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Coupled vibrations and frequency shift of compound system consisting of quartz crystal resonator in thickness-shear motions and micro-beam array immersed in liquid*

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Abstract The dynamic characteristics of a quartz crystal resonator (QCR) in thickness-shear modes (TSM) with the upper surface covered by an array of micro-beams immersed in liquid are studied. The liquid is assumed to be inviscid and incompressible for simplicity. Dynamic equations of the coupled system are established. The added mass effect of liquid on micro-beams is discussed in detail. Characteristics of frequency shift are clarified for different liquid depths. Modal analysis shows that a drag effect of liquid has resulted in the change of phase of interaction (surface shear force), thus changing the system resonant frequency. The obtained results are useful in resonator design and applications.

Key words quartz crystal resonator (QCR), micro-beam, modal analysis, frequency shift, added mass of liquid

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1 Introduction

A plate-like quartz crystal resonator (QCR) vibrating in thickness-shear modes (TSM) has been widely used for resonators. The models of QCR in TSM motion carrying different surface attached micro-structures on either or both sides of the crystal plates have gradually been proposed. Unlike a uniform thin film or mass layer, the vibration modes of micro-structures may produce great frequency and mode effects on the dynamic behaviors of the QCR[1]. The effects are fundamental to the improvement of existing acoustic wave devices, or to the design of new acoustic wave devices, especially new sensors based on these effects[2].

As a kind of micro-structures, micro- or nano-scale beam arrays have great potentials for new devices including dynamic tuning of surface wetting, dry adhesives that mimic gecko foot fibrillars, efficient micro-needles in drug delivery, substrates for sensing cell response, and micro-electro mechanical systems (MEMS) actuators[3–7]. Recently, some researchers have been trying

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to use a QCR in TSM carrying an array of micro-beams to investigate the possibility of characterizing the geometric/physical properties of these fibers from their frequency effects on the QCR. Liu et al.[8] studied the micro-beams using the Euler-Bernoulli theory of bending and obtained some useful results of dynamic characteristics of a quartz plate covered by micro-beams. Zhang et al.[9] treated the beams as rigid and considered the effect of moments existing on beam/QCR interface. The research clarified the influence of surface bending moment on QCR vibrations.

In this paper, we establish a liquid-solid coupled dynamic model to study the vibration characteristics of a compound QCR system consisting of a QCR and micro-beam array immersed in liquid. The general solutions for the quartz plate under TSM vibrations are expressed in Section 2. The governing equations for the micro-beams/liquid system are established and the solution procedure is discussed in Section 3. Two solutions obtained in Sections 2 and 3 are coupled as a whole in Section 4, and the dynamic characteristics of the compound QCR system are numerically analyzed in detail in Section 5. Finally, a few useful conclusions are drawn in Section 6.

2 Equations and fields of quartz crystal plate under TSM vibrations

Consider an AT-cut electroded quartz plate whose lower surface is traction-free, while the upper surface covered with an array of elastic beams is immersed in liquid (see Fig. 1). For thickness-shear vibrations of a quartz crystal plate, the governing equations are[10]

$$\begin{cases} T_{21,2} = \rho_{\mathrm{Q}} \ddot{u}_1, \quad D_{2,2} = 0, \\ T_{21} = c_{66} u_{1,2} - e_{26} E_2, \quad D_2 = e_{26} u_{1,2} + \varepsilon_{22} E_2, \end{cases} \tag{1}$$

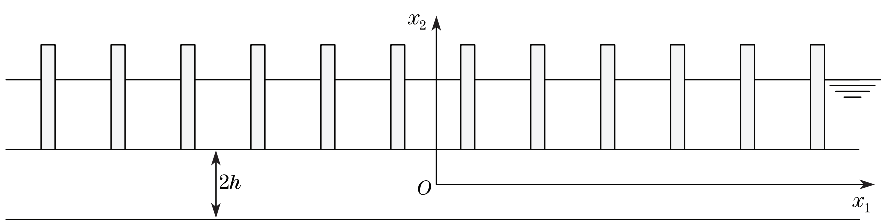

Fig. 1 Quartz crystal plate with array of elastic beams immersed in liquid

where the electric field $E_2 = -\varphi_{,2}$, φ denotes the electric potential, T_{21} is the stress component, u_1 is the mechanical displacement, and D_2 is the electric displacement. ρ_{Q} stands for the mass density of QCR. c_{66}, e_{26}, and ε_{22} denote the elastic, piezoelectric, and dielectric constants, respectively. Because the crystal plate is under harmonic vibrations, the time-harmonic factor, $\exp(\mathrm{i}\omega t)$, will be dropped hereafter for simplicity. Thus, we get the general solution for (1) as

