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# General solutions of plane problem in one-dimensional quasicrystal piezoelectric materials and its application on fracture mechanics*

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**Abstract** Based on the fundamental equations of piezoelasticity of quasicrystals (QCs), with the symmetry operations of point groups, the plane piezoelasticity theory of one-dimensional (1D) QCs with all point groups is investigated systematically. The governing equations of the piezoelasticity problem for 1D QCs including monoclinic QCs, orthorhombic QCs, tetragonal QCs, and hexagonal QCs are deduced rigorously. The general solutions of the piezoelasticity problem for these QCs are derived by the operator method and the complex variable function method. As an application, an antiplane crack problem is further considered by the semi-inverse method, and the closed-form solutions of the phonon, phason, and electric fields near the crack tip are obtained. The path-independent integral derived from the conservation integral equals the energy release rate.

**Key words** quasicrystals (QCs), piezoelasticity, fracture mechanics, crack, complex variable method

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## 1 Introduction

Quasicrystals (QCs) as a new structure of solid matter were discovered first on April 8, 1982, and were first reported by Shechtman et al.[1], who won the Nobel's Prize in 2011. This discovery has brought a significant breakthrough for condensed matter physics in recent years, because QCs possess both quasi-periodic long-range translational symmetry and noncrystallographic rotational symmetry. According to the cut-and-projection method, a three-dimensional (3D) quasilattice can be obtained by the selected projection of the respective six-dimensional (6D) periodical lattice[2–3]. The one-dimensional (1D) (or two-dimensional (2D)) QCs are the ones in which the atomic structures of the materials are quasiperiodic in one direction or two directions, while are periodic in the other two directions or one direction. The 3D QCs show quasiperiodicity in all the three directions. Since the discovery of QCs, great progress has been

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made in the elastic theory for many years[4–8]. To solve the boundary value problems of elasticity for QCs, the governing equations and the general solutions are of great importance. Liu et al.[9] investigated systematically the governing equations of the plane elasticity problems for 1D QCs with all point groups, and obtained the general solutions. With the differential operator matrix method, Chen et al.[10], Wang[11], Wang and Pan[12], and Gao et al.[13] analyzed the general solutions of 3D elastic problems for 1D hexagonal QCs. For the plane elasticity problem of 2D QCs with noncrystal rotational symmetry, Liu et al.[14] presented the general solutions of different point groups including dodecagonal, pentagonal, decagonal, and octagonal systems. By virtue of the operator method, Gao et al.[15] studied a theory of general solutions of the plane problems for 2D octagonal QCs. Gao and Zhao[16] obtained the general solutions of the 3D problems for 2D QCs by introducing the displacement functions and using the operator analysis technique. For the elastic problems of 3D QCs, Fan and Guo[17] derived the final governing equation and the fundamental solution of plane elasticity of icosahedral QCs. Based on the stress potential function, Li and Fan[18] presented the general solution of plane elasticity for icosahedral QCs. Gao and Zhao[19] made a general treatment of 3D elasticity for QCs by the operator method. Gao[20] further simplified the governing equation of cubic QCs by introducing a displacement function, and established the general solutions through an operator method.

QCs are sensitive to mechanical, thermal, electrical, magnetic, and optical effects. The physical properties of QCs have been investigated intensively[21–29]. The independent and non-vanishing first-order piezoelectric, piezomagnetic, pyromagenetic, photoelastic, and magnetoelectric coefficients are obtained[30–32]. The development of QCs, such as the material properties, the theories of elasticity, and some applications, has been addressed[33–36].

Rao et al.[32] studied the electric effects of QCs on the piezoelasticity in QCs. Altay and Dökmeci[37] developed the 3D fundamental equations of piezoelasticity of QCs. As mentioned above, only the elastic problems of QCs have been concerned. Recently, Li et al.[38] addressed the 3D general solutions to static problems of 1D hexagonal piezoelectric QCs by introducing two displacement functions and utilizing the rigorous operator theory. By introducing four potential functions, Zhang et al.[39] obtained the general solutions of the plane problems in 1D orthorhombic QCs with the piezoelectric effect. However, the governing equations of the plane piezoelasticity theory of other 1D QCs with all point groups and the general solutions have not been done up to now. It is well-known that the governing equations and the general solutions play an important role in solving the boundary value problems of the piezoelasticity of QCs, because they not only have theoretical merits themselves, but also test the validity of various approximate methods such as the finite element method and the boundary element method. Meanwhile, they pave the way to the forthcoming study of dislocation, fracture, interface, and similar problems for the piezoelasticity of QCs. Therefore, it is the purpose of this work to investigate systematically the governing equations of the plane piezoelasticity of 1D QCs with all point groups. With the help of the decomposition and superposition principles, the general solutions are derived by the operator method and the complex variable function method.

## 2 Basic equations for piezoelasticity of QCs

In a fixed rectangular coordinate system $x_i$ ($i = 1, 2, 3$), the basic equations for the piezoelasticity of QCs presented by Altay and Dökmeci[37] are as follows. The equilibrium equations are

$$\sigma_{ij,i} = 0, \quad H_{ij,i} = 0, \quad D_{i,i} = 0. \tag{1}$$

The gradient equations are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \omega_{ij} = w_{i,j}, \quad E_i = -\phi_{,i}. \tag{2}$$

The constitutive equations are

$$\begin{cases} \sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}\omega_{kl} - e_{kij}E_k, \\ H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}\omega_{kl} - d_{kij}E_k, \\ D_i = e_{kij}\varepsilon_{jk} + d_{kij}\omega_{jk} + \lambda_{ij}E_j. \end{cases} \tag{3}$$

In the above equations, a comma denotes partial differentiation, and the repeated indices represent summation. $\sigma_{ij}$, $\varepsilon_{ij}$, and $u_i$ are the stress, the strain, and the displacement of the phonon field, respectively. $H_{ij}$, $\omega_{ij}$, and $w_i$ are the stress, the strain, and the displacement of the phason field, respectively. $D_i$, $E_i$, and $\phi$ stand for the electric displacement, the electric field, and the electric potential, respectively. $C_{ijkl}$, $K_{ijkl}$, $R_{ijkl}$, $e_{ijk}$, and $d_{ijk}$ stand for the phonon elastic, the phason elastic, and the phonon-phason coupling moduli, respectively. $\lambda_{ij}$ stands for the dielectric permittivity. The following reciprocal symmetry conditions hold:

$$\begin{cases} C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad R_{ijkl} = R_{jikl}, \quad K_{ijkl} = K_{klij}, \\ e_{ijk} = e_{ikj}, \quad d_{ijk} = d_{ikj}, \quad \lambda_{ij} = \lambda_{ji}. \end{cases} \tag{4}$$

For stable materials, $C_{ijkl}$, $K_{ijkl}$, and $\lambda_{ij}$ satisfy the positive-semidefinite conditions as follows:

$$C_{ijkl}\eta_{ij}\eta_{kl} \geqslant 0, \quad K_{ijkl}\eta_{ij}\eta_{kl} \geqslant 0, \quad \lambda_{ij}\eta_i\eta_j \geqslant 0 \tag{5}$$

for non-zero vector $\eta_i$ and non-zero tensor $\eta_{ij}$.

For the piezoelasticity problems of 1D QCs, there are non-zero phonon displacements $u_x$, $u_y$, and $u_z$, phason displacement $w_z$ ($w_x = w_y = 0$), and electric potential $\phi$. Therefore, the corresponding strains and electric fields are

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z}, \\ \varepsilon_{yz} = \dfrac{1}{2}\Big(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}\Big), \quad \varepsilon_{zx} = \dfrac{1}{2}\Big(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}\Big), \quad \varepsilon_{xy} = \dfrac{1}{2}\Big(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\Big), \\ \omega_{zy} = \dfrac{\partial w_z}{\partial y}, \quad \omega_{zx} = \dfrac{\partial w_z}{\partial x}, \quad \omega_{zz} = \dfrac{\partial w_z}{\partial z}, \\ E_x = -\dfrac{\partial \phi}{\partial x}, \quad E_y = -\dfrac{\partial \phi}{\partial y}, \quad E_z = -\dfrac{\partial \phi}{\partial z}. \end{cases} \tag{6}$$

The above equation holds for the piezoelasticity of all 1D QCs. In this paper, we only consider the piezoelasticity of 1D QCs, because among various QCs, 1D QCs are of particular interest for researchers after the success of Merlin et al.[40] in growing model systems, in which quasiperiodicity was built up.

According to the symmetry operations of point groups[41], for the piezoelasticity of monoclinic QCs with the point group m and the $z$-axis as a symmetry axis, we can obtain the constitutive equations as follows:

$$
\begin{aligned}
&(\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{yz}\ \sigma_{zx}\ \sigma_{xy}\ H_{zz}\ H_{yz}\ H_{zx}\ D_x\ D_y\ D_z)^{\mathrm{T}}\\
&=\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} & R_1 & 0 & 0 & e_{11} & 0 & e_{31}\\
 & C_{22} & C_{23} & 0 & 0 & C_{26} & R_2 & 0 & 0 & e_{12} & 0 & e_{31}\\
 & & C_{33} & 0 & 0 & C_{36} & R_3 & 0 & 0 & e_{13} & 0 & e_{33}\\
 & & & C_{44} & C_{45} & 0 & 0 & R_5 & R_4 & 0 & e_{24} & 0\\
 & & & & C_{55} & 0 & 0 & R_7 & R_6 & e_{15} & 0 & e_{35}\\
 & & & & & C_{66} & R_8 & 0 & 0 & 0 & e_{26} & 0\\
 & & & & & & K_3 & 0 & 0 & d_{13} & 0 & d_{33}\\
 & & \text{Symmetry} & & & & & K_2 & K_4 & 0 & d_{24} & 0\\
 & & & & & & & & K_1 & d_{15} & 0 & d_{35}\\
 & & & & & & & & & -\kappa_{11} & 0 & -\kappa_{13}\\
 & & & & & & & & & & -\kappa_{22} & 0\\
 & & & & & & & & & & & -\kappa_{33}
\end{pmatrix}\\
&\cdot(\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ 2\varepsilon_{yz}\ 2\varepsilon_{zx}\ 2\varepsilon_{xy}\ \omega_{zz}\ \omega_{yz}\ \omega_{zx}\ -E_x\ -E_y\ -E_z)^{\mathrm{T}},
\end{aligned}\tag{7}
$$

where short notations are used for the phonon elastic constant tensors, i.e.,

$$11\to 1,\quad 22\to 2,\quad 33\to 3,\quad 23\to 4,\quad 31\to 5,\quad 12\to 6,$$

and $C_{ijkl}$ is denoted as $C_{pq}$ accordingly. There are 13 independent phonon elastic constants, i.e.,

$$
\begin{cases}
C_{11}=C_{1111},\quad C_{22}=C_{2222},\quad C_{33}=C_{3333},\\
C_{12}=C_{1122},\quad C_{13}=C_{1133},\quad C_{23}=C_{2233},\\
C_{44}=C_{2323},\quad C_{55}=C_{3131},\quad C_{66}=C_{1212},\\
C_{45}=C_{2331},\quad C_{16}=C_{1112},\quad C_{26}=C_{2212},\quad C_{36}=C_{3312};
\end{cases}
$$

