Appl. Math. Mech. -Engl. Ed., **36**(12), 1639–1650 (2015)
DOI 10.1007/s10483-015-2009-6

Applied Mathematics and Mechanics (English Edition)

Stability and boundary equilibrium bifurcations of modified Chua's circuit with smooth degree of 3*

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Abstract Chua's circuit is a well-known nonlinear electronic model, having complicated nonsmooth dynamic behaviors. The stability and boundary equilibrium bifurcations for a modified Chua's circuit system with the smooth degree of 3 are studied. The parametric areas of stability are specified in detail. It is found that the bifurcation graphs of the supercritical and irregular pitchfork bifurcations are similar to those of the piecewise-smooth continuous (PWSC) systems caused by piecewise smoothness. However, the bifurcation graph of the supercritical Hopf bifurcation is similar to those of smooth systems. Therefore, the boundary equilibrium bifurcations of the non-smooth systems with the smooth degree of 3 should receive more attention due to their special features.

Key words modified Chua's circuit, boundary equilibrium point, stability, bifurcation

Chinese Library Classification O361.3
2010 Mathematics Subject Classification 34F10, 39A28

1 Introduction

Chua's circuit is a well-known electronic oscillator model, having complicated nonlinear dynamic behaviors. It is a typical non-smooth system with an absolute value term. Over years, some authors mainly pay attention to the dynamics of Chua's circuit, and have obtained many achievements[1–5]. Freire et al.[1] found a possible degeneration of the focus-center-limit cycle bifurcation. Dana et al.[2] reported some experimental results of the Shil'nikov-type homoclinic chaos in asymmetry-induced Chua's oscillators. Zhang and Bi[3] observed that the trajectories of Chua's circuit passed across both the two switching boundaries, and predicted the occurrence of discontinuous bifurcations. Chua's circuit also has several modified mathematical models. Tang et al.[6] introduced a modified Chua's circuit with the piecewise-smooth quadratic function $x|x|$. Tang and Wang[7] investigated the adaptive active control problem of the modified Chua's circuit introduced in Ref. [6].

The bifurcation theory is very important in understanding the qualitative change in the dynamical behavior. The bifurcations of smooth dynamical systems, usually called the classical bifurcations, are well developed, and can be treated by analytical or topological approaches[8–11].

∗ Received Jan. 31, 2014 / Revised Jun. 24, 2015
Project supported by the National Natural Science Foundation of China (Nos. U1204106, 11372282, 11272024, and 11371046) and the National Basic Research Program of China (973 Program) (Nos. 2012CB821200 and 2012CB821202)
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For non-smooth dynamical systems, stresses are laid on non-smooth bifurcations or discontinuity-induced bifurcations. For piecewise-smooth continuous (PWSC) systems, Leine et al.[12] pointed that non-smooth bifurcations of an equilibrium point occurred when the eigenvalues were set-valued. Di Bernardo and Budd[13] claimed that a boundary equilibrium bifurcation occurred when the left and right Jacobian matrices were unequal at the boundary equilibrium point.

In recent years, much attention in the research on non-smooth dynamical systems has been directed towards impact systems, Filippov systems (e.g., dry friction systems), and PWSC systems. Di Bernardo and Budd[13], Di Bernardo and Hogan[14], and Di Bernardo et al.[15] defined the smooth degree of an equilibrium point for the classification of nonsmooth systems, and pointed that the smooth degree was equal to zero for impact systems, one for Filippov systems, and two for PWSC systems. In fact, there still exist the systems with the smooth degree of 3. The modified Chua's circuit with the function $x|x|$ has the smooth degree of 3. Tang et al.[6] briefly analyzed the classical bifurcations of this model for some parameters. However, there is no research on the bifurcations of the boundary equilibrium points locating on the switching interfaces. Therefore, it is necessary to explore the boundary equilibrium bifurcations of this modified Chua's circuit in greater depth and breadth.

The organization of this paper is given as follows. In Section 2, we introduce some concepts about the boundary equilibrium bifurcation and the smooth degree. In Sections 3 and 4, we investigate the stability and bifurcations of some boundary equilibrium points of the modified Chua's circuit, respectively. Finally, the conclusions are drawn in Section 5.