$$\begin{cases} u_1(x_2) = P_1 \sin(k_{\mathrm{Q}} x_2) + P_2 \cos(k_{\mathrm{Q}} x_2), \\ \varphi(x_2) = \dfrac{e_{26}}{\varepsilon_{22}} \left(P_1 \sin(k_{\mathrm{Q}} x_2) + P_2 \cos(k_{\mathrm{Q}} x_2)\right) + P_3 x_2 + P_4, \\ T_{21}(x_2) = c_{\mathrm{Q}} \left(P_1 k_{\mathrm{Q}} \cos(k_{\mathrm{Q}} x_2) - P_2 k_{\mathrm{Q}} \sin(k_{\mathrm{Q}} x_2)\right) + e_{26} P_3, \end{cases} \tag{2}$$

where P_1 , P_2 , P_3, and P_4 are constants to be determined, and

$$c_{\mathrm{Q}} = \bar{c}_{66} = c_{66}\Big(1 + \frac{e_{26}^2}{(\varepsilon_{22} c_{66})}\Big), \quad k_{\mathrm{Q}} = \omega \Big(\frac{\rho_{\mathrm{Q}}}{c_{\mathrm{Q}}}\Big)^{1/2}. \tag{3}$$

Under the case without surface beams and liquid, the boundary conditions of QCR under TSM vibrations are (assuming that the electrodes are shorted and thin enough)

$$\varphi|_{x_2=-h} = \varphi|_{x_2=h}, \quad T_{21}|_{x_2=\pm h} = 0. \tag{4}$$

We are only interested in the anti-symmetric modes of the QCR which are excitable by a thickness electric field[10]. Thus, the frequency equation of the anti-symmetric modes can be obtained from (2) and (4) as follows:

$$\frac{e_{26}^2}{c_{\mathrm{Q}}\varepsilon_{22}} - \frac{k_{\mathrm{Q}}h}{\tan(k_{\mathrm{Q}}h)} = 0. \tag{5}$$

It has the same expression as the work by some previous researchers, such as Tiersten[10], Yang et al.[11], and Liu et al.[12].

3 Analysis on liquid fields and beam-liquid coupling vibrations

Figure 2(a) shows a beam immersed in liquid with the bottom fixed on a substrate, i.e., the QCR upper-surface, which possesses a horizontal oscillation motion, $u_1(h)$, in the y-direction. Obviously, the beam displacement Y consists of two parts: the bending deflection w in the yz-plane (see Fig. 2(b)) and a rigid-body displacement $u_1(h)$. Referred to (A5) in Appendix A, the deflection w can be expressed through the natural modes, $W_n(\bar{z})$, of micro-beams, $\bar{z} = z/L$. Thus, the normalized beam displacement $\overline{Y}$ $(= Y/R)$ can be written as

$$\overline{Y} = \frac{u_1(h)}{R}\Big(1 + \sum_{n=1}^{N} D_n W_n(\bar{z})\Big). \tag{6}$$

It is noteworthy that for the low modes of the beam in our following numerical calculation, a modest N, such as 100, is big enough to assure convergence.