4 independent phason elastic constants, i.e.,

$$K_1=K_{3131},\quad K_2=K_{3232},\quad K_3=K_{3333},\quad K_4=K_{3132};$$

8 phonon-phason coupling elastic constants, i.e.,

$$
\begin{cases}
R_1=R_{1133},\quad R_2=R_{2233},\quad R_3=R_{3333},\quad R_4=R_{2331},\\
R_5=R_{2332},\quad R_6=R_{3131},\quad R_7=R_{3132},\quad R_8=R_{1233};
\end{cases}
$$

15 independent piezoelastic constants, i.e.,

$$
\begin{cases}
e_{11}=e_{111},\quad e_{12}=e_{122},\quad e_{13}=e_{133},\quad e_{15}=e_{113},\\
e_{24}=e_{223},\quad e_{26}=e_{212},\quad e_{31}=e_{311},\quad e_{32}=e_{322},\\
e_{33}=e_{333},\quad e_{35}=e_{331},\quad d_{13}=d_{133},\quad d_{33}=d_{333},\\
d_{24}=d_{223},\quad d_{15}=d_{113},\quad d_{35}=d_{331};
\end{cases}
$$

and 4 independent dielectric permittivities, i.e., $\kappa_{11}$, $\kappa_{22}$, $\kappa_{33}$, and $\kappa_{13}$. Thus, for the 1D monoclinic QCs, there are 44 non-zero material constants in total.

From Eq. (7), we can obtain the corresponding stress-strain relations as follows:

$$\begin{cases}\sigma_{xx}=C_{11}\varepsilon_{xx}+C_{12}\varepsilon_{yy}+C_{13}\varepsilon_{zz}+2C_{16}\varepsilon_{xy}+R_1\omega_{zz}-e_{11}E_x-e_{31}E_z,\\ \sigma_{yy}=C_{12}\varepsilon_{xx}+C_{22}\varepsilon_{yy}+C_{23}\varepsilon_{zz}+2C_{26}\varepsilon_{xy}+R_2\omega_{zz}-e_{12}E_x-e_{31}E_z,\\ \sigma_{zz}=C_{13}\varepsilon_{xx}+C_{23}\varepsilon_{yy}+C_{33}\varepsilon_{zz}+2C_{36}\varepsilon_{xy}+R_3\omega_{zz}-e_{13}E_x-e_{33}E_z,\\ \sigma_{zy}=2C_{44}\varepsilon_{zy}+2C_{45}\varepsilon_{zx}+R_5\omega_{zy}+R_4\omega_{zx}-e_{24}E_y,\\ \sigma_{zx}=2C_{45}\varepsilon_{zy}+2C_{55}\varepsilon_{zx}+R_7\omega_{zy}+R_6\omega_{zx}-e_{15}E_x-e_{35}E_z,\\ \sigma_{xy}=C_{16}\varepsilon_{xx}+C_{26}\varepsilon_{yy}+C_{36}\varepsilon_{zz}+2C_{66}\varepsilon_{xy}+R_8\omega_{zz}-e_{26}E_y,\\ H_{zz}=R_1\varepsilon_{xx}+R_2\varepsilon_{yy}+R_3\varepsilon_{zz}+2R_8\varepsilon_{xy}+K_3\omega_{zz}-d_{13}E_x-d_{33}E_z,\\ H_{zy}=2R_5\varepsilon_{zy}+2R_7\varepsilon_{zx}+K_2\omega_{zy}+K_4\omega_{zx}-d_{24}E_y,\\ H_{zx}=2R_4\varepsilon_{zy}+2R_6\varepsilon_{zx}+K_4\omega_{zy}+K_1\omega_{zx}-d_{15}E_x-d_{35}E_z,\\ D_x=e_{11}\varepsilon_{xx}+e_{12}\varepsilon_{yy}+e_{13}\varepsilon_{zz}+2e_{15}\varepsilon_{zx}+d_{13}\omega_{zz}+d_{15}\omega_{zx}+\kappa_{11}E_x+\kappa_{13}E_z,\\ D_y=2e_{24}\varepsilon_{zy}+2e_{26}\varepsilon_{xy}+d_{24}\omega_{zy}+\kappa_{22}E_y,\\ D_z=e_{31}\varepsilon_{xx}+e_{32}\varepsilon_{yy}+e_{33}\varepsilon_{zz}+2e_{35}\varepsilon_{zx}+d_{33}\omega_{zz}+d_{35}\omega_{zx}+\kappa_{13}E_x+\kappa_{33}E_z.\end{cases}\tag{8}$$

The corresponding equilibrium equations to Eq. (1) are

$$\begin{cases}\dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{xy}}{\partial y}+\dfrac{\partial\sigma_{xz}}{\partial z}=0,\\ \dfrac{\partial\sigma_{yx}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+\dfrac{\partial\sigma_{yz}}{\partial z}=0,\\ \dfrac{\partial\sigma_{zx}}{\partial x}+\dfrac{\partial\sigma_{zy}}{\partial y}+\dfrac{\partial\sigma_{zz}}{\partial z}=0,\\ \dfrac{\partial H_{zx}}{\partial x}+\dfrac{\partial H_{zy}}{\partial y}+\dfrac{\partial H_{zz}}{\partial z}=0,\\ \dfrac{\partial D_x}{\partial x}+\dfrac{\partial D_y}{\partial y}+\dfrac{\partial D_z}{\partial z}=0.\end{cases}\tag{9}$$

It is found from Eqs. (6), (8), and (9) that there are 29 equations and 29 field variables including 4 displacements, 9 strains, 9 stresses, 3 electric fields, 3 electric displacements, and one electric potential. Thus, the elastic equilibrium problem of piezoelasticity of 1D monoclinic QCs is more complicated than that of 3D classic elasticity, 1D monoclinic QC elasticity and piezoelectric materials. We will present a rigorous treatment of the problem in this work.

## 3 Governing equations of plane piezoelasticity of QC systems

### 3.1 Monoclinic QC

If there is a straight dislocation or a Griffith crack along the direction of the atom quasiperiodic arrangement and the polarized direction of the electric field along the $z$-axis, the deformation is independent of the $z$-axis, i.e.,

$$\frac{\partial(\cdot)}{\partial z} = 0. \tag{10}$$

Therefore, we have the following gradient equations and equilibrium equations in the absence of the body forces of phonon and phason fields and the electric density:

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \\ \varepsilon_{yz} = \dfrac{1}{2}\dfrac{\partial u_z}{\partial y}, \quad \varepsilon_{zx} = \dfrac{1}{2}\dfrac{\partial u_z}{\partial x}, \quad \varepsilon_{xy} = \dfrac{1}{2}\Big(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\Big), \\ \omega_{yz} = \dfrac{\partial w_z}{\partial y}, \quad \omega_{zx} = \dfrac{\partial w_z}{\partial x}, \\ E_x = -\dfrac{\partial \phi}{\partial x}, \quad E_y = -\dfrac{\partial \phi}{\partial y}, \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0, \quad \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0, \\ \dfrac{\partial \sigma_{zx}}{\partial x} + \dfrac{\partial \sigma_{zy}}{\partial y} = 0, \quad \dfrac{\partial H_{zx}}{\partial x} + \dfrac{\partial H_{zy}}{\partial y} = 0, \\ \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} = 0. \end{cases} \tag{12}$$

The constitutive equation (8) can be simplified as follows:

$$\begin{cases} \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + 2C_{16}\varepsilon_{xy} - e_{11}E_x, \\ \sigma_{yy} = C_{12}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + 2C_{26}\varepsilon_{xy} - e_{12}E_x, \\ \sigma_{zz} = C_{13}\varepsilon_{xx} + C_{23}\varepsilon_{yy} + 2C_{36}\varepsilon_{xy} - e_{13}E_x, \\ \sigma_{zy} = 2C_{44}\varepsilon_{zy} + 2C_{45}\varepsilon_{zx} + R_5\omega_{zy} + R_4\omega_{zx} - e_{24}E_y, \\ \sigma_{zx} = 2C_{45}\varepsilon_{zy} + 2C_{55}\varepsilon_{zx} + R_7\omega_{zy} + R_6\omega_{zx} - e_{15}E_x, \\ \sigma_{xy} = C_{16}\varepsilon_{xx} + C_{26}\varepsilon_{yy} + 2C_{66}\varepsilon_{xy} - e_{26}E_y, \\ H_{zz} = R_1\varepsilon_{xx} + R_2\varepsilon_{yy} + 2R_8\varepsilon_{xy} - d_{13}E_x, \\ H_{zy} = 2R_5\varepsilon_{zy} + 2R_7\varepsilon_{zx} + K_2\omega_{zy} + K_4\omega_{zx} - d_{24}E_y, \\ H_{zx} = 2R_4\varepsilon_{zy} + 2R_6\varepsilon_{zx} + K_4\omega_{zy} + K_1\omega_{zx} - d_{15}E_x, \\ D_x = e_{11}\varepsilon_{xx} + e_{12}\varepsilon_{yy} + 2e_{15}\varepsilon_{zx} + d_{15}\omega_{zx} + \kappa_{11}E_x, \\ D_y = 2e_{24}\varepsilon_{zy} + 2e_{26}\varepsilon_{xy} + d_{24}\omega_{zy} + \kappa_{22}E_y, \\ D_z = e_{31}\varepsilon_{xx} + e_{32}\varepsilon_{yy} + 2e_{35}\varepsilon_{zx} + d_{35}\omega_{zx} + \kappa_{13}E_x. \end{cases} \tag{13}$$