2 Preliminaries

Consider the following piecewise smooth system with the parameter μ:

$$\dot{X} = f(X, \mu) = \begin{cases} f_1(X, \mu), & h(X) \geqslant 0, \\ f_2(X, \mu), & h(X) < 0, \end{cases} \tag{1}$$

where $X \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ is the parameter, $f_1, f_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}$. Let

$$\begin{cases} v_- = \{(x_1 \cdots x_n) | h(X) < 0\}, \\ \Sigma = \{(x_1 \cdots x_n) | h(X) = 0\}, \\ v_+ = \{(x_1 \cdots x_n) | h(X) > 0\}, \end{cases}$$

where Σ is the switching boundary. It is assumed that f is first-order differentiable at the boundary point, i.e., if X^* is a boundary point satisfying $h(X^*) = 0$, then

$$f(X^*, \mu) = f_1|_{X=X^*} = f_2|_{X=X^*},$$

and the Jacobian matrix J can be defined by

$$J(X^*, \mu) = \frac{\partial f_1}{\partial X}\Big|_{X=X^*} = \frac{\partial f_2}{\partial X}\Big|_{X=X^*}.$$

Definition 1 *A point X^* is an admissible equilibrium point of* (1) *if $f_1|_{X=X^*} = 0$, $h(X^*) > 0$ or $f_2|_{X=X^*} = 0$, $h(X^*) < 0$. It is a boundary equilibrium point if $f_1|_{X=X^*} = 0$, $f_2|_{X=X^*} = 0$, and $h(X^*) = 0$.*

Assume that both the vector fields f_1 and f_2 are defined over the entire local region of the phase space under consideration, i.e., on both sides of Σ. Thus, the flows φ_i $(i = 1, 2)$ generated by each of the vector fields can be defined as the quantities satisfying

$$\frac{\partial \varphi_i(X, t)}{\partial t} = f_i(\varphi_i(X, t)), \quad \varphi_i(X, 0) = X.$$

Here, we assume that such flows can be expanded as a Taylor series about the switching manifold.

Definition 2[14] *The smoothness degree of a system at a boundary point X^* is equal to r if the Taylor series expansions of $\varphi_1(X^*, t)$ and $\varphi_2(X^*, t)$ with respect to t, evaluated at $t = 0$, agree up to the terms of $o(t^{r-1})$. That is, the first non-zero partial derivative with respect to t of the difference $(\varphi_1(X^*, t) - \varphi_2(X^*, t))|_{t=0}$ is of the order r.*

Taking account of that

$$f_1|_{X=X^*} = f_2|_{X=X^*}, \quad \frac{\partial f_1}{\partial X}\Big|_{X=X^*} = \frac{\partial f_2}{\partial X}\Big|_{X=X^*},$$

we have

$$\frac{\partial^2 \varphi_1(X,t)}{\partial t^2}\Big|_{X=X^*} = \frac{\partial^2 \varphi_2(X,t)}{\partial t^2}\Big|_{X=X^*},$$

and (1) has the smooth degree of 3 at X^* by the above definition. However, we cannot investigate the boundary equilibrium bifurcation of (1) by the classical bifurcation theory, where the vector field of the smooth systems is sufficiently differentiable everywhere. Although there is a distinguishable difference between (1) and the smooth systems, some relations between them still exist. They have the well-defined Jacobian matrix everywhere. Therefore, it is possible to define the boundary equilibrium bifurcations of (1) by means of the eigenvalues of the Jacobian matrix.

Definition 3 *The piecewise smooth system* (1) *may undergo a bifurcation at a boundary equilibrium point $X = X^*$ if there exists $\mu = \mu^*$ such that*

(i) $f(X^*, \mu^*) = 0$,

(ii) $h(X^*, \mu^*) = 0$,

(iii) $\mathrm{Re}(\mathrm{eig}(J(X^*, \mu^*))) = 0$, *i.e., the real parts of the eigenvalues of the Jacobian matrix $J(X^*, \mu^*)$ are zero.*

There may exist various boundary equilibrium bifurcations for system (1). This definition is the necessary condition, and is somewhat similar to the classical equilibrium bifurcation of smooth systems. Nevertheless, (1) is not higher order differentiable, and the sufficient conditions for the equilibrium bifurcation of smooth systems are not applicable here. Therefore, we will see later that there are different features in the boundary equilibrium bifurcation of (1) due to the smoothness of the function f at the switching boundary.

