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Three-dimensional elastostatic solutions for transversely isotropic functionally graded material plates containing elastic inclusion*

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Abstract Based on the generalized England-Spencer plate theory, the equilibrium of a transversely isotropic functionally graded plate containing an elastic inclusion is studied. The general solutions of the governing equations are expressed by four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$, and $\psi(\zeta)$ when no transverse forces are acting on the surfaces of the plate. Axisymmetric problems of a functionally graded circular plate and an infinite functionally graded plate containing a circular hole subject to loads applied on the cylindrical boundaries of the plate are firstly investigated. On this basis, the three-dimensional (3D) elasticity solutions are then obtained for a functionally graded infinite plate containing an elastic circular inclusion. When the material is degenerated into the homogeneous one, the present elasticity solutions are exactly the same as the ones obtained based on the plane stress elasticity, thus validating the present analysis in a certain sense.

Key words functionally graded plate, elastic circular inclusion, elasticity solution

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1 Introduction

Elastic response analysis of homogeneous plates inserted with an elastic circular inclusion is always one of important classical research topics in elasticity. For example, Muskhelishvili[1], Savin[2], and Lekhnitskii[3] obtained a series of plane elasticity solutions and classical plate theory solutions by the complex variable method.

Functionally graded materials (FGMs) are a new type of inhomogeneous materials which can ensure the continuous distribution of stresses and thus avoid some problems appearing in conventional laminated materials, such as stress concentration and interfacial debonding. Therefore, FGMs have shown significant application prospects in many fields. Many theoretical studies have been carried out on static analysis of FGM plates. For instance, Ramirez et al.[4] obtained approximate static solutions for two types of FGM plates by a discrete layer model in

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conjunction with the Ritz method. Li et al.[5] derived the elasticity solutions for transversely isotropic FGM circular plates subject to axisymmetric transverse loads. Based on the fourth-order shear deformation plate theory, the axisymmetric bending of FGM circular plates subject to a uniform transverse load was studied by Sahraee and Saidi[6]. More recent works on FGM plate theories and their applications can be found in the review paper of Jha et al.[7].

It is noted that based on the three-dimensional (3D) theory of elasticity, Mian and Spencer[8] obtained a class of 3D solutions for isotropic FGM plates with tractions-free surfaces, in which the material properties were assumed to vary arbitrarily with the thickness-coordinate. Using the complex function theory, England[9] generalized Mian and Spencer's method[8] to the case involving the effect of the top-surface pressure, which satisfies the biharmonic equation or higher-order ones. Hereinafter, this complex formulation will be referred to as the England-Spencer plate theory. With this formulation, England[10] studied the equilibrium problem of an isotropic FGM annular plate containing a rigid inclusion. Yang et al.[11–12] extended England's method[9] to the transversely isotropic FGM plates and obtained the elasticity solutions of an FGM rectangular plate with opposite edges simply supported and subject to a special family of biharmonic polynomial loads (totally 12 different types) as well as those of an FGM annular plate subject to biharmonic loads under different boundary conditions.

To the best knowledge of the authors, no work is available in the literature on the subject of the equilibrium problem of an FGM plate containing an elastic circular inclusion. The purpose of this paper is to investigate 3D equilibrium problems of a transversely isotropic FGM plate containing an elastic inclusion based on the authors' previous work[11–12].

2 England-Spencer plate theory

Consider a transversely isotropic FGM plate bounded by the planes $z = \pm h/2$ in the Cartesian coordinates $(x,\ y,\ z)$. The isotropic planes of the material are parallel with the xy-plane that coincides with the mid-plane of the plate and are perpendicular to the z-axis that is vertically upward. The plate is free from the shear tractions on the upper and lower surfaces and subject to a normal biharmonic pressure $p\,(x, y)$ only on the upper surface. Thus, we have

$$\sigma_z = -p(x,y), \quad \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at} \quad z = h/2,$$
$$\sigma_z = \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at} \quad z = -h/2.$$

