0, \\ D_{\mathrm{c}}^{\alpha} e_{q_i} = D_{\mathrm{c}}^{\alpha}(q_i - p_i) = \gamma_i (g(x_i))^{\mathrm{T}} e_i, \ \ \gamma_i > 0. \end{cases} \tag{13}$$

Theorem 1 *Suppose that Assumption* 1 *holds. Then, the network* (4) *is synchronized under the controller* (12) *with the adaptive law* (13), *that is, the tracking error* (11) *is stable.*

Proof Construct a scalar function $J(t)$ as follows:

$$\begin{aligned} J(t) &= Y^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} Y \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} e_i + \sum_{i=1}^{N} \frac{1}{\gamma_i} e_{q_i}^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} e_{q_i} + \sum_{i=1}^{N} \frac{1}{\varepsilon_i} (k_i - k_i^*)^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} (k_i - k_i^*), \end{aligned} \tag{14}$$

where $Y = (e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \cdots, e_N^{\mathrm{T}}, \frac{1}{\sqrt{\gamma_1}} e_{q_1}^{\mathrm{T}}, \frac{1}{\sqrt{\gamma_2}} e_{q_2}^{\mathrm{T}}, \cdots, \frac{1}{\sqrt{\gamma_N}} e_{q_N}^{\mathrm{T}}, \frac{1}{\sqrt{\varepsilon_1}} k_1^{\mathrm{T}} - \frac{1}{\sqrt{\varepsilon_1}} k_1^{*\mathrm{T}}, \frac{1}{\sqrt{\varepsilon_2}} k_2^{\mathrm{T}} - \frac{1}{\sqrt{\varepsilon_2}} k_2^{*\mathrm{T}}, \cdots, \frac{1}{\sqrt{\varepsilon_N}} k_N^{\mathrm{T}} - \frac{1}{\sqrt{\varepsilon_N}} k_N^{*\mathrm{T}})^{\mathrm{T}}$, $k_i = (k_{i1}, k_{i2}, \cdots, k_{in})^{\mathrm{T}}$, and $k_i^* = (k_{i1}^*, k_{i2}^*, \cdots, k_{in}^*)^{\mathrm{T}}$ $(i = 1, 2, \cdots, N)$. Note that k_{il}^* $(i = 1, 2, \cdots, N;\ l = 1, 2, \cdots, n)$ are constants to be determined.

Combining (11), (12), and (13), we get

$$\begin{aligned} J(t) &= Y^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} Y \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} e_i + \sum_{i=1}^{N} \frac{1}{\gamma_i} e_{q_i}^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} e_{q_i} + \sum_{i=1}^{N} \frac{1}{\varepsilon_i} (k_i - k_i^*)^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} (k_i - k_i^*) \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} \Big(f(x_i) + g(x_i) p_i + \sum_{j=1}^{N} c_{ij} H(x_j - x_i) - K_i e_i - g(x_i) q_i - G(x_1, x_2, \cdots, x_N) \Big) \\ &\quad + \sum_{i=1}^{N} e_{q_i}^{\mathrm{T}} (g(x_i))^{\mathrm{T}} e_i + \sum_{i=1}^{N} \sum_{l=1}^{n} (k_{il} - k_{il}^*) e_i^{\mathrm{T}} e_i \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} \Big(f(x_i) - g(x_i) e_{q_i} + \sum_{j=1}^{N} c_{ij} H(e_j - e_i) - K_i e_i - G(x_1, x_2 \cdots, x_N) \Big) \\ &\quad + \sum_{i=1}^{N} e_{q_i}^{\mathrm{T}} (g(x_i))^{\mathrm{T}} e_i + \sum_{i=1}^{N} \sum_{l=1}^{n} (k_{il} - k_{il}^*) e_i^{\mathrm{T}} e_i \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} (f(x_i) - f(\overline{X})) + \sum_{i=1}^{N} e_i^{\mathrm{T}} f(\overline{X}) + \sum_{i=1}^{N} e_i^{\mathrm{T}} \sum_{j=1}^{N} c_{ij} H(e_j - e_i) \\ &\quad - \sum_{i=1}^{N} e_i^{\mathrm{T}} K_i e_i - \sum_{i=1}^{N} e_i^{\mathrm{T}} G(x_1, x_2, \cdots, x_N) + \sum_{i=1}^{N} \sum_{l=1}^{n} (k_{il} - k_{il}^*) e_i^{\mathrm{T}} e_i. \end{aligned} \tag{15}$$