Substituting the gradient equations in Eq. (11) into the constitutive equations in Eq. (13), and then into the equilibrium equations in Eq. (12), we have the final governing equations in terms of the displacements and electric potential as follows:

$$\begin{cases}\Big(C_{11}\dfrac{\partial^2}{\partial x^2}+C_{66}\dfrac{\partial^2}{\partial y^2}+2C_{16}\dfrac{\partial^2}{\partial x\partial y}\Big)u_x\\[2mm] +\Big(C_{16}\dfrac{\partial^2}{\partial x^2}+C_{26}\dfrac{\partial^2}{\partial y^2}+(C_{12}+C_{66})\dfrac{\partial^2}{\partial x\partial y}\Big)u_y+\Big(e_{11}\dfrac{\partial^2}{\partial x^2}+e_{26}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0,\\[2mm] \Big(C_{16}\dfrac{\partial^2}{\partial x^2}+C_{26}\dfrac{\partial^2}{\partial y^2}+(C_{12}+C_{66})\dfrac{\partial^2}{\partial x\partial y}\Big)u_x\\[2mm] +\Big(C_{66}\dfrac{\partial^2}{\partial x^2}+C_{22}\dfrac{\partial^2}{\partial y^2}+2C_{26}\dfrac{\partial^2}{\partial x\partial y}\Big)u_y+(e_{12}+e_{26})\dfrac{\partial^2\phi}{\partial x\partial y}=0,\\[2mm] \Big(C_{55}\dfrac{\partial^2}{\partial x^2}+C_{44}\dfrac{\partial^2}{\partial y^2}+2C_{45}\dfrac{\partial^2}{\partial x\partial y}\Big)u_z\\[2mm] +\Big(R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2}+(R_4+R_7)\dfrac{\partial^2}{\partial x\partial y}\Big)w_z+\Big(e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0,\\[2mm] \Big(R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2}+(R_4+R_7)\dfrac{\partial^2}{\partial x\partial y}\Big)u_z\\[2mm] +\Big(K_1\dfrac{\partial^2}{\partial x^2}+K_2\dfrac{\partial^2}{\partial y^2}+2K_4\dfrac{\partial^2}{\partial x\partial y}\Big)w_z+\Big(d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0,\\[2mm] \Big(e_{11}\dfrac{\partial^2}{\partial x^2}+e_{26}\dfrac{\partial^2}{\partial y^2}\Big)u_x+(e_{12}+e_{26})\dfrac{\partial^2 u_y}{\partial x\partial y}+\Big(e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2}\Big)u_z\\[2mm] +\Big(d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2}\Big)w_z-\Big(\kappa_{11}\dfrac{\partial^2}{\partial x^2}+\kappa_{22}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0.\end{cases}\tag{14}$$

This is a phonon-phason-electric coupling elasticity problem, involving the displacements $u_x$, $u_y$, $u_z$, $w_z$ and the electric potential $\phi$.

### 3.2 Orthorhombic QC

For the orthorhombic QC with the point group 2mm, the increase in the symmetric elements leads to

$$\begin{cases}C_{16}=C_{26}=C_{36}=C_{45}=0,\quad R_4=R_7=R_8=0,\quad K_4=0,\\ e_{11}=e_{12}=e_{13}=e_{26}=e_{35}=0,\quad d_{13}=d_{35}=0,\quad \kappa_{13}=0.\end{cases}\tag{15}$$

Therefore, the number of the non-zero independent electro-elastic constants of 1D orthorhombic QCs reduces to 28, i.e.,

$$C_{11},\ C_{22},\ C_{33},\ C_{12},\ C_{13},\ C_{23},\ C_{44},\ C_{55},\ C_{66}$$

for the phonon elastic constants, $K_1$, $K_2$, and $K_3$ for the phason elastic constants, $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, and $R_6$ for the phonon-phason coupling elastic constants,

$$e_{15},\ e_{24},\ e_{31},\ e_{32},\ e_{33},\ d_{33},\ d_{24},\ d_{15}$$

for the piezoelastic constants, and $\kappa_{11}$, $\kappa_{22}$, and $\kappa_{33}$ for the independent dielectric permittivities. With the superposition principle, we can decompose Eqs. (11)–(13) and (15) into the following two uncoupled problems.

Problem I

$$
\begin{cases}
\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy}, \\
\sigma_{yy} = C_{12}\varepsilon_{xx} + C_{22}\varepsilon_{yy}, \\
\sigma_{zz} = C_{13}\varepsilon_{xx} + C_{23}\varepsilon_{yy}, \\
\sigma_{xy} = 2C_{66}\varepsilon_{xy}, \\
H_{zz} = R_1\varepsilon_{xx} + R_2\varepsilon_{yy}, \\
D_z = e_{31}\varepsilon_{xx} + e_{32}\varepsilon_{yy}, \\
\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0, \quad \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0, \\
\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \dfrac{1}{2}\Big(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\Big).
\end{cases}
\tag{16}
$$

It is similar to a plane strain problem for orthorhombic crystals. The solution of Problem I agrees well with that of the classical elasticity theory. Therefore, it is not given in this work.

Problem II

$$
\begin{cases}
\sigma_{zy} = 2C_{44}\varepsilon_{zy} + R_5\omega_{zy} - e_{24}E_y, \\
\sigma_{zx} = 2C_{55}\varepsilon_{zx} + R_6\omega_{zx} - e_{15}E_x, \\
H_{zy} = 2R_5\varepsilon_{zy} + K_2\omega_{zy} - d_{24}E_y, \\
H_{zx} = 2R_6\varepsilon_{zx} + K_1\omega_{zx} - d_{15}E_x, \\
D_x = 2e_{15}\varepsilon_{zx} + d_{15}\omega_{zx} + \kappa_{11}E_x, \\
D_y = 2e_{24}\varepsilon_{zy} + d_{24}\omega_{zy} + \kappa_{22}E_y, \\
\dfrac{\partial \sigma_{zx}}{\partial x} + \dfrac{\partial \sigma_{zy}}{\partial y} = 0, \quad \dfrac{\partial H_{zx}}{\partial x} + \dfrac{\partial H_{zy}}{\partial y} = 0, \quad \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} = 0, \\
\varepsilon_{yz} = \dfrac{1}{2}\dfrac{\partial u_z}{\partial y}, \quad \varepsilon_{zx} = \dfrac{1}{2}\dfrac{\partial u_z}{\partial x}, \\
\omega_{yz} = \dfrac{\partial w_z}{\partial y}, \quad \omega_{zx} = \dfrac{\partial w_z}{\partial x}, \\
E_x = -\dfrac{\partial \phi}{\partial x}, \quad E_y = -\dfrac{\partial \phi}{\partial y}.
\end{cases}
\tag{17}
$$

It is an anti-plane phonon-phason-electric coupling elasticity problem, involving only the displacements $u_z$ and $w_z$ and the electric potential $\phi$.

For Problem II, the governing equations turn into

$$\begin{cases}\Big(C_{55}\dfrac{\partial^2}{\partial x^2}+C_{44}\dfrac{\partial^2}{\partial y^2}\Big)u_z+\Big(R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2}\Big)w_z+\Big(e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0,\\[2ex] \Big(R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2}\Big)u_z+\Big(K_1\dfrac{\partial^2}{\partial x^2}+K_2\dfrac{\partial^2}{\partial y^2}\Big)w_z+\Big(d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0,\\[2ex] \Big(e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2}\Big)u_z+\Big(d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2}\Big)w_z-\Big(\kappa_{11}\dfrac{\partial^2}{\partial x^2}+\kappa_{22}\dfrac{\partial^2}{\partial y^2}\Big)\phi=0.\end{cases} \tag{18}$$

### 3.3 Tetragonal QC

For the tetragonal QC with the point group 4mm, besides Eq. (15), the number of new symmetrical elements increases, i.e.,

$$\begin{cases}C_{11}=C_{22}, \quad C_{13}=C_{23}, \quad C_{44}=C_{55}, \quad R_1=R_2, \quad R_5=R_6,\\ K_1=K_2, \quad e_{31}=e_{32}, \quad e_{15}=e_{24}, \quad d_{15}=d_{24}, \quad \kappa_{11}=\kappa_{22}.\end{cases} \tag{19}$$

For Problem II, from Eq. (18), we can simplify the governing equations as follows:

$$\begin{cases}C_{44}\nabla^2 u_z+R_5\nabla^2 w_z+e_{15}\nabla^2\phi=0,\\ R_5\nabla^2 u_z+K_1\nabla^2 w_z+d_{15}\nabla^2\phi=0,\\ e_{15}\nabla^2 u_z+d_{15}\nabla^2 w_z-\kappa_{11}\nabla^2\phi=0,\end{cases} \tag{20}$$

where $\nabla^2$ is the Laplace operator defined by

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

### 3.4 Hexagonal QC

For the hexagonal QCs with the point group 6mm, we further have

$$C_{66}=\frac{1}{2}(C_{11}-C_{12}). \tag{21}$$

For Problem II, the governing equations are the same as Eq. (20).