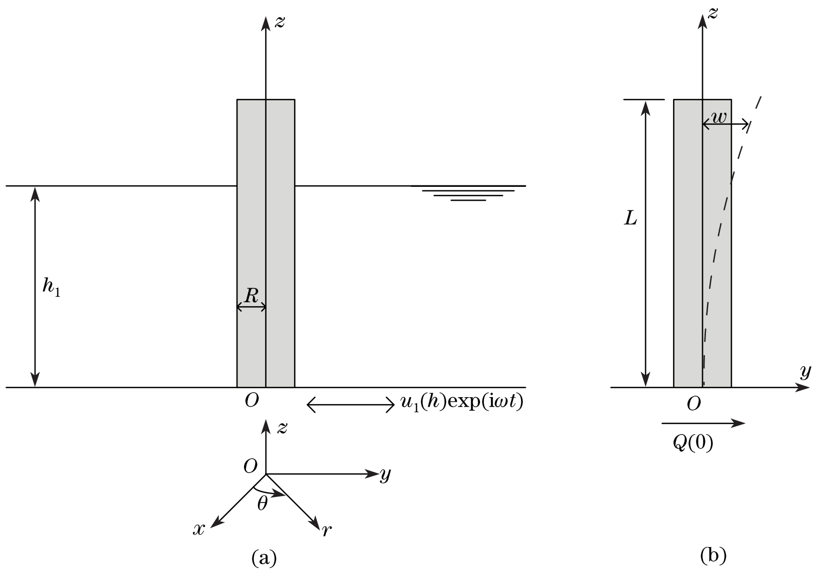

Fig. 2 (a) Notation and coordinate system for liquid domain and (b) configuration and coordinate system for beam bending

As for liquid fields, we assume that the liquid is inviscid and incompressible. For the time-harmonic motions of liquid in the cylindrical coordinate system in Fig. 2(a), the velocity potential can be described as follows[13]:

$$\Phi\left(r,\theta,z\right) = \varphi\left(r,\theta,z\right), \quad r > R, \quad 0 \leqslant z \leqslant h_1, \quad 0 \leqslant \theta < 2\pi, \tag{7}$$

where the time-harmonic factor $\exp(\mathrm{i}\omega t)$ has been dropped as above. The potential φ should satisfy the Laplace equation

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{8}$$

The pressure on the free liquid surface at $z=h_1$ vanishes, and we obtain the following equation:

$$\varphi|_{z=h_1}=0,\quad r>R,\quad 0\leqslant\theta<2\pi. \tag{9}$$

The condition without separation of liquid from QCR surface requires

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0}=0,\quad r>R,\quad 0\leqslant\theta<2\pi. \tag{10}$$

In addition, the continuity of velocity at the interface $r=R$ yields

$$\frac{\partial\varphi}{\partial r}=\mathrm{i}\omega Y\sin\theta,\quad 0\leqslant z\leqslant L,\quad 0\leqslant\theta<2\pi. \tag{11}$$

By separation of variables, the potential φ can be expressed as[13]

$$\varphi\left(r,\theta,z\right)=\mathrm{i}\omega\sum_{s=1}^{M}B_s\mathrm{K}_{1s}\left(\xi_s r\right)\cos\left(\xi_s z\right)\sin\theta,\quad r\geqslant R,\quad 0\leqslant z\leqslant h_1,\quad 0\leqslant\theta<2\pi, \tag{12}$$

where $\xi_s=\pi(2s-1)/(2h_1)$. M denotes the truncation order in the numerical analysis. It should be noted that M is set as 100 in the below calculation. K_{1s} is the modified Bessel functions of first-order which equates null at infinity.

Multiplying both sides of (11) by $\cos(\xi_s z)$ and integrating with respect to z over $(0, h_1)$, we obtain

$$B_s\mathrm{K}_{1s}\left(\xi_s R\right)=k_s\int_0^{h_1}Y\left(z\right)\cos\left(\xi_s z\right)\mathrm{d}z, \tag{13}$$

where

$$k_s=\frac{4}{\pi\left(2s-1\right)}\frac{\left.\mathrm{K}_{1s}\left(\xi_s r\right)\right|_{r=R}}{\left.\mathrm{K}'_{1s}\left(\xi_s r\right)\right|_{r=R}}. \tag{14}$$