3 Stability of boundary equilibrium points of modified Chua's circuit

The modified Chua's circuit is described by[6–7]

$$\dot{x} = \alpha(y - g(x)), \quad \dot{y} = x - y + z, \quad \dot{z} = -\beta y, \tag{2}$$

where b and a are parameters, and

$$g(x) = ax + bx|x|, \quad \alpha > 0, \quad \beta > 0, \quad b > 0.$$

At this time, we have

$$v_- = \{(x, y, z)|h(x) = x < 0\}, \quad \Sigma = \{(x, y, z)|h(x) = x = 0\}, \quad v_+ = \{(x, y, z)|h(x) = x > 0\}.$$

Taking account of the term $x|x|$, we can see that $g(x)$ is first-order differentiable at $x = 0$, and the modified Chua's circuit has the smooth degree of 3.

Note that (2) has only one boundary equilibrium point $E_0 = (0\,0\,0)$ for $a \geqslant 0$, while it has 3 equilibrium points for $a < 0$, including a boundary equilibrium point E_0 and two admissible equilibrium points

$$E_- = \Big(\frac{a}{b} \quad 0 \quad -\frac{a}{b}\Big), \quad E_+ = \Big(-\frac{a}{b} \quad 0 \quad \frac{a}{b}\Big).$$

In what follows, we will analyze the stability of the equilibrium points first. Let

$$\begin{cases} a_1 = \dfrac{1-\alpha+\sqrt{(1+\alpha)^2-4\beta}}{2\alpha}, \quad a_2 = \dfrac{1-\alpha-\sqrt{(1+\alpha)^2-4\beta}}{2\alpha}, \\ a_3 = \dfrac{\alpha-1+\sqrt{(1+\alpha)^2-4\beta}}{2\alpha}, \quad a_4 = \dfrac{\alpha-1-\sqrt{(1+\alpha)^2-4\beta}}{2\alpha}. \end{cases}$$

Two cases of the stability of equilibrium points of the modified Chua's circuit (2) are considered for negative a and non-negative a, respectively.

Theorem 1 *Assume that $a < 0$. The boundary equilibrium point E_0 is unstable. The other two equilibrium points $E_\mp$ are asymptotically stable if one of the following conditions holds*:

$$\text{(C1)} \quad \alpha > 1, \quad \beta > \frac{(1+\alpha)^2}{4}, \tag{3}$$

$$\text{(C2)} \quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a < a_2, \tag{4}$$

$$\text{(C3)} \quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a > a_1, \tag{5}$$

$$\text{(C4)} \quad \alpha > 1, \quad 0 < \beta \leqslant \alpha, \quad a < a_2, \tag{6}$$

$$\text{(C5)} \quad 0 < \alpha \leqslant 1, \quad \beta \geqslant \alpha, \tag{7}$$

$$\text{(C6)} \quad 0 < \alpha \leqslant 1, \quad 0 < \beta < \alpha, \quad a \neq a_2, \tag{8}$$

while $E_\mp$ are unstable if one of the following conditions holds:

$$\text{(C7)} \quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a_2 < a < a_1, \tag{9}$$

$$\text{(C8)} \quad \alpha > 1, \quad 0 < \beta \leqslant \alpha, \quad a > a_2, \tag{10}$$

$$\text{(C9)} \quad \alpha \leqslant 1, \quad 0 < \beta < \alpha, \quad a > a_2. \tag{11}$$

Proof First, we consider the stability of the boundary equilibrium point E_0. The Jacobian matrix at E_0 is

$$J_0 = \begin{pmatrix} -a\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix},$$

which results in the characteristic polynomial

$$p_0(\lambda) = \lambda^3 + (1+a\alpha)\lambda^2 + (\beta-\alpha+a\alpha)\lambda + \alpha\beta a = 0.$$

Its roots have non-negative real parts by the Routh-Hurwitz criterion. Assume that $\alpha\beta a < 0$ for $\alpha > 0$, $\beta > 0$, and $a < 0$. Then, all roots should be nonzero. Moreover, the polynomial $P_0(\lambda)$

is of the degree 3, and has at least one real root. Therefore, $P_0(\lambda)$ has at least one positive real root, and E_0 is unstable.