## 4 General solutions of QC systems

### 4.1 Monoclinic QC system

The governing equation (14) can be rewritten as the following matrix equation:

$$\boldsymbol{D}\boldsymbol{V}=0, \tag{22}$$

where $\boldsymbol{V}=(u_x,\ u_y,\ u_z,\ w_z,\ \phi)^{\mathrm{T}}$, and $\boldsymbol{D}$ is a $5\times 5$ differential operator matrix expressed by

$$\boldsymbol{D}=\begin{pmatrix}D_{11} & D_{12} & D_{13} & D_{14} & D_{15}\\ & D_{22} & D_{23} & D_{24} & D_{25}\\ & & D_{33} & D_{34} & D_{35}\\ & \text{Symmetry} & & D_{44} & D_{45}\\ & & & & D_{55}\end{pmatrix}. \tag{23}$$

The elements in Eq. (23) are

$$\begin{cases} D_{11}=C_{11}\dfrac{\partial^2}{\partial x^2}+C_{66}\dfrac{\partial^2}{\partial y^2}+2C_{16}\dfrac{\partial^2}{\partial x\partial y}, \\ D_{12}=C_{16}\dfrac{\partial^2}{\partial x^2}+C_{26}\dfrac{\partial^2}{\partial y^2}+(C_{12}+C_{66})\dfrac{\partial^2}{\partial x\partial y}, \\ D_{13}=D_{14}=D_{23}=D_{24}=0, \\ D_{15}=e_{11}\dfrac{\partial^2}{\partial x^2}+e_{26}\dfrac{\partial^2}{\partial y^2}, \quad D_{25}=(e_{12}+e_{26})\dfrac{\partial^2}{\partial x\partial y}, \\ D_{22}=C_{66}\dfrac{\partial^2}{\partial x^2}+C_{22}\dfrac{\partial^2}{\partial y^2}+2C_{26}\dfrac{\partial^2}{\partial x\partial y}, \\ D_{33}=C_{55}\dfrac{\partial^2}{\partial x^2}+C_{44}\dfrac{\partial^2}{\partial y^2}+2C_{45}\dfrac{\partial^2}{\partial x\partial y}, \\ D_{34}=R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2}+(R_4+R_7)\dfrac{\partial^2}{\partial x\partial y}, \\ D_{35}=e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2}, \quad D_{44}=K_1\dfrac{\partial^2}{\partial x^2}+K_2\dfrac{\partial^2}{\partial y^2}+2K_4\dfrac{\partial^2}{\partial x\partial y}, \\ D_{45}=d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2}, \quad D_{55}=-\kappa_{11}\dfrac{\partial^2}{\partial x^2}-\kappa_{22}\dfrac{\partial^2}{\partial y^2}. \end{cases} \tag{24}$$

The general solutions of Eq. (22) can be obtained by the operator method developed by Gao and Zhao[19], Wang and Wang[42], and Wang and Shi[43]. Due to the complexity, we do not give them here. Among various 1D QCs, the special QCs including orthorhombic QCs, tetragonal QCs, and hexagonal QCs are of particular interest for researchers. Therefore, we will give the general solutions of these special QCs by the operator method or the complex variable function method as follows.

### 4.2 Orthorhombic QC system

The solutions of Problem II are given as follows by using the operator method. The governing equation (18) is rewritten as the following matrix equation:

$$\boldsymbol{A}\boldsymbol{U}=\boldsymbol{0}, \tag{25}$$

where $\boldsymbol{U}=(u_z,\, w_z,\, \varphi)^{\mathrm{T}}$, and $\boldsymbol{A}$ is a $3\times 3$ differential operator matrix expressed by

$$\boldsymbol{A}=\begin{pmatrix} C_{55}\dfrac{\partial^2}{\partial x^2}+C_{44}\dfrac{\partial^2}{\partial y^2} & R_6\dfrac{\partial^2}{\partial x^2}+R_5\dfrac{\partial^2}{\partial y^2} & e_{15}\dfrac{\partial^2}{\partial x^2}+e_{24}\dfrac{\partial^2}{\partial y^2} \\ & K_1\dfrac{\partial^2}{\partial x^2}+K_2\dfrac{\partial^2}{\partial y^2} & d_{15}\dfrac{\partial^2}{\partial x^2}+d_{24}\dfrac{\partial^2}{\partial y^2} \\ \text{Symmetry} & & -\kappa_{11}\dfrac{\partial^2}{\partial x^2}-\kappa_{22}\dfrac{\partial^2}{\partial y^2} \end{pmatrix}. \tag{26}$$

Let us introduce a $3\times 3$ differential operator matrix $\boldsymbol{B}$ as the "adjoint matrix" of $\boldsymbol{A}$ such that

$$\boldsymbol{A}\boldsymbol{B}=\boldsymbol{B}\boldsymbol{A}=\det(\boldsymbol{A})\boldsymbol{I}, \tag{27}$$

where the components $B_{ij}$ of $\boldsymbol{B}$ are "algebraic complement minors" of $\boldsymbol{A}$, i.e.,

$$\begin{cases} B_{11} = -\Big(K_1\dfrac{\partial^2}{\partial x^2} + K_2\dfrac{\partial^2}{\partial y^2}\Big)\Big(\kappa_{11}\dfrac{\partial^2}{\partial x^2} + \kappa_{22}\dfrac{\partial^2}{\partial y^2}\Big) - \Big(d_{15}\dfrac{\partial^2}{\partial x^2} + d_{24}\dfrac{\partial^2}{\partial y^2}\Big)^2, \\ B_{12} = B_{21} = \Big(R_6\dfrac{\partial^2}{\partial x^2} + R_5\dfrac{\partial^2}{\partial y^2}\Big)\Big(\kappa_{11}\dfrac{\partial^2}{\partial x^2} + \kappa_{22}\dfrac{\partial^2}{\partial y^2}\Big) \\ \qquad + \Big(e_{15}\dfrac{\partial^2}{\partial x^2} + e_{24}\dfrac{\partial^2}{\partial y^2}\Big)\Big(d_{15}\dfrac{\partial^2}{\partial x^2} + d_{24}\dfrac{\partial^2}{\partial y^2}\Big), \\ B_{13} = B_{31} = \Big(R_6\dfrac{\partial^2}{\partial x^2} + R_5\dfrac{\partial^2}{\partial y^2}\Big)\Big(d_{15}\dfrac{\partial^2}{\partial x^2} + d_{24}\dfrac{\partial^2}{\partial y^2}\Big) \\ \qquad - \Big(K_1\dfrac{\partial^2}{\partial x^2} + K_2\dfrac{\partial^2}{\partial y^2}\Big)\Big(e_{15}\dfrac{\partial^2}{\partial x^2} + e_{24}\dfrac{\partial^2}{\partial y^2}\Big), \\ B_{22} = -\Big(C_{55}\dfrac{\partial^2}{\partial x^2} + C_{44}\dfrac{\partial^2}{\partial y^2}\Big)\Big(\kappa_{11}\dfrac{\partial^2}{\partial x^2} + \kappa_{22}\dfrac{\partial^2}{\partial y^2}\Big) - \Big(e_{15}\dfrac{\partial^2}{\partial x^2} + e_{24}\dfrac{\partial^2}{\partial y^2}\Big)^2, \\ B_{23} = B_{32} = -\Big(C_{55}\dfrac{\partial^2}{\partial x^2} + C_{44}\dfrac{\partial^2}{\partial y^2}\Big)\Big(d_{15}\dfrac{\partial^2}{\partial x^2} + d_{24}\dfrac{\partial^2}{\partial y^2}\Big) \\ \qquad + \Big(R_6\dfrac{\partial^2}{\partial x^2} + R_5\dfrac{\partial^2}{\partial y^2}\Big)\Big(e_{15}\dfrac{\partial^2}{\partial x^2} + e_{24}\dfrac{\partial^2}{\partial y^2}\Big), \\ B_{33} = \Big(C_{55}\dfrac{\partial^2}{\partial x^2} + C_{44}\dfrac{\partial^2}{\partial y^2}\Big)\Big(K_1\dfrac{\partial^2}{\partial x^2} + K_2\dfrac{\partial^2}{\partial y^2}\Big) - \Big(R_6\dfrac{\partial^2}{\partial x^2} + R_5\dfrac{\partial^2}{\partial y^2}\Big)^2. \end{cases} \tag{28}$$

The determinant of $\boldsymbol{A}$ is defined by

$$\det(\boldsymbol{A}) = a\frac{\partial^6}{\partial y^6} + b\frac{\partial^6}{\partial x^2\partial y^4} + c\frac{\partial^6}{\partial x^4\partial y^2} + d\frac{\partial^6}{\partial x^6}, \tag{29}$$

where

$$\begin{cases} a = -C_{44}K_2\kappa_{22} + 2R_5d_{24}e_{24} - K_2e_{24}^2 + R_5^2\kappa_{22} - C_{44}d_{24}^2, \\ b = -C_{44}K_2\kappa_{11} - C_{55}K_2\kappa_{22} - C_{44}K_1\kappa_{22} + 2R_5d_{24}e_{15} + 2R_6d_{24}e_{24} + 2R_5d_{15}e_{24} \\ \qquad - 2K_2e_{24}e_{15} - K_1e_{24}^2 + R_5^2\kappa_{11} - 2R_6R_5\kappa_{22} - C_{55}d_{24}^2 - 2C_{44}d_{24}d_{15}, \\ c = -C_{55}K_2\kappa_{11} - C_{44}K_1\kappa_{11} - C_{55}K_1\kappa_{22} + 2R_6d_{24}e_{15} + R_5d_{15}e_{15} + R_6d_{15}e_{24} \\ \qquad + 2R_6d_{24}e_{24} + R_5d_{24}e_{15} - K_2e_{15}^2 - 2K_1e_{15}e_{24} + 2R_6R_5\kappa_{11} + R_6^2\kappa_{22} \\ \qquad - 2C_{55}d_{24}d_{15} - C_{44}d_{15}^2, \\ d = -C_{55}K_1\kappa_{11} + 2R_6d_{15}e_{15} - K_1e_{15}^2 + R_6^2\kappa_{11} - C_{55}d_{15}^2. \end{cases} \tag{30}$$

Let us introduce a displacement function $F$, which satisfies

$$\nabla_1^2\nabla_2^2\nabla_3^2F = 0, \tag{31}$$

where $\nabla_i^2$ can be written as follows:

$$\nabla_i^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{s_i^2}\frac{\partial^2}{\partial y^2}, \quad i = 1,\ 2,\ 3. \tag{32}$$

In Eq. (32), $s_i^2$ $(i = 1,\ 2,\ 3)$ are three characteristic roots of the following cubic algebra equation of $s^2$:

$$as^6 - bs^4 + cs^2 - d = 0. \tag{33}$$

The three roots expressed by $a$, $b$, $c$, and $d$ exist a real one among them. Assume that $s_1^2$ is the real root without loss of generality. Moreover, we further assume $\mathrm{Re}(s_i^2) > 0$.