By generalizing the England-Spencer plate theory[9], we take the following forms of the displacement field[11–12]:

$$\begin{cases} u + \mathrm{i}v = \overline{u} + \mathrm{i}\overline{v} + 2\dfrac{\partial}{\partial\overline{\zeta}}(R_1\Delta + R_0\overline{w} + R_2\nabla^2\overline{w} + R_3\nabla^4\overline{w} + R_4\nabla^6\overline{w}), \\ w = \overline{w} + T_1\Delta + T_2\nabla^2\overline{w} + T_3\nabla^4\overline{w} + T_4\nabla^6\overline{w}, \end{cases} \tag{1}$$

where $R_0, R_1, \cdots, R_4$ and $T_1, T_2, \cdots, T_4$ are functions of z, $\overline{u} = \overline{u}(x,y)$, $\overline{v} = \overline{v}(x,y)$, and $\overline{w} = \overline{w}(x,y)$ are the mid-plane displacements, and

$$\begin{cases} \Delta = \overline{u}_{,x} + \overline{v}_{,y}, \quad \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}, \quad \zeta = x + \mathrm{i}y, \quad \overline{\zeta} = x - \mathrm{i}y, \\ 2\dfrac{\partial}{\partial\zeta} = \dfrac{\partial}{\partial x} - \mathrm{i}\dfrac{\partial}{\partial y}, \quad 2\dfrac{\partial}{\partial\overline{\zeta}} = \dfrac{\partial}{\partial x} + \mathrm{i}\dfrac{\partial}{\partial y}. \end{cases} \tag{2}$$

The expressions of functions $R_j\ (j = 0, 1, \cdots, 4)$ and $T_k\ (k = 1, 2, \cdots, 4)$ can be determined by invoking the stress boundary conditions on the upper and lower surfaces of the plate, along with the following equations governing $\overline{w}(x,y)$ and $\Omega(x,y)=\overline{v}_{,x}-\overline{u}_{,y}$:

$$\frac{\partial}{\partial\overline{\zeta}}(\kappa_1\Delta+\kappa_2\nabla^2\overline{w}+\kappa_3\nabla^4\overline{w}+\kappa_4\nabla^6\overline{w}+\mathrm{i}\Omega(x,y))=0, \tag{3}$$

$$S_1(h/2)\nabla^4\overline{w}=-p(x,y)+S_{21}\nabla^2 p(x,y), \tag{4}$$

where the expressions of constants κ_1, κ_2, κ_3, κ_4, $S_1(h/2)$, and S_{21} were given by Yang et al.[11].

If $p(x,\ y)=0$, Eqs. (4), (3), and (1) can be immediately simplified as

$$\begin{cases}\nabla^4\overline{w}=0, \quad \dfrac{\partial}{\partial\overline{\zeta}}(\kappa_1\Delta+\kappa_2\nabla^2\overline{w}+\mathrm{i}\Omega(x,y))=0,\\ u+\mathrm{i}v=\overline{u}+\mathrm{i}\overline{v}+2\dfrac{\partial}{\partial\overline{\zeta}}(R_1\Delta+R_0\overline{w}+R_2\nabla^2\overline{w}),\\ w=\overline{w}+T_1\Delta+T_2\nabla^2\overline{w}.\end{cases} \tag{5}$$

It is then shown that the mid-plane displacements and the resultant forces can be expressed as follows:

$$\overline{w}=\alpha(\zeta)+\overline{\alpha(\zeta)}+\overline{\zeta}\beta(\zeta)+\zeta\overline{\beta(\zeta)}, \tag{6}$$

$$D=\overline{u}+\mathrm{i}\overline{v}=\frac{\kappa_1+1}{\kappa_1-1}\phi(\zeta)-\zeta\overline{\phi'(\zeta)}-\overline{\psi(\zeta)}-2\frac{\kappa_2}{\kappa_1}(\beta(\zeta)+\zeta\overline{\beta'(\zeta)}), \tag{7}$$