Next, we consider the asymptotic stability of the boundary equilibrium points $E_\pm$. The Jacobian matrix at $E_\pm$ can be written as follows:

$$J_\pm = \begin{pmatrix} a\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix},$$

which leads to the characteristic polynomial

$$p_\pm(\lambda) = \lambda^3 + (1 - a\alpha)\lambda^2 + (\beta - \alpha - a\alpha)\lambda - \alpha\beta a = 0.$$

Therefore, from the Hurwitz criterion, we can obtain that $E_\pm$ are asymptotically stable if

$$-\alpha\beta a > 0, \quad 1 - \alpha a > 0, \quad \alpha^2 a^2 + (\alpha^2 - \alpha)a - \alpha + \beta > 0.$$

For $\alpha > 0$, $\beta > 0$, and $a < 0$, we certainly have

$$-\alpha\beta a > 0, \quad 1 - \alpha a > 0.$$

Therefore, we only need to prove

$$\alpha^2 a^2 + (\alpha^2 - \alpha)a - \alpha + \beta > 0$$

for Conditions (C1)–(C6). Define two functions as follows:

$$y = \alpha^2 a^2 + (\alpha^2 - \alpha)a - \alpha, \tag{12}$$

$$y = -\beta. \tag{13}$$

In the following, we will obtain that (12) is greater than (13), where (12) is a quadratic function of a, and (13) is a constant function when α and β are constant. Moreover, we will give the conclusions in two cases, i.e., $\alpha > 1$ and $\alpha \leqslant 1$.

For $\alpha > 1$, we present the curves of the functions (12) and (13) in Fig. 1 for different α and β. The peak of the quadratic function (12) is

$$\frac{1-\alpha}{2\alpha} - \frac{(1+\alpha)^2}{4},$$

and the curves of the functions (12) and (13) intersect at

$$a_{1,2} = \frac{1 - \alpha \pm \sqrt{(1+\alpha)^2 - 4\beta}}{2\alpha},$$

which are the roots of

$$\alpha^2 a^2 + (\alpha^2 - \alpha)a - \alpha + \beta = 0.$$

If

$$\beta > \frac{(1+\alpha)^2}{4}, \quad -\beta < -\frac{(1+\alpha)^2}{4},$$

then the curve of (12) locates above that of (13) (see Fig. 1(a)). Therefore, (12) is greater than (13) when

$$\alpha > 1, \quad \beta > \frac{(1+\alpha)^2}{4},$$

i.e., Condition (C1) is satisfied. If

$$\alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a_1 = \frac{1-\alpha+\sqrt{(1+\alpha)^2-4\beta}}{2\alpha}$$

when

$$-((\alpha-1)^2-(1+\alpha)^2+4\beta) = -(\beta-\alpha) < 0$$

and

$$a_2 = \frac{1-\alpha-\sqrt{(1+\alpha)^2-4\beta}}{2\alpha} < 0$$

when $\alpha > 1$, then the relative position of (12) and (13) is shown in Fig. 1(b). Obviously, (12) is greater than (13) when

$$\alpha > 1, \quad \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a < a_2 \quad \text{or} \quad a > a_1,$$

i.e., Conditions (C2) and (C3) are satisfied. If

$$\beta \leqslant \alpha, \quad a_1 = \frac{1-\alpha+\sqrt{(1+\alpha)^2-4\beta}}{2\alpha} > 0$$

when

$$\begin{cases} (1+\alpha)^2-4\beta \geqslant (1+\beta)^2-4\beta = (1-\beta)^2 \geqslant 0, \\ a_2 = \dfrac{1-\alpha-\sqrt{(1+\alpha)^2-4\beta}}{2\alpha} < 0, \end{cases}$$

then the relative position of (12) and (13) is given in Fig. 1(c). It is evident that the curve of (12) is above that of (13) when $a < a_2$ or $a > a_1$. However, we only take care of $a < 0$. Therefore, the cases when $a > a_1 > 0$ are rejected here. Therefore, when Condition (C4) holds, (12) is greater than (13), and $E_{\mp}$ are stable. It is similar to prove that $E_{\pm}$ are stable when Condition (C5) or (C6) holds.