Therefore, the general solutions of Eq. (25) can be obtained as follows:

$$u_z = B_{i1}F, \quad w_z = B_{i2}F, \quad \phi = B_{i3}F. \tag{34}$$

Take one of the general solutions of Eq. (34) as an example, i.e.,

$$u_z = B_{21}F, \quad w_z = B_{22}F, \quad \phi = B_{23}F, \tag{35}$$

or

$$\begin{cases} u_z = a_{11}\dfrac{\partial^4 F}{\partial x^4} + a_{12}\dfrac{\partial^4 F}{\partial x^2\partial y^2} + a_{13}\dfrac{\partial^4 F}{\partial y^4}, \\ w_z = a_{21}\dfrac{\partial^4 F}{\partial x^4} + a_{22}\dfrac{\partial^4 F}{\partial x^2\partial y^2} + a_{23}\dfrac{\partial^4 F}{\partial y^4}, \\ \phi = a_{31}\dfrac{\partial^4 F}{\partial x^4} + a_{32}\dfrac{\partial^4 F}{\partial x^2\partial y^2} + a_{33}\dfrac{\partial^4 F}{\partial y^4}, \end{cases} \tag{36}$$

where

$$\begin{cases} a_{11} = R_6\kappa_{11} + e_{15}d_{15}, \\ a_{12} = R_6\kappa_{22} + R_5\kappa_{11} + e_{15}d_{24} + e_{24}d_{15}, \\ a_{13} = R_5\kappa_{22} + e_{24}d_{24}, \\ a_{21} = -C_{55}\kappa_{11} - e_{15}^2, \quad a_{23} = -C_{44}\kappa_{22} - e_{24}^2, \\ a_{22} = -C_{55}\kappa_{22} + C_{44}\kappa_{11} - 2e_{15}e_{24}, \\ a_{31} = -C_{55}d_{15} + R_6e_{15}, \quad a_{33} = -C_{44}d_{24} + R_5e_{24}, \\ a_{32} = -C_{55}d_{24} - C_{44}d_{15} + R_6e_{24} + R_5e_{15}. \end{cases} \tag{37}$$

From the work of Gao and Zhao[16], it can be proved that the above-mentioned general solutions are complete in any limited domain in $E^3$[20]. According to the theorem, if the domain $\Omega$ is $y$-convex and $F$ follows that

$$\nabla_1^2\nabla_2^2\nabla_3^2 F = 0, \quad F \in \Omega, \tag{38}$$

then there exist displacement functions $F_i$ $(i = 1, 2, 3)$ in the three forms as follows:

Case 1

$$F = F_1 + F_2 + F_3, \quad s_1^2 \neq s_2^2 \neq s_3^2. \tag{39}$$

Case 2

$$F = F_1 + yF_2 + F_3, \quad s_1^2 = s_2^2 \neq s_3^2. \tag{40}$$

Case 3

$$F = F_1 + yF_2 + y^2F_3, \quad s_1^2 = s_2^2 = s_3^2. \tag{41}$$

In the above equations, $F_i$ $(i = 1, 2, 3)$ satisfy the following second-order governing equations:

$$\nabla_i^2 F_i = 0. \tag{42}$$

It is clearly seen that the complicated governing equations (18) can be simplified into several partial differential equations of lower order by the operator method. We will deduce three different forms of the general solutions of the anti-plane problem for the orthorhombic QC system by considering the different cases of three characteristic roots.

For Case 1, when

$$s_1^2 \neq s_2^2 \neq s_3^2 \neq s_1^2,$$

Eqs. (32), (39), and (36) yield

$$u_z = \alpha_i \frac{\partial^4 F_i}{\partial y^4}, \quad w_z = \beta_i \frac{\partial^4 F_i}{\partial y^4}, \quad \phi = \gamma_i \frac{\partial^4 F_i}{\partial y^4}, \tag{43}$$

where

$$\begin{cases} \alpha_i = a_{11}\dfrac{1}{s_i^4} - a_{12}\dfrac{1}{s_i^2} + a_{13}, \\[2ex] \beta_i = a_{21}\dfrac{1}{s_i^4} - a_{22}\dfrac{1}{s_i^2} + a_{23}, \\[2ex] \gamma_i = a_{31}\dfrac{1}{s_i^4} - a_{32}\dfrac{1}{s_i^2} + a_{33}. \end{cases} \tag{44}$$

For Case 2, when

$$s_1^2 = s_2^2 \neq s_3^2,$$

Eqs. (32), (36), and (40) lead to

$$\begin{cases} u_z = \alpha_1 \dfrac{\partial^4 F_1}{\partial y^4} + y\alpha_1 \dfrac{\partial^4 F_2}{\partial y^4} + \alpha_3 \dfrac{\partial^4 F_3}{\partial y^4} + \Big(4a_{13} - \dfrac{2a_{12}}{s_1^2}\Big)\dfrac{\partial^3 F_2}{\partial y^3}, \\[3ex] w_z = \beta_1 \dfrac{\partial^4 F_1}{\partial y^4} + y\beta_1 \dfrac{\partial^4 F_2}{\partial y^4} + \beta_3 \dfrac{\partial^4 F_3}{\partial y^4} + \Big(4a_{23} - \dfrac{2a_{22}}{s_1^2}\Big)\dfrac{\partial^3 F_2}{\partial y^3}, \\[3ex] \phi = \gamma_1 \dfrac{\partial^4 F_1}{\partial y^4} + y\gamma_1 \dfrac{\partial^4 F_2}{\partial y^4} + \gamma_3 \dfrac{\partial^4 F_3}{\partial y^4} + \Big(4a_{33} - \dfrac{2a_{32}}{s_1^2}\Big)\dfrac{\partial^3 F_2}{\partial y^3}. \end{cases} \tag{45}$$

For Case 3, when

$$s_1^2 = s_2^2 = s_3^2,$$

Eqs. (32), (36), and (41) result in

$$
\left\{
\begin{aligned}
&u_z = \alpha_1 \frac{\partial^4 F_1}{\partial y^4} + y\alpha_1 \frac{\partial^4 F_2}{\partial y^4} + y^2 \alpha_1 \frac{\partial^4 F_3}{\partial y^4} + \Big(4a_{13} - \frac{2a_{12}}{s_1^2}\Big)\frac{\partial^3 F_2}{\partial y^3} \\
&\qquad + \Big(8a_{13}y - \frac{4a_{12}y}{s_1^2}\Big)\frac{\partial^3 F_3}{\partial y^3} + \Big(12a_{13} - \frac{2a_{12}}{s_1^2}\Big)\frac{\partial^2 F_3}{\partial y^2}, \\
&w_z = \beta_1 \frac{\partial^4 F_1}{\partial y^4} + y\beta_1 \frac{\partial^4 F_2}{\partial y^4} + y^2 \beta_1 \frac{\partial^4 F_3}{\partial y^4} + \Big(4a_{23} - \frac{2a_{22}}{s_1^2}\Big)\frac{\partial^3 F_2}{\partial y^3} \\
&\qquad + \Big(8a_{23}y - \frac{4a_{22}y}{s_1^2}\Big)\frac{\partial^3 F_3}{\partial y^3} + \Big(12a_{23} - \frac{2a_{22}}{s_1^2}\Big)\frac{\partial^2 F_3}{\partial y^2}, \\
&\varphi = \gamma_1 \frac{\partial^4 F_1}{\partial y^4} + y\gamma_1 \frac{\partial^4 F_2}{\partial y^4} + y^2 \gamma_1 \frac{\partial^4 F_3}{\partial y^4} + \Big(4a_{33} - \frac{2a_{32}}{s_1^2}\Big)\frac{\partial^3 F_2}{\partial y^3} \\
&\qquad + \Big(8a_{33}y - \frac{4a_{32}y}{s_1^2}\Big)\frac{\partial^3 F_3}{\partial y^3} + \Big(12a_{33} - \frac{2a_{32}}{s_1^2}\Big)\frac{\partial^2 F_3}{\partial y^2}.
\end{aligned}
\right. \tag{46}
$$

### 4.3 Tetragonal QC system

For Problem II, Eq. (20) is satisfied if $u_z$, $w_z$, and $\phi$ are harmonic functions. This can be achieved by letting $u_z$, $w_z$, and $\phi$ be the imaginary parts of the analytic functions $U(z)$, $W(z)$, and $\Phi(z)$, respectively, such that

$$
u_z = \mathrm{Im}U(z), \quad w_z = \mathrm{Im}W(z), \quad \phi = \mathrm{Im}\Phi(z), \tag{47}
$$

where $z = x + \mathrm{i}y$, and Im denotes the imaginary part of the complex function. The stresses of the phonon field, the phason field, and the electric displacements can then be expressed as follows:

$$
\left\{
\begin{aligned}
&\sigma_{xz} = \mathrm{Im}(C_{44}U'(z) + R_5 W'(z) + e_{15}\Phi'(z)), \\
&\sigma_{yz} = \mathrm{Re}(C_{44}U'(z) + R_5 W'(z) + e_{15}\Phi'(z)), \\
&H_{zx} = \mathrm{Im}(R_5 U'(z) + K_1 W'(z) + d_{15}\Phi'(z)), \\
&H_{zy} = \mathrm{Re}(R_5 U'(z) + K_1 W'(z) + d_{15}\Phi'(z)), \\
&D_x = \mathrm{Im}(e_{15}U'(z) + d_{15}W'(z) - \kappa_{11}\Phi'(z)), \\
&D_y = \mathrm{Re}(e_{15}U'(z) + d_{15}W'(z) - \kappa_{11}\Phi'(z)),
\end{aligned}
\right. \tag{48}
$$

where Re is the real part of the complex function, and the prime indicates differentiation with respect to the complex variable $z$.

It is found that the solution to Problem II of the hexagonal QC system is the same as that of the tetragonal QC system.

## 5 Fracture mechanics of Griffith crack

To our interest, the phonon-phason-electric coupling anti-plane elasticity problem described by Eq. (47) may bring new insight into the piezoelasticity scope of QCs.