$$N_x+N_y=a_1(\phi'(\zeta)+\overline{\phi'(\zeta)})+4a_2(\beta'(\zeta)+\overline{\beta'(\zeta)}), \tag{8}$$

$$\begin{aligned}N_y-N_x+2\mathrm{i}N_{xy}=a_1(\overline{\zeta}\phi''(\zeta)+\psi'(\zeta))-a_5\phi'''(\zeta)+4a_2\overline{\zeta}\beta''(\zeta)\\ +2a_6\alpha''(\zeta)-a_7\beta'''(\zeta),\end{aligned} \tag{9}$$

$$M_x+M_y=-b_1(\phi'(\zeta)+\overline{\phi'(\zeta)})+4b_2(\beta'(\zeta)+\overline{\beta'(\zeta)}), \tag{10}$$

$$\begin{aligned}M_y-M_x+2\mathrm{i}M_{xy}=a_6(\overline{\zeta}\phi''(\zeta)+\psi'(\zeta))-b_5\phi'''(\zeta)+b_6\overline{\zeta}\beta''(\zeta)\\ +b_7\alpha''(\zeta)-b_8\beta'''(\zeta),\end{aligned} \tag{11}$$

$$Q_{xz}-\mathrm{i}Q_{yz}=-(b_1+a_6)\phi''(\zeta)+(4b_2-b_6)\beta''(\zeta), \tag{12}$$

where $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$, and $\psi(\zeta)$ are four analytic functions of the complex variable ζ, and a_1, a_2, a_5, a_6, a_7, b_1, b_2, b_5, b_6, b_7, and b_8 are real constants that can be found in Ref. [11].

3 FGM annular plate

Let us consider an FGM annular plate with the inner radius a and the outer radius b, subject to the uniform radial forces N_r^a, N_r^b and the uniform bending moments M_r^a, M_r^b acting on the inner and outer cylindrical boundaries of the plate. This is an axisymmetric problem. We assume that the four analytic functions take the following simple forms:

$$\phi(\zeta)=\dot{\phi_1}\zeta, \quad \beta(\zeta)=\dot{\beta_1}\zeta, \quad \alpha(\zeta)=A\ln\zeta, \quad \psi(\zeta)=\dot{\psi}_{-1}\zeta^{-1}, \tag{13}$$

where $\phi^{\cdot}_1$, $\beta^{\cdot}_1$, A, and $\psi^{\cdot}_{-1}$ are real constants to be determined. Substituting Eq. (13) into Eqs. (6) and (7) yields

$$\overline{w} = 2A\ln r + 2\beta^{\cdot}_1 r^2, \tag{14}$$

$$\begin{aligned}\overline{u}_r + \mathrm{i}\overline{u}_\theta &= (\overline{u} + \mathrm{i}\overline{v})\mathrm{e}^{-\mathrm{i}\theta}\\ &= 2\Big(\frac{1}{\kappa_1 - 1}\phi^{\cdot}_1 - 2\frac{\kappa_2}{\kappa_1}\beta^{\cdot}_1\Big)r - \psi^{\cdot}_{-1}\frac{1}{r}.\end{aligned} \tag{15}$$

It can be found from Eq. (15) that

$$\begin{cases}\overline{u}_r = 2\Big(\dfrac{1}{\kappa_1 - 1}\phi^{\cdot}_1 - 2\dfrac{\kappa_2}{\kappa_1}\beta^{\cdot}_1\Big)r - \psi^{\cdot}_{-1}\dfrac{1}{r},\\ \overline{u}_\theta = 0.\end{cases} \tag{16}$$