It follows from (12) that the liquid force per unit length acted on the beam $(0\leqslant z\leqslant h_1)$ is

$$q_{\mathrm{f}}\left(z\right)=-\rho_{\mathrm{l}}\int_0^{2\pi}\mathrm{i}\omega\varphi\left(R,\theta,z\right)\left(-\sin\theta\right)R\mathrm{d}\theta=-\pi\rho_{\mathrm{l}}R\omega^2\sum_{s=1}^{M}B_s\mathrm{K}_{1s}\left(\xi_s R\right)\cos\left(\xi_s z\right), \tag{15}$$

where ρ_{l} is the mass density of liquid. Then, the normalized dynamic equation of beams coupled with liquid is

$$\frac{\mathrm{d}^4\overline{Y}}{\mathrm{d}\bar{z}^4}-\alpha^4\overline{Y}-\frac{L^4}{EIR}q_{\mathrm{f}}H\left(\bar{h}_1-\bar{z}\right)=0, \tag{16}$$

where $H(\cdot)$ stands for the Heaviside step function and $\overline{h}_1=h_1/L$. Substituting (6), (13), and (15) into (16) yields

$$\begin{aligned}&\sum_{n=1}^{N}(\omega_{0n}^2-\omega^2)D_nW_n(\bar{z})-\omega^2\\&+\mu\omega^2\sum_{s=1}^{M}\Big(f_s+\sum_{n=1}^{N}k_sD_nM_{ns}\Big)\cos(\bar{\xi}_s\bar{z})H(\bar{h}_1-\bar{z})=0,\end{aligned} \tag{17}$$

where

$$\mu = \frac{\rho_l L}{\rho_b R}, \quad f_s = (-1)^{s-1} k_s/\bar{\xi}_s, \quad M_{ns} = \int_0^{h_1/L} W_n(\bar{z}) \cos(\bar{\xi}_s \bar{z})\, d\bar{z}, \quad \bar{\xi}_s = \xi_s L. \tag{18}$$

Expanding both sides of (17) with $W_i(\bar{z})$ as the expanded base-functions, we obtain a set of linear equations to determine the unknowns D_n as follows:

$$\begin{aligned} &(\omega_{0i}^2 - \omega^2) D_i \int_0^1 (W_i(\bar{z}))^2\, d\bar{z} - \omega^2 \int_0^1 W_i(\bar{z})\, d\bar{z} \\ &+ \mu\omega^2 \sum_{s=1}^{M} \Big(f_s + \sum_{n=1}^{N} k_s D_n M_{ns}\Big) M_{is} = 0, \quad i = 1, 2, \cdots, N. \end{aligned} \tag{19}$$

4 Coupled vibrations of QCR over-coated by micro-beams immersed in liquid

When the coupled beams-liquid structure appears on the upper-surface of QCR, the boundary conditions (4) become

$$\begin{cases} T_{21}|_{x_2=h} = -\overline{N} Q(0), \quad u_1(h) = Y(0), \\ \varphi|_{x_2=-h} = \varphi|_{x_2=h}, \quad T_{21}|_{x_2=-h} = 0, \end{cases} \tag{20}$$

where $\overline{N}$ is the number density of beams per unit area of crystal surface, and $Q(0)$ denotes the shear force at the beam bottom cross-section. It is important to point out that for sparse beam arrays, the interaction of adjacent beams has been ignored. Substituting (24) into (2), we obtain the frequency equation of the compound QCR system consisting of liquid-beams and QCR as follows:

$$\frac{e_{26}^2}{c_Q \varepsilon_{22}} - \frac{k_Q h}{\tan(k_Q h)} = -\vartheta (k_Q h)^2 \Big(1 + \frac{e_{26}^2}{c_Q \varepsilon_{22}} \frac{\cot(k_Q h)}{k_Q h} - \cot^2(k_Q h)\Big), \tag{21}$$

in which

$$\vartheta = \nu\varsigma, \quad \nu = \overline{N}\pi R^2 \Big(\frac{\rho_b}{\rho_Q} \frac{L}{2h}\Big), \quad \varsigma = 2 \sum_{n=1}^{N} \frac{D_n}{\alpha_n} \Big(\frac{\omega_n}{\omega}\Big)^2, \tag{22}$$

which is similar to the result obtained by Hu et al.[1]

It follows from (21) and (22) that ϑ may be called the effective mass ratio of the beam-array immersed in liquid to QCR[8]. ν is the actual mass ratio between the beams and the elastic plate. Therefore, ς can be called the proportional coefficient between the effective mass ratio ϑ and the actual mass ratio ν. Obviously, ς will be affected by many factors related to micro-beams and liquid, e.g., beam elastic modulus, aspect ratio, deflection modes, liquid density, and depth h_1. It is readily found that the effective mass ratio may be either positive or negative, depending on the positive/negative definiteness of the proportional coefficient ς, which implies that the effective mass ratio ϑ is frequency-dependent.