Consider a mode III fracture problem, for which a Griffith crack with the length $2a$ is embedded in an infinite 1D hexagonal QC subjected to far-field phonon, phason, and electrical loads (see Fig. 1). The boundary conditions on the upper and lower surfaces of the crack are free of the surface traction and the surface charge, i.e.,

$$\sigma_{yz} = 0, \quad H_{yz} = 0, \quad D_y = 0, \quad |x| < a, \quad y = 0. \tag{49}$$

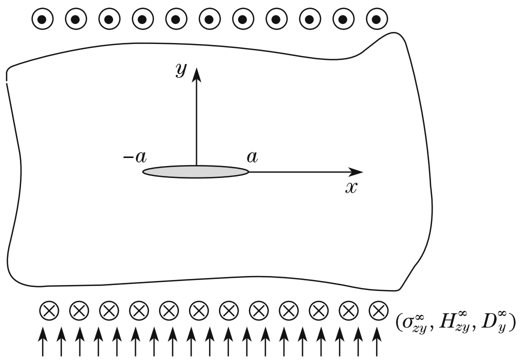


**Fig. 1** Griffith crack in 1D hexagonal QCs subjected to far-field mechanical and electrical loads

Then, with Eq. (47), we can obtain the stresses and the electric displacements as follows:

$$\begin{cases} \sigma_{xz} = \mathrm{Im}(C_{44}U'(z) + R_3W'(z) + e_{15}\Phi'(z)), \\ \sigma_{yz} = \mathrm{Re}(C_{44}U'(z) + R_3W'(z) + e_{15}\Phi'(z)), \\ H_{zx} = \mathrm{Im}(R_3U'(z) + K_2W'(z) + d_{15}\Phi'(z)), \\ H_{zy} = \mathrm{Re}(R_3U'(z) + K_2W'(z) + d_{15}\Phi'(z)), \\ D_x = \mathrm{Im}(e_{15}U'(z) + d_{15}W'(z) - \lambda_{11}\Phi'(z)), \\ D_y = \mathrm{Re}(e_{15}U'(z) + d_{15}W'(z) - \lambda_{11}\Phi'(z)), \end{cases} \tag{50}$$

where Re is the real part of the complex function, and the prime indicates differentiation with respect to the complex variable $z$. If the medium is loaded uniformly at infinity, we take a semi-inverse method by assuming $U(z)$, $W(z)$, and $\Phi(z)$ to be

$$U(z) = A\sqrt{z^2 - a^2}, \quad W(z) = B\sqrt{z^2 - a^2}, \quad \Phi(z) = -C\sqrt{z^2 - a^2}. \tag{51}$$

It can be seen that Eqs. (47) and (49) are satisfied. The unknown real constants $A$, $B$ and $C$ will be determined from the far-field loading conditions. Substituting Eq. (51) into Eqs. (47) and (50) yields

$$\begin{cases} u_z = A\sqrt{r_1 r_2}\sin\Big(\dfrac{\theta_1 + \theta_2}{2}\Big), \\ w_z = B\sqrt{r_1 r_2}\sin\Big(\dfrac{\theta_1 + \theta_2}{2}\Big), \\ \phi = -C\sqrt{r_1 r_2}\sin\Big(\dfrac{\theta_1 + \theta_2}{2}\Big), \end{cases} \tag{52}$$

$$\begin{cases} \sigma_{xz} = (C_{44}A + R_3B - e_{15}C)\dfrac{r}{\sqrt{r_1r_2}} \sin\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \\ \sigma_{yz} = (C_{44}A + R_3B - e_{15}C)\dfrac{r}{\sqrt{r_1r_2}} \cos\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \\ H_{zx} = (R_3A + K_2B - d_{15}C)\dfrac{r}{\sqrt{r_1r_2}} \sin\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \\ H_{zy} = (R_3A + K_2B - d_{15}C)\dfrac{r}{\sqrt{r_1r_2}} \cos\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \\ D_x = (e_{15}A + d_{15}B + \lambda_{11}C)\dfrac{r}{\sqrt{r_1r_2}} \sin\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \\ D_y = (e_{15}A + d_{15}B + \lambda_{11}C)\dfrac{r}{\sqrt{r_1r_2}} \cos\Big(\theta - \dfrac{\theta_1+\theta_2}{2}\Big), \end{cases} \tag{53}$$

where $r$ and $\theta$ are the coordinates defined in Fig. 2.

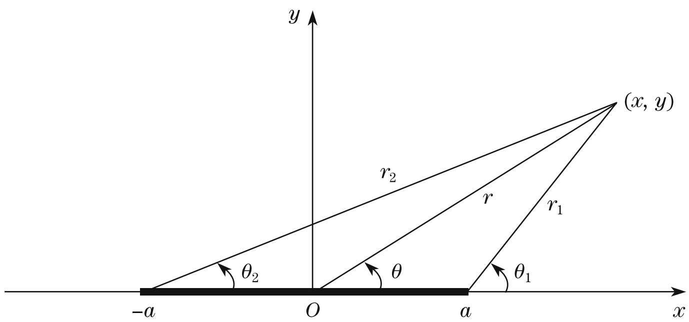


**Fig. 2** Coordinate system at crack tip

By applying the far-field loading conditions, the constants $A$, $B$, and $C$ are obtained for the following possible boundary conditions at infinity:

Case 1 $\sigma_{yz}^{\infty} = \tau^{\infty},\ H_{yz}^{\infty} = H^{\infty},\ D_y^{\infty} = D^{\infty}$ as $x^2 + y^2 \to \infty$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} C_{44} & R_3 & -e_{15} \\ R_3 & K_2 & -d_{15} \\ e_{15} & d_{15} & \lambda_{11} \end{pmatrix}^{-1} \begin{pmatrix} \tau^{\infty} \\ H^{\infty} \\ D^{\infty} \end{pmatrix}. \tag{54}$$

Case 2 $\sigma_{yz}^{\infty} = \tau^{\infty},\ H_{yz}^{\infty} = H^{\infty},\ E_y^{\infty} = E^{\infty}$ as $x^2 + y^2 \to \infty$

$$C = E^{\infty}, \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C_{44} & R_3 \\ R_3 & K_2 \end{pmatrix}^{-1} \left( \begin{pmatrix} \tau^{\infty} \\ H^{\infty} \end{pmatrix} + \begin{pmatrix} e_{15} \\ d_{15} \end{pmatrix} E^{\infty} \right). \tag{55}$$

Case 3 $\sigma_{yz}^{\infty} = \tau^{\infty},\ \omega_{yz}^{\infty} = \omega^{\infty},\ D_y^{\infty} = D^{\infty}$ as $x^2 + y^2 \to \infty$

$$B = \omega^{\infty}, \quad \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} C_{44} & -e_{15} \\ e_{15} & \lambda_{11} \end{pmatrix}^{-1} \left( \begin{pmatrix} \tau^{\infty} \\ D^{\infty} \end{pmatrix} - \begin{pmatrix} R_3 \\ d_{15} \end{pmatrix} \omega^{\infty} \right). \tag{56}$$

Case 4 $\sigma_{yz}^{\infty}=\tau^{\infty},\ \omega_{yz}^{\infty}=\omega^{\infty},\ E_y^{\infty}=E^{\infty}$ as $x^2+y^2\to\infty$

$$B=\omega^{\infty},\quad C=E^{\infty},\quad A=\frac{1}{C_{44}}\left(\tau^{\infty}-R_3\omega^{\infty}+e_{15}E^{\infty}\right). \tag{57}$$

Case 5 $\gamma_{yz}^{\infty}=2\varepsilon_{yz}^{\infty}=\gamma^{\infty},\ H_{yz}^{\infty}=H^{\infty},\ D_y^{\infty}=D^{\infty}$ as $x^2+y^2\to\infty$

$$A=\gamma^{\infty},\quad \begin{pmatrix} B \\ C \end{pmatrix}=\begin{pmatrix} K_2 & -d_{15} \\ d_{15} & \lambda_{11} \end{pmatrix}^{-1}\left(\begin{pmatrix} H^{\infty} \\ D^{\infty} \end{pmatrix}-\begin{pmatrix} R_3 \\ e_{15} \end{pmatrix}\gamma^{\infty}\right). \tag{58}$$

Case 6 $\gamma_{yz}^{\infty}=2\varepsilon_{yz}^{\infty}=\gamma^{\infty},\ H_{yz}^{\infty}=H^{\infty},\ E_y^{\infty}=E^{\infty}$ as $x^2+y^2\to\infty$

$$A=\gamma^{\infty},\quad C=E^{\infty},\quad B=\frac{1}{K_2}\left(H^{\infty}-R_3\gamma^{\infty}+d_{15}E^{\infty}\right). \tag{59}$$

Case 7 $\gamma_{yz}^{\infty}=2\varepsilon_{yz}^{\infty}=\gamma^{\infty},\ \omega_{yz}^{\infty}=\omega^{\infty},\ D_y^{\infty}=D^{\infty}$ as $x^2+y^2\to\infty$

$$A=\gamma^{\infty},\quad B=\omega^{\infty},\quad C=\frac{1}{\lambda_{11}}\left(D^{\infty}-e_{15}\gamma^{\infty}-d_{15}\omega^{\infty}\right). \tag{60}$$

Case 8 $\gamma_{yz}^{\infty}=2\varepsilon_{yz}^{\infty}=\gamma^{\infty},\ \omega_{yz}^{\infty}=\omega^{\infty},\ E_y^{\infty}=E^{\infty}$ as $x^2+y^2\to\infty$

$$A=\gamma^{\infty},\quad B=\omega^{\infty},\quad C=E^{\infty}. \tag{61}$$

Evaluating the solution (53) near the right crack tip and extending the traditional concept of stress intensity factors to other field variables, we have

$$\begin{cases} \gamma_{xz}=-\dfrac{K_{\parallel}^{\mathrm{S}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & \gamma_{yz}=\dfrac{K_{\parallel}^{\mathrm{S}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \\ \omega_{xz}=-\dfrac{K_{\perp}^{\mathrm{S}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & \omega_{yz}=\dfrac{K_{\perp}^{\mathrm{S}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \\ E_x=-\dfrac{K_{\mathrm{E}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & E_y=\dfrac{K_{\mathrm{E}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \\ \sigma_{xz}=-\dfrac{K_{\parallel}^{\mathrm{III}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & \sigma_{yz}=\dfrac{K_{\parallel}^{\mathrm{III}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \\ H_{zx}=-\dfrac{K_{\perp}^{\mathrm{III}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & H_{zy}=\dfrac{K_{\perp}^{\mathrm{III}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \\ D_x=-\dfrac{K_{\mathrm{D}}}{\sqrt{2\pi r_1}}\sin\dfrac{\theta_1}{2}, & D_y=\dfrac{K_{\mathrm{D}}}{\sqrt{2\pi r_1}}\cos\dfrac{\theta_1}{2}, \end{cases} \tag{62}$$

where $K_{\parallel}^{\mathrm{S}}$ and $K_{\parallel}^{\mathrm{III}}$ denote the strain factor and the stress intensity factor of the phonon field, respectively. $K_{\perp}^{\mathrm{S}}$ and $K_{\perp}^{\mathrm{III}}$ stand for the strain factor and the stress intensity factor of the phason field, respectively. $K_{\mathrm{E}}$ and $K_{\mathrm{D}}$ are the electric field factor and the electric displacement intensity factor, respectively. For this problem, these field intensity factors have the following forms:

$$\begin{cases} K_{\parallel}^{\mathrm{S}} = A\sqrt{\pi a} = \gamma^{\infty}\sqrt{\pi a}, \\ K_{\perp}^{\mathrm{S}} = B\sqrt{\pi a} = \omega^{\infty}\sqrt{\pi a}, \quad K_{\mathrm{E}} = C\sqrt{\pi a} = E^{\infty}\sqrt{\pi a}, \\ K_{\parallel}^{\mathrm{III}} = C_{44}K_{\parallel}^{\mathrm{S}} + R_3K_{\perp}^{\mathrm{S}} - e_{15}K_{\mathrm{E}} = \tau^{\infty}\sqrt{\pi a}, \\ K_{\perp}^{\mathrm{III}} = R_3K_{\parallel}^{\mathrm{S}} + K_2K_{\perp}^{\mathrm{S}} - d_{15}K_{\mathrm{E}} = H^{\infty}\sqrt{\pi a}, \\ K_{\mathrm{D}} = e_{15}K_{\parallel}^{\mathrm{S}} + d_{15}K_{\perp}^{\mathrm{S}} + \lambda_{11}K_{\mathrm{E}} = D^{\infty}\sqrt{\pi a}. \end{cases} \tag{63}$$