Finally, we consider the instability of the equilibrium points $E_{\pm}$. If Condition (C7) or (C8) holds, we conclude that (12) is equal to or less than (13), i.e.,

$$\alpha^2 a^2 + (\alpha^2-\alpha)a - \alpha + \beta \leqslant 0.$$

Then, we can obtain that the characteristic polynomial $P_{\pm}(\lambda)$ have non-negative real parts by the Routh-Hurwitz criterion. Furthermore, it is shown that $P_{+}(\lambda)$ or $P_{-}(\lambda)$ has at least one positive real root eigenvalue due to $\alpha\beta a < 0$ when $\alpha > 0$, $\beta > 0$, and $a < 0$. Therefore, both $E_{\pm}$ are unstable. It is similar to prove the corresponding result if Condition (C9) holds. The proof is completed.

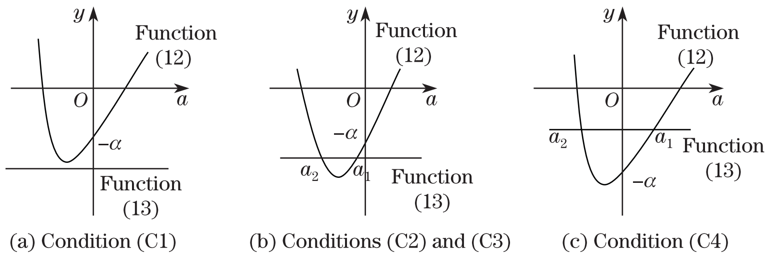

Fig. 1 Plots of functions (12) and (13) when $a < 0$ and $\alpha > 1$

Theorem 2 *Assume that $a \geqslant 0$. The only boundary equilibrium point E_0 is asymptotically stable if one of the following conditions holds*:

$$\text{(D1)}\quad \alpha > 1, \quad \beta > \frac{(1+\alpha)^2}{4}, \tag{14}$$

$$\text{(D2)}\quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a < a_4, \tag{15}$$

$$\text{(D3)}\quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a > a_3, \tag{16}$$

$$\text{(D4)}\quad \alpha > 1, \quad 0 < \beta \leqslant \alpha, \quad a > a_3, \tag{17}$$

$$\text{(D5)}\quad 0 < \alpha \leqslant 1, \quad \beta \geqslant \alpha, \tag{18}$$

$$\text{(D6)}\quad 0 < \alpha \leqslant 1, \quad \beta < \alpha, \quad a > a_3, \tag{19}$$

while is unstable if one of the following conditions holds:

$$\text{(D7)}\quad 1 < \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}, \quad a_4 < a < a_3, \tag{20}$$

$$\text{(D8)}\quad \alpha > 1, \quad 0 < \beta \leqslant \alpha, \quad a < a_3, \tag{21}$$

$$\text{(D9)}\quad 0 < \beta < \alpha \leqslant 1, \quad a < a_3. \tag{22}$$

The proof can be referred to that of Theorem 1.

To facilitate the research, we divide the planar (a, β) into some parts when $\alpha > 1$ and $\alpha \leqslant 1$ according to Theorems 1 and 2. The division is shown in Fig. 2, where

$$D_1 = \left\{(a,\beta)|a > 0, \beta > \frac{(1+\alpha)^2}{4}\right\},$$

$$D_1^{'} = \left\{(a,\beta)|a < 0, \beta > \frac{(1+\alpha)^2}{4}\right\},$$

$$D_2 = \left\{(a,\beta)|0 < a < a_4, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_2^{'} = \left\{(a,\beta)|a_1 < a < 0, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_3 = \left\{(a,\beta)|a_3 < a, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_3' = \left\{(a,\beta)|a < a_2, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_4 = \{(a,\beta)|a_3 < a, 0 < \beta \leqslant \alpha\},$$

$$D_4' = \{(a,\beta)|a < a_2, 0 < \beta \leqslant \alpha\},$$

$$D_5 = \left\{(a,\beta)|a_4 < a < a_3, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_5' = \left\{(a,\beta)|a_2 < a < a_1, \alpha < \beta \leqslant \frac{(1+\alpha)^2}{4}\right\},$$