5 Results and discussion

We consider an AT-cut quartz crystal plate with c_{66}=29.01×10^9 N·m^2, $e_{26} = -0.095$ C/m^2, ε_{22}=39.82×10^{-12} C/Vm, ρ_Q=2649 kg/m^3[14], and h=0.165 mm. For the micro-beam array, ρ_b= 2 500 kg/m^3, D=2R=0.1 μm, L=2 μm, and $\overline{N}$=1×10^{12} m^{-2}, unless otherwise stated. For liquid, ρ_l=1 000 kg/m^3. It is readily obtained that the fundamental frequency of QCR is f_0= 5 MHz.

Figure 3 shows dependence of frequency shift versus the normalized elastic modulus log (E/E_0) from 0.0 to 3.0 for h_1/L=0.0, 0.5, 0.8, 1.0, respectively, where E_0 =1×10^7 Pa. The picture exhibits a series cycles, and the frequency shift varies from minimum to maximum inside each cycle, which is easily understood because micro-beams with larger elastic modulus produce stronger constraint on QCR for fixed density and fixed vibration mode (i.e., the node number of vibration mode keeps invariant). Further modal analysis shows that the right, middle, and the left jump points in Fig. 3, respectively, correspond to the first-, the second-, and the third-order natural frequencies of micro-beams. We can also find from Fig. 3 that the effect of liquid on frequency shift of the compound QCR system gradually becomes evident with the increase in liquid depth as expected. At those points where frequency shift appears jump, for example at point M, log(E/E_0)=1.117 for no-liquid case, the natural frequency of micro-beams is just equal to 5 MHz, which results in resonance.

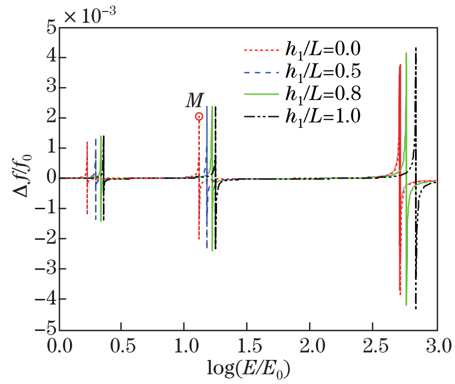

Fig. 3 Dependence of frequency shift upon normalized elastic modulus log(E/E_0)

Figure 4 shows the displacement distributions of micro-beams at two sides of point M for h_1/L=0.0, 0.5, 0.8, 1.0, respectively. At the left of point M, log(E/E_0)=1.083, the node numbers of beam deflections are totally the same for the four cases (see Fig. 4(a)). However, the appearance of liquid makes successive decrease in the vibration amplitude with increasing liquid depth. This is easily understood that the effect of liquid on the vibrations is equivalent to some added mass to the beam. Thus, the rise in system mass (or inertia effect) will result in the decrease of frequency shift for fixed beam-stiffness. On the other hand, at the right of point M, log(E/E_0)=1.152, only one node exists for the beam deflection without liquid, while the other three cases are still with two nodes (see Fig. 4(b)). Calculating on $Q(0)=EIY'''(0)$, we find that the shear force acted on the QCR upper surface has become negative for null liquid, but the others still remain positive due to the liquid drag. A positive shear force $Q(0)$ produces constraint on QCR vibrations to induce the rise in resonant frequency, while a negative shear force $Q(0)$ increases system inertia effect.

Figure 5 shows dependence of frequency shift upon aspect ratio of beams under E=5×10^6 Pa and L=2 μm. The dependence exhibits a series of cycles. Likewise, the appearance of liquid leads to some left-shift of the picture as expected.