For this particular problem, the field variables have the same crack-tip behavior as the classical mode III fracture problem. If all electrical quantities are made to vanish, the present solutions can be reduced to the solutions of 1D hexagonal QCs[36]. If all the phason field quantities vanish, the present solutions can be reduced to the solutions of piezoelectric materials[44]. It can be seen from Eqs. (62) and (63) that the stresses of the phonon field, the phason field, and the electric displacement are uncoupled with each other. Further, the field intensity factors corresponding to the field variables used at infinity are independent of the material constants, and are uncoupled with each other. Therefore, we will consider the energy release rate in the characterizing defects subjected to more than one field loading. For this, we firstly derive a conservative integral. Mariano and Planas[45] pointed out that the phason tractions at the surface of the QC are null because we do not know any loading device that is able to use non-zero phason tractions at the external boundary of a quasicrystalline body. In this work, we only consider the phason tractions similar to the phonon tractions from the theoretical point of view.

For an antiplane shear crack in other 1D QCs, the solutions can be obtained to utilize the rigorous operator theory[38–39].

## 6 Conservative integral

Let us define an energy function $F$ defined by

$$F = \frac{1}{2}(\sigma_{ij}\varepsilon_{ij} + H_{ij}\omega_{ij} - D_iE_i). \tag{64}$$

Substituting Eq. (3) into Eq. (64), we have

$$\begin{aligned} F = {} & \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}K_{ijkl}\omega_{ij}\omega_{kl} - \frac{1}{2}\lambda_{ij}E_iE_j \\ & + R_{ijkl}\varepsilon_{ij}\omega_{kl} - e_{kij}\varepsilon_{ij}E_k - d_{kij}\omega_{ij}E_k. \end{aligned} \tag{65}$$

If the QC material is homogeneous and is free of any defects, then Eq. (65) becomes

$$\begin{aligned} F_{,p} = {} & C_{ijkl}u_{i,jp}u_{k,l} + K_{ijkl}w_{i,jp}w_{k,l} - \lambda_{ij}E_{i,p}E_j + R_{ijkl}u_{i,jp}w_{k,l} \\ & + R_{ijkl}u_{i,j}w_{k,lp} - e_{kij}u_{i,jp}E_k - e_{kij}u_{i,j}E_{k,p} \\ & - d_{kij}w_{i,jp}E_k - d_{kij}w_{i,j}E_{k,p}. \end{aligned} \tag{66}$$

With the help of Eq. (3), Eq. (66) can be further written as

$$F_{,p} = \sigma_{ij}u_{i,jp} + H_{ij}w_{i,jp} - D_iE_{i,p}. \tag{67}$$

Now, let us consider the following integral:

$$I_p = \int_{\Omega}(F\delta_{jp} - \sigma_{ij}u_{i,p} - H_{ij}w_{i,p} + D_jE_p)_{,j}\mathrm{d}V, \tag{68}$$

where $I_p$ denotes the $p$th component of the conservation integral, and $\delta_{jp}$ is the Kronecker delta.

With Eq. (1), we can rewrite Eq. (68) as follows:

$$I_p = \int_\Omega (F_{,p} - \sigma_{ij}u_{i,pj} - H_{ij}w_{i,pj} + D_jE_{p,j})\mathrm{d}V. \tag{69}$$

Moreover, according to the divergence theorem, Eq. (68) turns into

$$I_p = \int_S (F\delta_{jp} - \sigma_{ij}u_{i,p} - H_{ij}w_{i,p} + D_jE_p)n_j\mathrm{d}S, \tag{70}$$

where $n_j$ denotes the $n_j$-direction component of the unit normal vector $\boldsymbol{n}$ of a closed surface $S$ ($\boldsymbol{n}$ directs towards the environment). It is found from Eqs. (68) and (70) that $I_p = 0$. Thus, for any closed surface $S$ in a homogeneous material, the following relation holds:

$$I_p = \int_S (F\delta_{jp} - \sigma_{ij}u_{i,p} - H_{ij}w_{i,p} + D_jE_p)n_j\mathrm{d}S = 0. \tag{71}$$

Therefore, Eq. (71) is called the path-independent integral for the linear piezoelasticity of QCs. The $x$-component of $I_p$ is the $J$-integral of fracture mechanics of the piezoelasticity of QCs.

It is seen from Eq. (65) that the energy function $F$ is not positive definite. We further define an internal energy density $W$ that represents the internal energy per unit volume as follows:

$$\begin{aligned} W &= \frac{1}{2}(\sigma_{ij}\varepsilon_{ij} + H_{ij}w_{ij} + D_iE_i) \\ &= F + D_iE_i \\ &= \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}K_{ijkl}w_{ij}w_{kl} + \frac{1}{2}\lambda_{ij}E_iE_j + R_{ijkl}\varepsilon_{ij}w_{kl}. \end{aligned} \tag{72}$$

## 7 Energy release rate

The path-independent integral derived earlier can be used to obtain the energy release rate for the mode III piezoelasticity fracture problem of QCs. Denote $J$ to be the $x$-component of the conservation integral $I_p$. Then, the path-independent integral takes the following form:

$$J = I_x = \int_S (Fn_x - \sigma_{ij}n_ju_{i,x} - H_{ij}n_jw_{i,x} + D_jn_jE_x)\mathrm{d}S. \tag{73}$$

Using the solution obtained previously, the $J$-integral can be obtained by evaluating Eq. (73) on a vanishingly small contour at a crack tip as follows:

$$J = \frac{K_\parallel^{\mathrm{III}}K_\parallel^{\mathrm{S}} + K_\perp^{\mathrm{III}}K_\perp^{\mathrm{S}} - K_{\mathrm{D}}K_{\mathrm{E}}}{2}. \tag{74}$$

This result can also be obtained by considering the virtual crack closure integral as follows:

$$\begin{aligned} G &= \lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta (\sigma_{yz}(x+a,\ 0)u_z(x+a-\delta,\ 0) \\ &\quad + H_{yz}(x+a,\ 0)w_z(x+a-\delta,\ 0) + D_y(x+a,\ 0)\varphi(x+a-\delta,\ 0))\mathrm{d}x \\ &= \frac{1}{2}(K_\parallel^{\mathrm{III}}K_\parallel^{\mathrm{S}} + K_\perp^{\mathrm{III}}K_\perp^{\mathrm{S}} - K_{\mathrm{D}}K_{\mathrm{E}}). \end{aligned} \tag{75}$$

It is clear that due to the linear piezoelasticity of QCs, for the purely elastic and piezoelectroelastic cases, the value of the $J$-integral is identical to the energy release rate $G$, i.e.,

$$J = G. \tag{76}$$

Substituting Eq. (63) into Eq. (75), the energy release rate can be expressed by the field intensity factors as follows:

$$\begin{aligned} G &= \frac{1}{2}(K_{\parallel}^{\mathrm{S}},\ K_{\perp}^{\mathrm{S}},\ K_{\mathrm{E}}) \begin{pmatrix} C_{44} & R_3 & -e_{15} \\ R_3 & K_2 & -d_{15} \\ -e_{15} & -d_{15} & -\lambda_{11} \end{pmatrix} (K_{\parallel}^{\mathrm{S}},\ K_{\perp}^{\mathrm{S}},\ K_{\mathrm{E}})^{\mathrm{T}} \\ &= \frac{1}{2}(K_{\parallel}^{\mathrm{III}},\ K_{\perp}^{\mathrm{III}},\ K_{\mathrm{D}}) \begin{pmatrix} C_{44} & R_3 & e_{15} \\ R_3 & K_2 & d_{15} \\ e_{15} & d_{15} & -\lambda_{11} \end{pmatrix}^{-1} (K_{\parallel}^{\mathrm{III}},\ K_{\perp}^{\mathrm{III}},\ K_{\mathrm{D}})^{\mathrm{T}}, \end{aligned} \tag{77}$$

which indicates that the energy release rate depends not only on the field intensity factors but also on the material constants. Thus, the energy release rate can be used as the fracture criterion for the piezoelasticity of QCs. If there is no applied electric loading at infinity, i.e., $D^{\infty} = 0$ or $E^{\infty} = 0$, Eq. (77) reduces to the results of 1D hexagonal QCs[36]. If there is no applied phason field at infinity, i.e., $H_{zy}^{\infty} = 0$ or $w_z^{\infty} = 0$, Eq. (77) reduces to the results of piezoelectric materials[44].

## 8 Conclusions

Based on the fundamental equations of piezoelasticity of QCs, the governing equations of plane piezoelasticity problems for 1D QCs with all point groups are investigated systematically with the symmetry operations of point groups. The equilibrium problem of piezoelasticity of 1D QCs is more complicated than that of 3D classical elasticity, 1D QC elasticity, and purely piezoelectric materials. When the electric field is neglected, the obtained governing equations in this paper are identical to the governing equations of the plane elasticity problems for the corresponding QCs[9]. If the phason field is not considered, the present governing equations can be reduced to the results of piezoelectric materials[36]. The general solutions of the plane piezoelasticity problems for 1D QCs with all point groups are derived by the operator method and the complex variable function method. For some special QC systems such as orthorhombic QCs, tetragonal QCs, and hexagonal QCs, the plane piezoelasticity problem can be decomposed into two uncoupled problems, i.e., the classical plane strain elasticity problem of conventional crystals and the phonon-phason-electric coupling anti-plane elasticity problem of 1D QCs.