It is shown from Eqs. (14) and (16) that the four analytic functions in Eq. (13) do give rise to a state of axisymmetric deformation. Substituting Eq. (13) into Eqs. (8)–(10) and (12) yields

$$N_r + N_\theta = N_x + N_y = 2a_1\phi^{\cdot}_1 + 8a_2\beta^{\cdot}_1, \tag{17}$$

$$M_r + M_\theta = M_x + M_y = -2b_1\phi^{\cdot}_1 + 8b_2\beta^{\cdot}_1, \tag{18}$$

$$\begin{aligned}N_\theta - N_r + 2\mathrm{i}N_{r\theta} &= (N_y - N_x + 2\mathrm{i}N_{xy})\mathrm{e}^{\mathrm{i}2\theta}\\ &= -(a_1\psi^{\cdot}_{-1} + 2a_6A)\frac{1}{r^2},\end{aligned} \tag{19}$$

$$Q_{rz} - \mathrm{i}Q_{\theta z} = (Q_{xz} - \mathrm{i}Q_{yz})\mathrm{e}^{\mathrm{i}\theta} = 0. \tag{20}$$

We obtain from Eqs. (17)–(19) that

$$N_r = a_1\phi^{\cdot}_1 + 4a_2\beta^{\cdot}_1 + (a_1\psi^{\cdot}_{-1} + 2a_6A)\frac{1}{2r^2}, \tag{21}$$

$$\begin{cases}N_\theta = a_1\phi^{\cdot}_1 + 4a_2\beta^{\cdot}_1 - (a_1\psi^{\cdot}_{-1} + 2a_6A)\dfrac{1}{2r^2},\\ N_{r\theta} = 0,\end{cases} \tag{22}$$

$$M_r = -b_1\phi^{\cdot}_1 + 4b_2\beta^{\cdot}_1 + (a_6\psi^{\cdot}_{-1} + b_7A)\frac{1}{2r^2}, \tag{23}$$

$$\begin{cases}M_\theta = -b_1\phi^{\cdot}_1 + 4b_2\beta^{\cdot}_1 - (a_6\psi^{\cdot}_{-1} + b_7A)\dfrac{1}{2r^2},\\ M_{r\theta} = 0.\end{cases} \tag{24}$$

The following conditions are satisfied on the cylindrical boundaries ($r = a,\ b$) of the annular plate:

$$N_r(a) = N_r^a, \quad M_r(a) = M_r^a, \quad N_{r\theta}(a) = 0, \quad Q_{rz}(a) + \frac{\partial M_{r\theta}(a)}{a\partial\theta} = 0, \tag{25}$$

$$N_r(b) = N_r^b, \quad M_r(b) = M_r^b, \quad N_{r\theta}(b) = 0, \quad Q_{rz}(b) + \frac{\partial M_{r\theta}(b)}{b\partial\theta} = 0. \tag{26}$$

Obviously, the last two equations in Eqs. (25) and (26) are automatically met. Substituting Eqs. (21) and (23) into the first two equations in Eqs. (25) and (26) yields

$$a_1\phi_1^{\cdot} + 4a_2\beta_1^{\cdot} + (a_1\psi_{-1}^{\cdot} + 2a_6 A)\frac{1}{2a^2} = N_r^a, \tag{27}$$

$$a_1\phi_1^{\cdot} + 4a_2\beta_1^{\cdot} + (a_1\psi_{-1}^{\cdot} + 2a_6 A)\frac{1}{2b^2} = N_r^b, \tag{28}$$

$$-b_1\phi_1^{\cdot} + 4b_2\beta_1^{\cdot} + (a_6\psi_{-1}^{\cdot} + b_7 A)\frac{1}{2a^2} = M_r^a, \tag{29}$$

$$-b_1\phi_1^{\cdot} + 4b_2\beta_1^{\cdot} + (a_6\psi_{-1}^{\cdot} + b_7 A)\frac{1}{2b^2} = M_r^b. \tag{30}$$