6 Conclusions

A coupled dynamic model of a micro-beam array immersed in liquid and a QCR vibrating in thickness-shear modes is established. Effect of liquid on dynamic performance of the compound QCR system is analyzed in detail. Dependence of frequency shift on the normalized elastic modulus/aspect ratio of micro-beams is calculated and the cyclical feature is revealed.

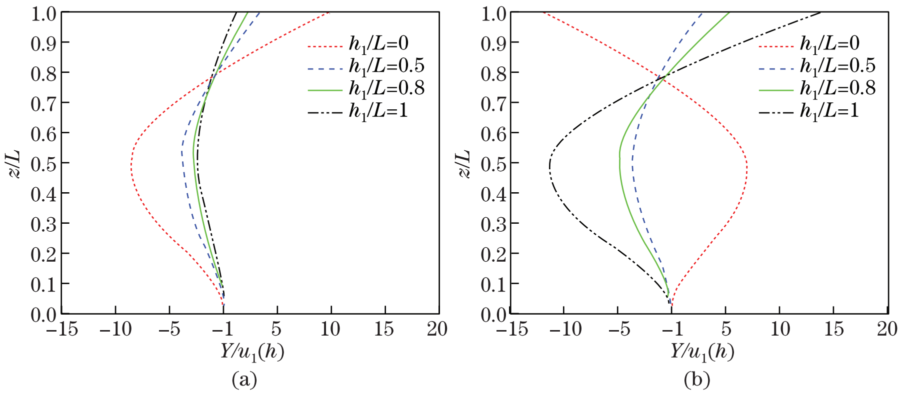

Fig. 4 Beam displacement distributions at two sides of point M: (a) at left; (b) at right

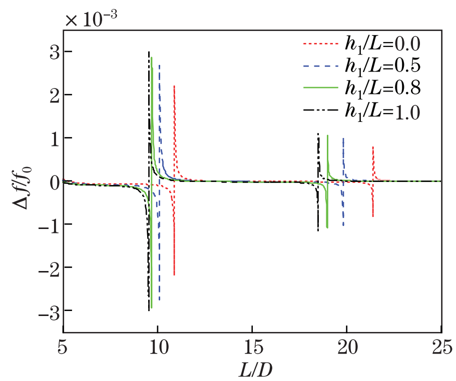

Fig. 5 Dependence of frequency shift upon aspect ratio of beams

Comparison between with liquid and without liquid shows that the frequency picture has been transferred a little as expected due to the liquid drag. The results obtained are useful to QCR design and applications.

References

[1] Hu, Y. T., Hu, H. L., Luo, B., Xue, H., Xie, J. M., and Wang, J. Frequency shift of a crystal quartz resonator in thickness-shear modes induced by an array of hemispherical material units. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, **60**, 1777–1782 (2013)

[2] Liu, N. *Vibrations of Crystal Plates with Surface Structures for Resonator and Sensor Applications*, Ph. D. dissertation, University of Nebraska-Lincoln (2012)

[3] Geim, A. K., Dubonos, S. V., Grigorieva, I. V., Novoselov, K. S., Zhukov, A. A., and Shapoval, S. Y. Microfabricated adhesive mimicking gecko foot-hair. *Nature Materials*, **2**, 461–463 (2003)

[4] Kim, K., Park, S., Lee, J. B., Manohara, H., Desta, Y., Murpphy, M., and Ahn, C. H. Rapid replication of polymeric and metallic high aspect ratio microstructures using PDMS and LIGA technology. *Microsystem Technologies*, **9**, 5–10 (2002)

[5] McAllister, D. V., Wang, P. M., Davis, S. P., Park, J. H., Canatella, P. J., Allen, M. G., and Prausnitz, M. R. Microfabricated needles for transdermal delivery of macromolecules and nanoparticles: fabrication methods and transport studies. *Proceedings of the National Academy of Sciences of the United States of America*, **100**, 13755–13760 (2003)

[6] Nomura, S., Kojima, H., Ohyabu, Y., Kuwabara, K., Miyauchi, A., and Uemura, T. Cell culture on nanopillar sheet: study of HeLa cells on nanopillar sheet. *Japanese Journal of Applied Physics*, **44**, L1184–L1186 (2005)