$$D_6 = \{(a,\beta)|0 < a < a_3, 0 < \beta \leqslant \alpha\},$$

$$D_6' = \{(a,\beta)|a_2 < a < 0, 0 < \beta \leqslant \alpha\},$$

$$D_7 = \{(a,\beta)|0 < a, \alpha \leqslant \beta\},$$

$$D_7' = \{(a,\beta)|a < 0, \alpha \leqslant \beta\},$$

$$D_8 = \{(a,\beta)|a > a_3, \beta < \alpha\},$$

$$D_8' = \{(a,\beta)|a_2 > a, \beta < \alpha\},$$

$$D_9 = \{(a,\beta)|0 < a < a_3, \beta < \alpha\},$$

$$D_9' = \{(a,\beta)|a_1 < a < 0, \beta < \alpha\}.$$

It is noted that one equilibrium point $E_0 = (0\ 0\ 0)$ exists in the first quadrant of Figs. 2(a) and 2(b) when $\alpha > 1$ and $\alpha \leqslant 1$, respectively. It is shown that E_0 is stable in the regions D_1, D_2, D_3, D_4, D_7, and D_8 (dashed regions), while is unstable in the regions D_5, D_6, and D_9. Three equilibrium points $E_0 = (0\ 0\ 0)$ and $E_{\mp} = (\pm\frac{a}{b}\ 0\ \mp\frac{a}{b})$ exist in the second quadrant of Fig. 2, where E_0 is unstable, while $E_{\mp}$ are stable in the regions D_1', D_2', D_3', D_4', D_7', and D_8' (dashed regions) and unstable in the regions D_5', D_6', and D_9'.

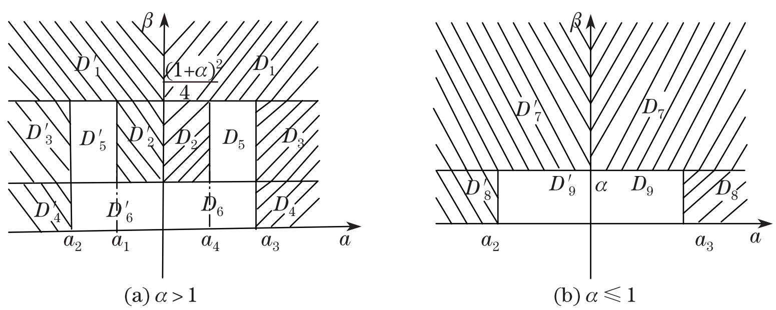

Fig. 2 Partition of (a,β)-plane

4 Boundary equilibrium bifurcations of modified Chua's circuit

Now, we consider several typical cases of the boundary equilibrium bifurcations of (2) as examples. We take a as the bifurcation parameter, and fix the values of other parameters.

Case I Supercritical pitchfork bifurcation at the boundary equilibrium point ($\alpha = 2 > 1$, $\beta = 3 \geqslant \frac{(1+\alpha)^2}{4}$, and $b = 1 > 0$)

When $a = 0$, the Jacobian matrix of E_0 has the following eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = \frac{-1 + \sqrt{1 - 4(\beta - \alpha)}}{2}, \quad \lambda_3 = \frac{-1 - \sqrt{1 - 4(\beta - \alpha)}}{2}.$$

Therefore, a boundary equilibrium bifurcation occurs when $a = 0$. Moreover, according to Theorems 1 and 2 or Fig. 2, there is one stable boundary equilibrium point E_0 when $a > 0$, while E_0 becomes unstable and two stable equilibrium points $E_\mp$ appear when $a < 0$. The bifurcation diagram is shown in Fig. 3. From the graphic structure in the neighborhood of the bifurcation point, it is similar to the supercritical pitchfork bifurcation in the PWSC systems due to the existence of the piecewise smooth function $x|x|$.