Since $\sum\limits_{i=1}^{N} e_i = 0$, one has

$$\begin{cases} \sum\limits_{i=1}^{N} e_i^{\mathrm{T}} f(\overline{X}) = 0, \\ \sum\limits_{i=1}^{N} e_i^{\mathrm{T}} G(x_1, x_2, \cdots, x_N) = 0. \end{cases} \tag{16}$$

Substituting (16) into (15) yields

$$\begin{aligned} J(t) &= \sum_{i=1}^{N} e_i^{\mathrm{T}} (f(x_i) - f(\overline{X})) + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} e_i^{\mathrm{T}} H(e_j - e_i) - \sum_{i=1}^{N} e_i^{\mathrm{T}} K_i e_i + \sum_{i=1}^{N}\sum_{l=1}^{n} (k_{il} - k_{il}^*) e_i^{\mathrm{T}} e_i \\ &\leqslant \sum_{i=1}^{N} e_i^{\mathrm{T}} l_i I_n e_i + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} e_i^{\mathrm{T}} H(e_j - e_i) - \sum_{i=1}^{N} e_i^{\mathrm{T}} K_i e_i + \sum_{i=1}^{N} e_i^{\mathrm{T}} (K_i - K_i^*) e_i \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} (l_i I_n - K_i^*) e_i + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} e_i^{\mathrm{T}} H(e_j - e_i) \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} \Big(l_i I_n - K_i^* - \sum_{j=1}^{N} c_{ij} H \Big) e_i + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} e_i^{\mathrm{T}} H e_j \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}} \Big(l_i I_n - K_i^* - \sum_{j=1}^{N} c_{ij} H \Big) e_i + e^{\mathrm{T}} (C \otimes H) e \\ &= -\Big(\sum_{i=1}^{N} e_i^{\mathrm{T}} \Big(K_i^* + \sum_{j=1}^{N} c_{ij} H - l_i I_n \Big) e_i - e^{\mathrm{T}} (C \otimes H) e \Big) \\ &= -\, e^{\mathrm{T}} Q e, \end{aligned} \tag{17}$$

where

$$e = (e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \cdots, e_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN},$$

$$K_i^* = \mathrm{diag}(k_{i1}^*, k_{i2}^*, \cdots, k_{in}^*), \quad K_i = \mathrm{diag}(k_{i1}, k_{i2}, \cdots, k_{in}), \quad i = 1, 2, \cdots, N,$$

$$C \otimes H = \begin{bmatrix} c_{11}H & \cdots & c_{1N}H \\ \vdots & & \vdots \\ c_{N1}H & \cdots & c_{NN}H \end{bmatrix},$$

$$Q = \begin{bmatrix} K_1^* + \sum_{j=1}^{N} c_{1j}H - l_1 I_n - c_{11}H & \cdots & -c_{1N}H \\ \vdots & & \vdots \\ -c_{N1}H & \cdots & K_N^* + \sum_{j=1}^{N} c_{Nj}H - l_N I_n - c_{NN}H \end{bmatrix} \in \mathbb{R}^{nN \times nN}.$$

By selecting suitable constants $k_{il}^* > 0$ $(i = 1, 2, \cdots, N; l = 1, 2, \cdots, n)$, we can achieve the aim that $\frac{1}{2}(Q + Q^{\mathrm{T}})$ is a positive definite matrix. Therefore, we have $J(t) \leqslant Y^{\mathrm{T}} D_{\mathrm{c}}^{\alpha} Y \leqslant -e^{\mathrm{T}} Q e \leqslant 0$. According to Lemma 1, the error system (11) is stable, indicating that all the nodes in a complex dynamic network are synchronized. This completes the proof.