As an application, a mode III piezoelasticity fracture is formulated, and the solutions of the phonon, phason, and electric fields near the crack tip are obtained by the semi-inverse method for the case of out-of-plane mechanical and in-plane electrical loadings. The stresses of the phonon field, the phason field, and the electric displacements at the crack tip show traditional square root singularities. Among the phonon field, the phason field, and the electrical field, any applied load alone cannot cause the singularities for the other two fields. The path-independent integral derived from the conservation integral equals the energy release rate, which can be used as the fracture criterion. The present results can be reduced to the earlier theories of elasticity of QCs and piezoelectric materials, which pave the way to the forthcoming study of dislocation, fracture, interface, and similar problems of both elasticity and piezoelasticity of QCs.

## References

[1] Shechtman, D., Blech, I., Gratias, D., and Cahn, J. W. Metallic phase with long-range orientational order and no translational symmetry. *Physical Review Letters*, **53**, 1951–1953 (1984)

[2] Elser, V. Comment on QCs: a new class of ordered structures. *Physical Review Letters*, **54**, 1730 (1985)

[3] Kramer, P. and Neri, R. On periodic and non-periodic space fillings obtained by projection. *Acta Crystallographica*, **A40**, 580–587 (1984)

[4] Ding, D. H., Yang, W. G., Hu, C. Z., and Wang, R. H. Generalized elasticity theory of QCs. *Physical Review B*, **48**, 7003–7009 (1993)

[5] Yang, W. G., Wang, R. H., Ding, D. H., and Hu, C. Z. Linear elasticity theory of cubic QCs. *Physical Review B*, **48**, 6999–7002 (1993)

[6] Hu, C. Z., Yang, W. G., Wang, R. H., and Ding, D. H. Point groups and elastic properties of two-dimensional QCs. *Acta Crystallographica*, **A52**, 251–256 (1996)

[7] Hu, C. Z., Wang, R. H., and Ding, D. H. Symmetry groups, physical property tensors, elasticity and dislocations in QCs. *Reports on Progress in Physics*, **63**, 1–39 (2000)

[8] Fan, T. Y. and Mai, Y. W. Elasticity theory, fracture mechanics, and some relevant thermal properties of quasi-crystalline materials. *Applied Mechanics Reviews*, **57**, 325–343 (2004)

[9] Liu, G. T., Fan, T. Y., and Guo, R. P. Governing equations and general solutions of plane elasticity of one-dimensional QCs. *International Journal of Solids and Structures*, **41**, 3949–3959 (2004)

[10] Chen, W. Q., Ma, Y. L., and Ding, H. J. On three-dimensional elastic problems of one-dimensional hexagonal quasicrystal bodies. *Mechanics Research Communications*, **31**, 633–641 (2004)

[11] Wang, X. The general solution of one-dimensional hexagonal quasicrystal. *Mechanics Research Communications*, **33**, 576–580 (2006)

[12] Wang, X. and Pan, E. Analytical solutions for some defect problems in 1D hexagonal and 2D octagonal quasicrystals. *Pramana - Journal of Physics*, **70**, 911–933 (2008)

[13] Gao, Y., Xu, S. P., and Zhao, B. S. General solutions of equilibrium equations for 1D hexagonal QCs. *Mechanics Research Communications*, **36**, 302–308 (2009)

[14] Liu, G. T., Guo, R. P., and Fan, T. Y. On the interaction between dislocations and cracks in one-dimensional hexagonal QCs. *Chinese Physics*, **2**, 1149–1155 (2003)

[15] Gao, Y., Zhao, B. S., and Xu, S. P. A theory of general solutions of plane problems in two-dimensional octagonal QCs. *Journal of Elasticity*, **93**, 263–277 (2008)

[16] Gao, Y. and Zhao, B. S. General solutions of three-dimensional problems for two-dimensional QCs. *Applied Mathematical Model*, **33**, 3382–3391 (2009)

[17] Fan, T. Y. and Guo, L. H. The final governing equation and fundamental solution of plane elasticity of icosahedral QCs. *Physics Letters A*, **341**, 235–239 (2005)

[18] Li, L. H. and Fan, T. Y. Final governing equation of plane elasticity of icosahedral QCs and general solution based on stress potential function. *Chinese Physics Letters*, **23**, 2519–2521 (2006)

[19] Gao, Y. and Zhao, B. S. A general treatment of three-dimensional elasticity of QCs by an operator method. *Physica Status Solidi*, *B*, *Basic Solid State Physics*, **243**, 4007–4019 (2006)

[20] Gao, Y. Governing equations and general solutions of plane elasticity of cubic QCs. *Physics Letters A*, **373**, 885–889 (2009)

[21] Thiel, P. A. and Dubois, J. M. QCs reaching maturity for technological applications. *Materials Today*, **2**, 3–7 (1999)

[22] Athanasiou, N. S., Politis, C., Spirlet, J. C., Baskoutas, S., and Kapaklis, V. The significance of valence electron concentration on the formation mechanism of some ternary aluminum-based QCs. *International Journal of Modern Physics*, **B16**, 4665–4683 (2002)

[23] Park, J. Y., Ogletree, D. F., Salmeron, M., Ribeiro, R. A., Canfield, P. C., Jenks, C. J., and Thiel, P. A. High frictional anisotropy of periodic and aperiodic directions on a quasicrystal surface. *Science*, **309**, 1354–1356 (2005)

[24] Park, J. Y., Sacha, G. M., Enachescu, M., Ogletree, D. F., Ribeiro, R. A., Canfield, P. C., Jenks, C. J., Thiel, P. A., Sáenz, J. J., and Salmeron, M. Sensing dipole fields at atomic steps with combined scanning tunneling and force microscopy. *Physical Review Letters*, **95**, 136802 (2005)

[25] Lipp, H., Engel, M., and Trebin, H. R. Phason dynamics in one-dimensional lattices. *Physical Review B*, **81**, 064302 (2010)

[26] Engel, M., Umezaki, M., and Trebin, H. R. Dynamics of particle flips in two-dimensional QCs. *Physical Review B*, **82**, 134206 (2010)

[27] Guo, J. H., Yu, J., and Si, R. A semi-inverse method of a Griffith crack in one-dimensional hexagonal QCs. *Applied Mathematics and Computation*, **219**, 7445–7449 (2013)

[28] Guo, J. H., Yu, J., and Xing, Y. M. Anti-plane analysis on a finite crack in a one-dimensional hexagonal quasicrystal strip. *Mechanics Research Communications*, **52**, 40–45 (2013)

[29] Jazbec, S., Vrtnik, S., Jagličić, Z., Kashimoto, S., Ivkov, J., Popčević, P., Smontara, A., Kim, H. J., Kim, J. G., and Dolinšek, J. Electronic density of states and metastability of icosahedral Au-Al-Yb quasicrystal. *Journal of Alloys and Compounds*, **586**, 343–348 (2014)

[30] Yang, W. G., Wang, R., Ding, D. H., and Hu, C. Z. Elastic strains induced by electric fields in QCs. *Journal of Physics*: *Condensed Matter*, **7**, L499–L502 (1995)

[31] Hu, C. Z., Wang, R., Ding, D. H., and Yang, W. Piezoelectric effects in QCs. *Physical Review B*, **56**, 2463–2468 (1997)

[32] Rao, R. M. K., Rao, H. P., and Chaitanya, B. S. K. Piezoelectricity in QCs: a group-theoretical study. *Pramana - Journal of Physics*, **68**, 481–487 (2007)

[33] Grimmer, H. The piezoelectric effect of second order in stress or strain: its form for crystals and QCs of any symmetry. *Acta Crystallographica A*, **63**, 441–446 (2007)

[34] Fujiwara, T. and Ishii, Y. *Quasicrystals*, *Handbook of Metal Physics*, Elsevier, Amsterdam (2008)

[35] Suck, J. B., Schreiber, M., and Haussler, P. *Quasicrystals*: *An Introduction to Structure*, *Physical Properties and Applications*, Springer-Verlag, Berlin (2010)

[36] Fan, T. Y. *Mathematical Theory of Elasticity of QCs and Its Applications*, Science Press, Beijing (2011)

[37] Altay, G. and Dökmeci, M. C. On the fundamental equations of piezoelasticity of quasicrystal media. *International Journal of Solids and Structures*, **49**, 3255–3262 (2012)

[38] Li, X. Y., Li, P. D., Wu, T. H., Shi, M. X., and Zhu, Z. W. Three-dimensional fundamental solutions for one-dimensional hexagonal quasicrystal with piezoelectric effect. *Physics Letters A*, **378**, 826–834 (2014)

[39] Zhang, L. L., Zhang, Y. M., and Gao, Y. General solutions of plane elasticity of one-dimensional orthorhombic quasicrystals with piezoelectric effect. *Physics Letters A*, **378**, 2768–2776 (2014)

[40] Merlin, R., Bajema, K., Clarke, R., Yuang, F., and Bhattacharya, P. K. Quasiperiodic GaAs-AlAs heterostructures. *Physical Review Letters*, **55**, 1768–1770 (1985)

[41] Wang, R. H., Yang, W. G., Hu, C. Z., and Ding, D. H. Point and space groups and elastic behaviours of one-dimensional quasicrystals. *Journal of Physics*: *Condensed Matter*, **9**, 2411–2422 (1997)

[42] Wang, M. Z. and Wang, W. Completeness and nonuniqueness of general solutions of transversely isotropic elasticity. *International Journal of Solids and Structures*, **32**, 501–513 (1995)

[43] Wang, W. and Shi, M. X. On the general solutions of transversely isotropic elasticity. *International Journal of Solids and Structures*, **35**, 3283–3297 (1998)

[44] Pak, Y. E. Crack extension force in a piezoelectric material. *Journal of Applied Physics*, **57**, 647–653 (1990)

[45] Mariano, P. M. and Planas, J. Phason self-actions in quasicrystals. *Physica D*: *Nonlinear Phenomena*, **249**, 46–57 (2013)