It can be found from Eqs. (27)–(30) that

$$\begin{bmatrix} \phi_1^{\cdot} \\ \beta_1^{\cdot} \end{bmatrix} = \frac{1}{(1-r_0)\,J_1}\begin{bmatrix} 4b_2 & -4a_2 \\ b_1 & a_1 \end{bmatrix}\begin{bmatrix} N_r^b - r_0 N_r^a \\ M_r^b - r_0 M_r^a \end{bmatrix}, \tag{31}$$

$$\begin{bmatrix} \psi_{-1}^{\cdot} \\ A \end{bmatrix} = \frac{2a^2}{(1-r_0)\,J_2}\begin{bmatrix} b_7 & -2a_6 \\ -a_6 & a_1 \end{bmatrix}\begin{bmatrix} N_r^a - N_r^b \\ M_r^a - M_r^b \end{bmatrix}, \tag{32}$$

where

$$r_0 = \frac{a^2}{b^2}, \quad J_1 = 4(a_1 b_2 + b_1 a_2), \quad J_2 = a_1 b_7 - 2a_6^2. \tag{33}$$

By substituting Eqs. (31) and (32) into Eqs. (21)–(24), we obtain

$$N_r = \frac{1}{1-r_0}\Big(N_r^b - r_0 N_r^a + \frac{a^2}{r^2}(N_r^a - N_r^b)\Big), \tag{34}$$

$$N_\theta = \frac{1}{1-r_0}\Big(N_r^b - r_0 N_r^a - \frac{a^2}{r^2}(N_r^a - N_r^b)\Big), \tag{35}$$

$$M_r = \frac{1}{1-r_0}\Big(M_r^b - r_0 M_r^a + \frac{a^2}{r^2}(M_r^a - M_r^b)\Big), \tag{36}$$

$$M_\theta = \frac{1}{1-r_0}\Big(M_r^b - r_0 M_r^a - \frac{a^2}{r^2}(M_r^a - M_r^b)\Big). \tag{37}$$

By substituting Eqs. (31) and (32) into Eqs. (16) and (14), we obtain

$$\begin{bmatrix} \overline{u}_r \\ \overline{w} \end{bmatrix} = \frac{1}{(1-r_0)\,J_1}\begin{bmatrix} 4\Big(\dfrac{2b_2}{\kappa_1-1} - b_1\dfrac{\kappa_2}{\kappa_1}\Big)r & -4\Big(\dfrac{2a_2}{\kappa_1-1} + a_1\dfrac{\kappa_2}{\kappa_1}\Big)r \\ 2b_1 r^2 & 2a_1 r^2 \end{bmatrix}\begin{bmatrix} N_r^b - r_0 N_r^a \\ M_r^b - r_0 M_r^a \end{bmatrix}$$
$$+ \frac{2a^2}{(1-r_0)\,J_2}\begin{bmatrix} -\dfrac{b_7}{r} & \dfrac{2a_6}{r} \\ -2a_6\ln r & 2a_1\ln r \end{bmatrix}\begin{bmatrix} N_r^a - N_r^b \\ M_r^a - M_r^b \end{bmatrix}. \tag{38}$$

4 Three cases

4.1 FGM circular plate

We have $r_0 = 0$ when $a \to 0$. In this case, the annular plate becomes a circular plate with radius b. It can be found from Eqs. (34)–(38) that

$$N_r = N_\theta = N_r^b, \quad M_r = M_\theta = M_r^b, \tag{39}$$

$$\begin{bmatrix} \overline{u}_r \\ \overline{w} \end{bmatrix} = \frac{2}{J_1} \begin{bmatrix} 2\Big(\dfrac{2b_2}{\kappa_1 - 1} - b_1 \dfrac{\kappa_2}{\kappa_1}\Big) r & -2\Big(\dfrac{2a_2}{\kappa_1 - 1} + a_1 \dfrac{\kappa_2}{\kappa_1}\Big) r \\ b_1 r^2 & a_1 r^2 \end{bmatrix} \begin{bmatrix} N_r^b \\ M_r^b \end{bmatrix}. \tag{40}$$