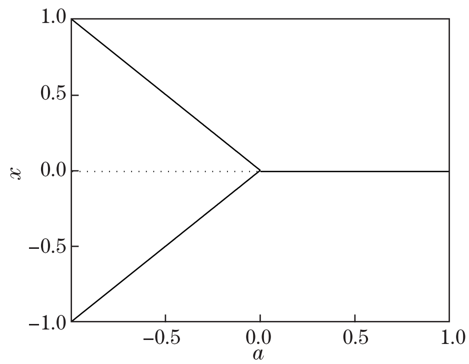

Fig. 3 Supercritical pitchfork bifurcation of boundary equilibrium point when $\alpha = 2$, $\beta = 3$, $b = 1$, and $a \in [-1, 1]$

Case II Irregular pitchfork bifurcation at the boundary equilibrium point ($\alpha = 2 > 1$, $\beta = 1.5 < \alpha$, and $b = 1 > 0$)

Similarly, a boundary equilibrium bifurcation occurs when $a = 0$. At this time, we have

$$a_2 = \frac{1 - \alpha - \sqrt{(1+\alpha)^2 - 4\beta}}{2\alpha} \approx -0.683, \quad a_3 = \frac{\alpha - 1 + \sqrt{(1+\alpha)^2 - 4\beta}}{2\alpha} \approx 0.683.$$

According to Theorems 1 and 2 or Fig. 2, the boundary equilibrium point E_0 is always unstable, and there are also two unstable equilibrium points $E_\mp$ when $a < 0$. The bifurcation diagram is shown in Fig. 4. As shown in the figure, all bifurcating branches are unstable. Therefore, it is called an irregular pitchfork bifurcation. This can be observed in the PWSC systems sometimes, but cannot be observed in smooth systems.

Remark 1 As we know, there is only one boundary equilibrium point when $a > 0$, but there are three equilibrium points E_0 and $E_\pm$ when $a < 0$. Therefore, $a = 0$ must be a pitchfork bifurcation point. There are several types of pitchfork bifurcations, which are determined by the stability of the equilibrium branches and depend on the values of the parameters α, β, and b.

Case III Hopf bifurcation at the boundary equilibrium point

The classical Hopf bifurcation occurs when a pair of complex eigenvalues of the Jacobian matrix cross the imaginary axis and a periodic orbit are generically created[16–17]. A similar result can be observed in the boundary equilibrium bifurcations of the modified Chua's circuit (2). However, the criterion for the classical Hopf bifurcation of smooth systems cannot be used since (2) is not a third-order differentiable system. Even so, we are still able to show the Hopf

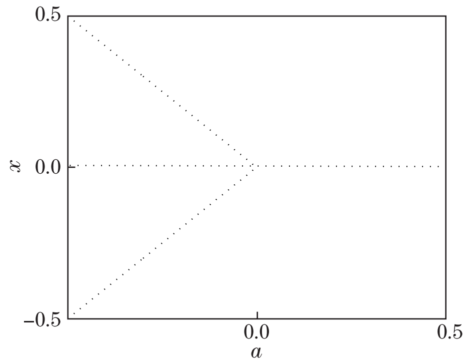

Fig. 4 Irregular pitchfork bifurcation of boundary equilibrium point when $\alpha = 2$, $\beta = 1.5$, $b = 1$, and $a \in [-0.5, 0.5]$

bifurcation of the boundary equilibrium points accompanied by the occurrence of the periodic solutions.

We fix α, β, and b, and take a as the bifurcation parameter. From the characteristic polynomial of the boundary point E_0, we have

$$p_0(\lambda) = \lambda^3 + (1 + a\alpha)\lambda^2 + (\beta - \alpha + a\alpha)\lambda + \alpha\beta a = 0.$$

If the parameter value a^* satisfies

$$\begin{cases} \beta - \alpha + \alpha a^* > 0, \quad \alpha\beta a^* > 0, \quad (1 + \alpha a^*)(\beta - \alpha + \alpha a^*) - \alpha\beta a^* = 0, \\ \alpha(2\alpha a^* - \alpha + 1) \neq 0, \end{cases} \tag{23}$$

then the Hopf bifurcation at the boundary point E_0 may occur when $a = a^*$[18]. The first expression in (23) leads to at least one negative eigenvalue. The second and third conditions are the existence conditions for a pair of pure imaginary eigenvalues, while the final expression ensures that this pair of complex eigenvalues passes transversally across the imaginary axis.

Based on the above analysis and some numerical simulations, we can study the necessary condition for the Hopf bifurcation of the boundary equilibrium point E_0. If

$$\alpha > 1, \quad \alpha < \beta < \frac{(1 + \alpha)^2}{4},$$

then the Hopf bifurcation of the boundary equilibrium point E_0 may occur when $a^* = a_3$ or a_4. If $0 < \beta < \alpha$, then the Hopf bifurcation may only occur when $a^* = a_3$. The proof is presented in what follows.