Remark 3 Lemma 1 can also be used to demonstrate that the tracking error systems between two complex networks pointed out in Ref. [25] converge to zero, and it is also unnecessary to require that the matrix C satisfies $\sum_{i=1}^{N} c_{ij} = 0$ $(j = 1, 2, \cdots, N)$.

4 Illustrative examples

In this section, we take the small-world and scale-free dynamic networks as examples to show effectiveness of the above proposed scheme through two illustrative examples and numerical simulations. The improved Adams-Bash forth-Moulton algorithm[26] is adopted to implement numerical simulations with a step-size of $h = 0.005$.

4.1 A small-world network of fractional-order Lorenz systems

Consider a network consisting of $N = 100$ identical nodes. In this case, the state of each node i is the fractional-order Lorenz system[27] depicted by

$$\begin{cases} D_{\mathrm{c}}^{\alpha} x_{i1} = p_{i1}(x_{i2} - x_{i1}), \\ D_{\mathrm{c}}^{\alpha} x_{i2} = x_{i1}(p_{i2} - x_{i3}) - x_{i2}, \\ D_{\mathrm{c}}^{\alpha} x_{i3} = x_{i1}x_{i2} - p_{i3}x_{i3}, \end{cases} \tag{18}$$

where p_{i1}, p_{i2}, and p_{i3} are uncertain parameters, and x_{il} $(i = 1, 2, \cdots, N;\ l = 1, 2, 3)$ are state variables of the system. When we set $\alpha = 0.997$, $p_{i1} = 10$, $p_{i2} = 28$, and $p_{i3} = 8/3$, the fractional-order Lorenz system exhibits chaotic attractor displayed in Fig. 1. The fractional-order complex network with $N = 100$ nodes is described as follows:

$$D_{\mathrm{c}}^{\alpha} x_i = \begin{bmatrix} 0 \\ -x_{i1}x_{i3} - x_{i2} \\ x_{i1}x_{i2} \end{bmatrix} + \begin{bmatrix} x_{i2} - x_{i1} & 0 & 0 \\ 0 & x_{i1} & 0 \\ 0 & 0 & -x_{i3} \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} + \sum_{j=1}^{N} c_{ij} H \left(x_j - x_i\right) + u_i \tag{19}$$

with

$$\begin{aligned} & u_i = -K_i e_i - g(x_i) q_i, \quad K_i = \mathrm{diag}(k_{i1}, k_{i2}, k_{i3}), \\ & D_{\mathrm{c}}^{\alpha} k_{il} = \varepsilon_i e_i^{\mathrm{T}} e_i, \quad l = 1, 2, 3, \quad \varepsilon_i > 0, \\ & D_{\mathrm{c}}^{\alpha} e_{q_i} = \gamma_i (g\left(x_i\right))^{\mathrm{T}} e_i, \quad \gamma_i > 0. \end{aligned}$$

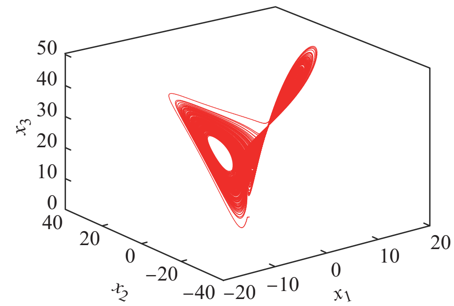

Fig. 1 Chaotic attractor of the fractional-order Lorenz system with order $\alpha = 0.997$

The inner-coupling matrix is $H = \mathrm{diag}(1, 1, 1, 1)$, $\varepsilon_i = 1$, $\gamma_i = 1$ $(i = 1, 2, \cdots, N)$, and the coupling configuration diffusive matrix is derived from the small-world network, which is constructed by the Watts-Strogatz mechanism from a regular ring lattice with rewiring probability 0.1. The initial values of dynamic network are selected as

$$\begin{cases} x_i(0) = \dfrac{1}{2}(0.3 - 0.01i, -0.4 + 0.01i, 0.2 - 0.01i)^{\mathrm{T}}, \\ k_i(0) = \dfrac{1}{2}(0.01i, 0.02i, 0.03i)^{\mathrm{T}}, \\ q_i(0) = \dfrac{1}{2}(0.31 - 0.01i, -0.3 + 0.01i, 0.5 - 0.01i)^{\mathrm{T}}, \end{cases} \tag{20}$$

where $i = 1, 2, \cdots, N$.