4.2 Infinite FGM plate with circular hole

We also have $r_0 = 0$ when $b \to \infty$. Then, the annular plate becomes an infinite plate with a circular hole of a radius a. Let $N_r^b = 0$ and $M_r^b = 0$. We find from Eqs. (34)–(38) that

$$N_r = \frac{a^2}{r^2} N_r^a = -N_\theta, \quad M_r = \frac{a^2}{r^2} M_r^a = -M_\theta, \tag{41}$$

$$\begin{bmatrix} \overline{u}_r \\ \overline{w} \end{bmatrix} = \frac{2a^2}{J_2} \begin{bmatrix} -\dfrac{b_7}{r} & \dfrac{2a_6}{r} \\ -2a_6 \ln r & 2a_1 \ln r \end{bmatrix} \begin{bmatrix} N_r^a \\ M_r^a \end{bmatrix}. \tag{42}$$

4.3 Infinite FGM plate containing elastic inclusion

To investigate the problem of a plate containing an elastic inclusion, we propose to insert an elastic inclusion of a radius $a + \varepsilon$ into the hole of a radius a in the infinite plate. Here, ε is a small quantity whose order is the same as some allowable displacement of the plate. The friction between the inclusion and the plate is ignored. This is still an axisymmetric problem.

In the following, the physical quantities of the inclusion are distinguished by the superscript (1) from those of the plate without any superscript. Thus, the radius of the elastic inclusion becomes $a + \varepsilon + \overline{u}_r^{(1)}$, and that of the plate is $a + \overline{u}_r$, when the elastic inclusion is inserted into the circular hole of the plate. Therefore,

$$\overline{u}_r - \overline{u}_r^{(1)} = \varepsilon, \tag{43}$$

while the transverse displacement is continuous. Thus,

$$\begin{bmatrix} \overline{u}_r \\ \overline{w} \end{bmatrix} - \begin{bmatrix} \overline{u}_r^{(1)} \\ \overline{w}^{(1)} \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}. \tag{44}$$

There are the resultant forces N_r^0 and M_r^0 acting on the interface between the inclusion and the plate. Let $r = a + \varepsilon$, $N_r^b = N_r^0$, and $M_r^b = M_r^0$ in Eq. (40). Let $r = a$, $N_r^a = N_r^0$, and $M_r^a = M_r^0$ in Eq. (42). Equation (44) can be further expressed as

$$\begin{aligned} &-\frac{2}{J_1^{(1)}} \begin{bmatrix} 2\Big(\dfrac{2b_2^{(1)}}{\kappa_1^{(1)} - 1} - b_1^{(1)} \dfrac{\kappa_2^{(1)}}{\kappa_1^{(1)}}\Big)(a+\varepsilon) & -2\Big(\dfrac{2a_2^{(1)}}{\kappa_1^{(1)} - 1} + a_1^{(1)} \dfrac{\kappa_2^{(1)}}{\kappa_1^{(1)}}\Big)(a+\varepsilon) \\ b_1^{(1)} (a+\varepsilon)^2 & a_1^{(1)} (a+\varepsilon)^2 \end{bmatrix} \begin{bmatrix} N_r^0 \\ M_r^0 \end{bmatrix} \\ &+ \frac{2a^2}{J_2} \begin{bmatrix} -\dfrac{b_7}{a} & \dfrac{2a_6}{a} \\ -2a_6 \ln a & 2a_1 \ln a \end{bmatrix} \begin{bmatrix} N_r^0 \\ M_r^0 \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}. \end{aligned} \tag{45}$$

The resultant forces N_r^0 and M_r^0 can be solved from Eq. (45). Replacing the resultant forces N_r^a and M_r^a with N_r^0 and M_r^0 in Eqs. (41) and (42), respectively, we can obtain the expressions of the mid-plane displacements and the resultant forces of the infinite plate and further the 3D displacement and stress fields.