If

$$\alpha > 1, \quad \alpha < \beta < \frac{(1 + \alpha)^2}{4},$$

we have

$$a_3 > 0, \quad a_4 > 0.$$

Therefore,

$$\alpha\beta a_3 > 0, \quad \alpha\beta a_4 > 0.$$

Furthermore, we have

$$\beta - \alpha + \alpha a_{3,4} > 0, \quad \alpha(2\alpha a_{3,4} - \alpha + 1) = \pm\alpha\sqrt{(1+\alpha)^2 - 4\beta} \neq 0.$$

Obviously, we have

$$(1 + \alpha a^*)(\beta - \alpha + \alpha a^*) - \alpha\beta a^* = \alpha^2 a^{*2} + (\alpha - \alpha^2)a^* - \alpha + \beta = 0$$

with the roots a_3 and a_4. Therefore, (23) holds. This indicates that the Hopf bifurcations of the boundary equilibrium point E_0 may occur when $a^* = a_3$ or a_4. If $0 < \beta < \alpha$, we have $a_3 > 0$ and $a_4 < 0$. Therefore, (23) is satisfied only when $a^* = a_3$. Therefore, the Hopf bifurcation of E_0 may occur only when $a^* = a_3$.

It is noted that the above condition is only the necessary condition, under which the Hopf bifurcation at E_0 may occur. At this time, we still need to consider the existence of the periodic solution by means of other methods (such as numerical simulation) to ensure the occurrence of the Hopf bifurcation. Furthermore, we will investigate the structure of the Hopf bifurcation of E_0. We take

$$\alpha = 2 > 1, \quad b = 2.1, \quad \alpha < \beta = 2.1 < \frac{(1+\alpha)^2}{4},$$

where

$$a_3 \approx 0.443\,6, \quad a_4 \approx 0.056\,4.$$

Now, we let a vary from 0.01 to 0.1 or from 0.6 to 0.3. If $a_3 < a < 0.6$, the system has only one stable boundary equilibrium point E_0 by Theorem 2 or Fig. 2. When a decreases from a_3 to 0.3, E_0 becomes unstable, and a limit cycle can be found (see Fig. 5).

Similarly, we can obtain the same bifurcation structure at a_4 when a varies from 0.01 to 0.1. Because the boundary equilibrium point E_0 changes the stability, the bifurcation is supercritical Hopf, which is similar to that of the smooth system.

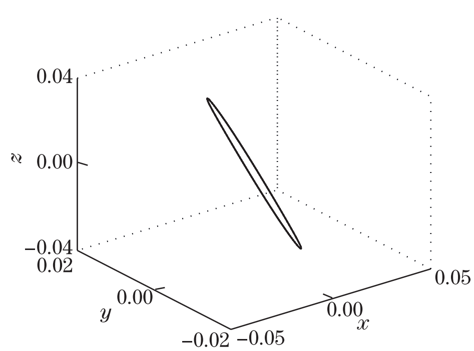

Fig. 5 Limit cycle in modified Chua's circuit when $\alpha = 2$, $\beta = 2.1$, $b = 2.1$, and $a = 0.4$

5 Conclusions

In the studies of stability and bifurcations of dynamical systems, smooth and PWSC systems are mostly concerned. The modified Chua's circuit with the function $x|x|$ differs from usual PWSC and smooth systems in some aspects. The vector field is piecewise smooth but up to first-order differentiable at the boundary point on the switching interface. Besides, the system has the smooth degree of 3 at the boundary equilibrium point. Therefore, some new phenomena in the boundary equilibrium bifurcations occur due to the smoothness property at the switching boundary. In this paper, we mainly investigate the stability and boundary equilibrium bifurcations of the modified Chua's circuit. It is found that the bifurcation graphs of the supercritical and irregular pitchfork bifurcations caused by piecewise smoothness are similar to those of the PWSC systems. However, the bifurcation graph of the supercritical Hopf bifurcation is similar to those of the smooth systems. More complicated dynamic phenomena in this system, including periodic and global dynamic behaviors, remain to be studied in the future.

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