Define the errors as $\|E_l\| = \frac{1}{N}\sqrt{\sum\limits_{i=1}^{N} e_{il}^2}$ $(l = 1, 2, 3)$. As displayed in Fig. 2, the trajectories of the error system for this dynamic network approach zero, which means that all nodes in the dynamic network are synchronized.

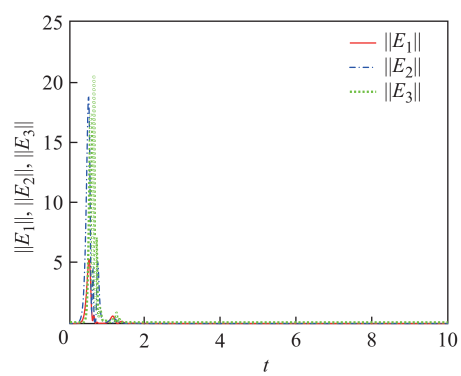

Fig. 2 Time history of synchronization errors of all nodes in the small-world network of fractional-order Lorenz systems with order $\alpha = 0.997$

4.2 A scale-free network of fractional-order hyperchaotic Chen systems

Set a network with $N = 100$ nodes and take the fractional-order hyperchaotic Chen system as the node dynamics. The fractional-order hyperchaotic Chen system[28] in the ith node is defined as follows:

$$\begin{cases} D_{\mathrm{c}}^{\alpha} x_{i1} = p_{i1}(x_{i2} - x_{i1}) + x_{i4}, \\ D_{\mathrm{c}}^{\alpha} x_{i2} = x_{i1}(p_{i2} - x_{i3}) + p_{i3} x_{i2}, \\ D_{\mathrm{c}}^{\alpha} x_{i3} = x_{i1} x_{i2} - p_{i4} x_{i3}, \\ D_{\mathrm{c}}^{\alpha} x_{i4} = x_{i2} x_{i3} + p_{i5} x_{i4}, \end{cases} \tag{21}$$

where $p_{i1},\ p_{i2},\ p_{i3},\ p_{i4}$, and p_{i5} are uncertain parameters, and x_{il} $(i = 1, 2, \cdots, N;\ l = 1, 2, 3, 4)$ are the state variables of the system. There exists a chaotic behavior shown in Fig. 3 with $\alpha = 0.98,\ p_{i1} = 35,\ p_{i2} = 7,\ p_{i3} = 12,\ p_{i4} = 3$, and $p_{i5} = 0.5$. The fractional-order complex network with $N = 100$ nodes is written as

$$D_{\mathrm{c}}^{\alpha} x_i = \begin{bmatrix} x_{i4} \\ -x_{i1} x_{i3} \\ x_{i1} x_{i2} \\ x_{i2} x_{i3} \end{bmatrix} + \begin{bmatrix} x_{i2} - x_{i1} & 0 & 0 & 0 & 0 \\ 0 & x_{i1} & x_{i2} & 0 & 0 \\ 0 & 0 & 0 & -x_{i3} & 0 \\ 0 & 0 & 0 & 0 & x_{i4} \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \\ p_{i4} \\ p_{i5} \end{bmatrix}$$
$$+ \sum_{j=1}^{N} c_{ij} H\left(x_j - x_i\right) + u_i, \tag{22}$$

where

$$\begin{aligned} & u_i = -K_i e_i - g(x_i) q_i, \quad K_i = \mathrm{diag}(k_{i1}, k_{i2}, \cdots, k_{i4}), \\ & D_{\mathrm{c}}^{\alpha} k_{il} = \varepsilon_i e_i^{\mathrm{T}} e_i, \quad l = 1, 2, 3, 4, \quad \varepsilon_i > 0, \\ & D_{\mathrm{c}}^{\alpha} e_{q_i} = \gamma_i (g(x_i))^{\mathrm{T}} e_i, \quad \gamma_i > 0. \end{aligned}$$