5 Degeneration analysis

When the material of the elastic inclusion and the plate degenerates to the transversely isotropic homogeneous one, we have

$$a_2 = 0, \quad a_6 = 0, \quad a_7 = 0, \quad b_1 = 0, \quad b_5 = 0, \quad k_2 = 0.$$

Hence,

$$J_1^{(1)} = 4a_1^{(1)} b_2^{(1)}, \quad J_2 = a_1 b_7. \tag{46}$$

If the material is isotropic and homogeneous, we have

$$a_1 = \frac{2Eh}{1+v} = 4Gh, \quad \kappa_1 = \frac{2}{1-v}, \quad \kappa_1 - 1 = \frac{1+v}{1-v}. \tag{47}$$

5.1 Homogeneous circular plate

It can be found from Eq. (40) that

$$\begin{cases} \overline{u}_r^{(1)} = \dfrac{\left(1-v^{(1)}\right) r}{2G^{(1)}\left(1+v^{(1)}\right)} \dfrac{N_r^0}{h}, \\[2ex] \overline{w}^{(1)} = -\dfrac{3\left(1-v^{(1)}\right) M_r^0}{G^{(1)}\left(1+v^{(1)}\right) h^3}. \end{cases} \tag{48}$$

By taking

$$K = \frac{3-v}{1+v}, \quad \mu = G, \quad \frac{N_r^0}{h} = -P,$$

we find that the expression of the radial displacement in Eq. (48) is exactly the same as that obtained by Muskhelishvili[1].

5.2 Infinite homogeneous plate with circular hole

It can be found from Eq. (42) that

$$\begin{cases} \overline{u}_r = -\dfrac{a^2}{2Gr} \dfrac{N_r^0}{h}, \\[2ex] \overline{w} = \dfrac{6a^2 \ln r}{Gh^3} M_r^0. \end{cases} \tag{49}$$

It can also be proved that the radial displacement given by Eq. (49) is identical to that obtained by Muskhelishvili[1] by letting $\mu = G$ and $N_r^0/h = -P$.

5.3 Infinite homogeneous plate containing elastic inclusion

Equation (45) can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} N_r^0 \\ M_r^0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}, \tag{50}$$

in which

$$\begin{cases} a_{11} = -\dfrac{2}{J_1^{(1)}}\left(\dfrac{2b_2^{(1)}}{\kappa_1^{(1)}-1} - b_1^{(1)}\dfrac{\kappa_2^{(1)}}{\kappa_1^{(1)}}\right)(a+\varepsilon) - \dfrac{ab_7}{J_2}, \\ a_{12} = \dfrac{2}{J_1^{(1)}}\left(\dfrac{2a_2^{(1)}}{\kappa_1^{(1)}-1} + a_1^{(1)}\dfrac{\kappa_2^{(1)}}{\kappa_1^{(1)}}\right)(a+\varepsilon) + \dfrac{2aa_6}{J_2}, \\ a_{21} = -\dfrac{b_1^{(1)}}{J_1^{(1)}}(a+\varepsilon)^2 - \dfrac{2a_6}{J_2}a^2\ln a, \\ a_{22} = -\dfrac{a_1^{(1)}}{J_1^{(1)}}(a+\varepsilon)^2 + \dfrac{2a_1}{J_2}a^2\ln a. \end{cases} \tag{51}$$

It can be obtained from Eq. (50) that

$$\begin{bmatrix} N_r^0 \\ M_r^0 \end{bmatrix} = \frac{\varepsilon}{2\left(a_{11}a_{22} - a_{12}a_{21}\right)} \begin{bmatrix} a_{22} \\ -a_{21} \end{bmatrix}. \tag{52}$$

When the material is homogeneous, we have

$$\begin{cases} a_{11} = -\dfrac{1}{a_1^{(1)}(\kappa_1^{(1)}-1)}(a+\varepsilon) - \dfrac{a}{a_1}, \\ a_{12} = 0, \quad a_{21} = 0, \\ a_{22} = -\dfrac{1}{4b_2^{(1)}}\left(a+\varepsilon\right)^2 + \dfrac{2}{b_7}a^2\ln a. \end{cases} \tag{53}$$