[7] Zhang, L., Zhou, Z. L., Cheng, B., DeSimone, J. M., and Samulski, E. T. Superhydrophobic behavior of a perfluoropolyether lotus-leaflike topography. *Langmuir*, **22**, 8576–8580 (2006)

[8] Liu, N., Yang, J. S., and Wang, J. Shear vibration of a crystal plate carrying an array of microbeams. *Philosophical Magazine Letters*, **91**, 572–581 (2011)

[9] Zhang, R. Y., Xie, J. M., Hu, Y. T., Yang, J. S., and Chen, X. D. Thickness-shear vibration of an elastic plate carrying an array of rigid microbeams with consideration of couple stresses. *International Journal of Engineering Science*, **51**, 179–189 (2012)

[10] Tiersten, H. F. Thickness vibrations of piezoelectric plates. *Journal of the Acoustical Society of America*, **35**, 53–58 (1963)

[11] Yang, J. S., Zhou, H. G., and Zhang, W. P. Thickness-shear vibration of rotated Y-cut quartz plates with relatively thick electrodes of unequal thickness. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, **52**, 918–922 (2005)

[12] Liu, N., Yang, J. S., and Jin, F. Transient thickness-shear vibration of a piezoelectric plate of monoclinic crystals. *International Journal of Applied Electromagnetics and Mechanics*, **38**, 27–37 (2012)

[13] Huang, Y. Y. *Structure Vibration Analysis* (in Chinese), Huazhong University of Science and Technology Press, Wuhan, 220–224 (1988)

[14] Yang, J. S. *An Introduction to the Theory of Piezoelectricity*, Springer, New York, 290–294 (2005)

[15] Meirovitch, L. *Analytical Methods in Vibrations*, Macmillan, New York (1967)

[16] Gere, J. M. *Mechanics of Materials*, Brooks/Cole, Pacific Grove (2001)

Appendix A

For flexural motions of beams, we use the Euler-Bernoulli theory of bending[15–16]. The time-harmonic factor, $\exp(\mathrm{i}\omega t)$, will be dropped as before in the following discussion.

The normalized governing equation of free-vibration beams is

$$\frac{\mathrm{d}^4 W}{\mathrm{d}\bar{z}^4} - \alpha^4 W = 0 \tag{A1}$$

with $\bar{z} = z/L$, $W = w/R$, and $\alpha=(\rho_{\mathrm{b}} A\omega^2 L^4/EI)^{1/4}$. E is Young's modulus of beams, ρ_{b} stands for the mass density, L is the beam length, and I is the moment of inertia. The beam cross-section is assumed circular with R as radius and A as area.

When the cantilever beam is under free flexural vibrations, the boundary conditions are

$$W(0) = W'(0) = W''(1) = W'''(1) = 0. \tag{A2}$$

It can be obtained from (A1) and (A2) that the natural frequencies and the corresponding natural modes of beams, respectively, are

$$\begin{cases} \omega_n = \left(\dfrac{\alpha_n}{L}\right)^2 \sqrt{\dfrac{EI}{\rho_{\mathrm{b}} A}}, \quad n = 1, 2, 3, \cdots, \\ W_n(\bar{z}) = \lambda_n (\cos(\alpha_n \bar{z}) - \mathrm{ch}(\alpha_n \bar{z})) + \sin(\alpha_n \bar{z}) - \mathrm{sh}(\alpha_n \bar{z}), \\ \lambda_n = -(\sin\alpha_n + \mathrm{sh}\,\alpha_n)/(\cos\alpha_n + \mathrm{ch}\,\alpha_n), \end{cases} \tag{A3}$$

where α_n are the real positive roots of the following equation:

$$\cos\alpha_n \mathrm{ch}\,\alpha_n + 1 = 0. \tag{A4}$$

Through the modal expansion, we can express the normalized deflection W of a cantilever beam as

$$W = \sum_{n=1}^{N} D_n W_n(\bar{z}), \tag{A5}$$

where D_n are the expansion constants to be determined, and N denotes a truncation order.