The inner-coupling matrix is $H = \mathrm{diag}(1, 1, 1, 1),\ \ \varepsilon_i = 1,\ \ \gamma_i = 1\ (i = 1, 2, \cdots, N)$, and the coupling configuration diffusive matrix is also selected from the scale-free network whose degree distribution follows a power law. We construct the scale-free network with the Barabási-Albert model, which assumes that the probability of a new node attaching to the existing nodes is proportional to their degrees. The initial values of the complex network are given as

$$\begin{cases} x_i(0) = \dfrac{1}{2}(0.3 - 0.01i, -0.3 + 0.01i, 0.2 - 0.01i, 0.4 - 0.01i)^{\mathrm{T}}, \\ k_i(0) = \dfrac{1}{2}(0.01i, 0.02i, 0.03i, 0.031i)^{\mathrm{T}}, \\ q_i(0) = \dfrac{1}{2}(0.41 - 0.01i, -0.4 + 0.01i, 0.3 - 0.01i, 0.5 - 0.01i, -0.3 + 0.01i)^{\mathrm{T}}, \end{cases}$$

where $i = 1, 2, \cdots, N$.

Figure 4 shows that the error system converges to zero with the errors defined as $\|E_l\| = \frac{1}{N}\sqrt{\sum\limits_{i=1}^{N} e_{il}^2}$ $(l = 1, 2, 3, 4)$.

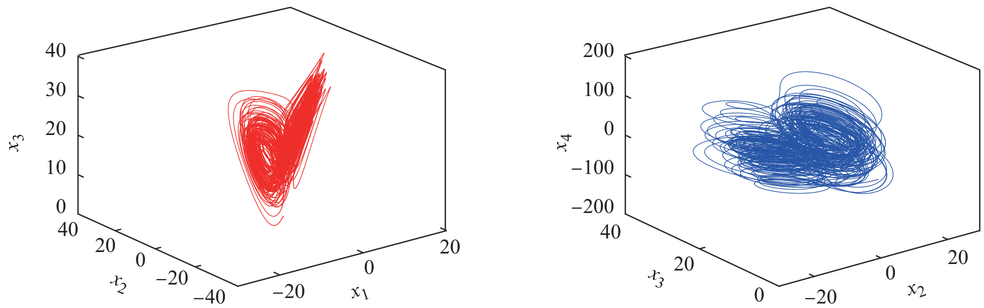

Fig. 3 Hyperchaotic attractor of the fractional-order Chen system with order $\alpha = 0.98$

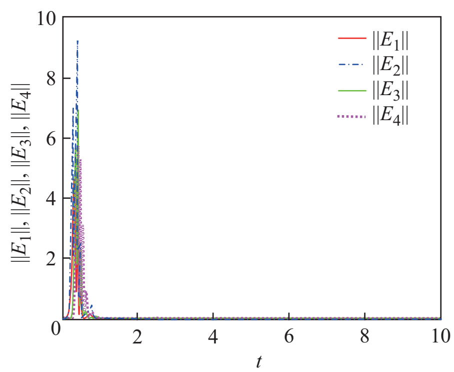

Fig. 4 Time history of synchronization errors of all nodes in the scale-free network of fractional-order hyperchaotic Chen systems with order $\alpha = 0.98$

5 Conclusions

In this work, synchronization of a fractional-order complex network is discussed. By means of the Lyapunov stability analysis, we introduce a fractional-order adaptive feedback controller and show that tracking errors of all nodes in a dynamic network converge to zero as time goes to infinity. In comparison with the existing methods, the assumptions of the coupling configuration are relaxed without requiring diffusivity, symmetry, balance, or irreducibility. Besides, we address synchronization between all nodes in a fractional-order dynamic network instead of synchronization between two complex networks, which is more meaningful and has more practical applications in physics and engineering. Moreover, the proposed scheme can be used to many networks such as small-world networks and scale-free networks. The simulation results of the small-world network of fractional-order Lorenz systems and the scale-free network of fractional-order hyperchaotic Chen systems are presented to illustrate effectiveness of the proposed method.

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