Therefore, we arrive at

$$M_r^0 = 0, \quad N_r^0 = \frac{\varepsilon}{2a_{11}}. \tag{54}$$

When the material is transversely isotropic, we have

$$\begin{cases} a_1^{(1)} = 4c_{66}^{(1)}h, \quad a_1 = 4c_{66}h, \quad \kappa_1 = \dfrac{c_1}{c_0}, \\ c_0 = \dfrac{c_{66}}{c_{44}}, \quad c_1 = \left(c_{11} - \dfrac{c_{13}^2}{c_{33}}\right)c_{44}^{-1}, \end{cases} \tag{55}$$

which become for the isotropic materials as follows:

$$\begin{cases} a_1^{(1)} = 4G^{(1)}h, \quad a_1 = 4Gh, \quad \kappa_1^{(1)} = \dfrac{2}{1-v^{(1)}}, \\ \kappa_1^{(1)} - 1 = \dfrac{1+v^{(1)}}{1-v^{(1)}}. \end{cases} \tag{56}$$

Hence, we have

$$2a_{11} = -\frac{(1-v^{(1)})G(a+\varepsilon) + G^{(1)}(1+v^{(1)})a}{2hGG^{(1)}(1+v^{(1)})}. \tag{57}$$

By substituting Eq. (57) into Eq. (54), we obtain

$$N_r^0 = \frac{-2hGG^{(1)}(1+v^{(1)})\varepsilon}{(1-v^{(1)})G(a+\varepsilon)+G^{(1)}(1+v^{(1)})a}. \tag{58}$$

The solution of this problem obtained by Muskhelishvili[1] is as follows:

$$P = \frac{4\mu\mu_0\varepsilon}{2\mu_0 R+\mu(\kappa_0-1)(R+\varepsilon)}. \tag{59}$$

Note that we have $\kappa_0 = (3-v_0)/(1+v_0)$ for the plane stress problem, in which v_0 is Poisson's ratio of inclusion. Therefore, Eq. (59) can be rewritten as

$$P = \frac{2\mu\mu_0(1+v_0)\varepsilon}{\mu_0(1+v_0)R+\mu(1-v_0)(R+\varepsilon)}. \tag{60}$$

Since $G=\mu$, $G^{(1)}=\mu_0$, $v^{(1)}=v_0$, $a=R$, and $N_r^0=-Ph$, we can find that Eq. (58) gives exactly the same result as Eq. (60).

We finally note that Eq. (52) is the solution of an FGM plate, and Eq. (54) corresponds to a homogeneous plate, for which $M_r^0 = 0$, indicating that it is a plane stress problem.

6 Conclusion

Based on a generalization of the England-Spencer plate theory for a transversely isotropic FGM plate, the elasticity solutions of FGM plates subject to loads applied on the cylindrical boundaries of the plates for three different cases are obtained. They include a circular plate, an infinite plate with a circular hole, and an infinite plate containing an elastic inclusion. In the analysis, the material coefficients are allowed to vary arbitrarily in a continuous fashion along the thickness of the plate. The analysis is based on the England-Spencer formulations in terms of four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$, and $\psi(\zeta)$. With these functions, the 3D displacement and stress fields for a boundary-value problem are completely determined.

The obtained analytical solutions for the three considered cases exactly satisfy the equilibrium equations of the plate and the traction boundary conditions on the faces/interfaces of the plate. Approximations are introduced only in the satisfaction of the boundary conditions around the circumferential edge of the plate. The elasticity solutions of a transversely isotropic (or isotropic) and homogeneous plate for the above three cases are also obtained through the degenerate analysis.

Because no simplified hypotheses about the stress and displacement fields are introduced, the present elasticity solutions can serve as a benchmark to access the validity and accuracy of various simplified plate theories or numerical methods that may be used in the analysis of such plates.

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