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Stretching surface in rotating viscoelastic fluid*

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Abstract The boundary layer flow over a stretching surface in a rotating viscoelastic fluid is considered. By applying a similarity transformation, the governing partial differential equations are converted into a system of nonlinear ordinary differential equations before being solved numerically by the Keller-box method. The effects of the viscoelastic and rotation parameters on the skin friction coefficients and the velocity profiles are thoroughly examined. The analysis reveals that the skin friction coefficients and the velocity in the x-direction increase as the viscoelastic parameter and the rotation parameter increase. Moreover, the velocity in the y-direction decreases as the viscoelastic parameter and the rotation parameter increase.

Key words boundary layer, stretching surface, rotating viscoelastic fluid, similarity transformation

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1 Introduction

The study of the boundary layer flow over a stretching surface has attracted the attention of many investigators since it was first introduced by Crane[1]. The astonishing development in this study is mainly due to its numerous applications in engineering and industrial processes. The cooling of an infinite metallic plate in a cooling bath, the aerodynamic extrusion of plastic sheets, the paper production, the metal spinning, and drawing a plastic film are examples of such flows in industry. The development of this problem was initiated analytically by Crane[1] when studying the two-dimensional stretching of a surface in a quiescent fluid. The three-dimensional case of this problem was solved by Wang[2]. Afterwards, many investigations of this type of flow are discussed in various aspects, and the closed analytical form was obtained as well as the numerical solutions. The problem of flow due to a stretching sheet has been continued by Zhu et al.[3], who obtained the analytical solutions using the homotopy analysis method (HAM). Wang[4] solved the steady two-dimensional stretching surface in a rotating fluid using a perturbation method to obtain self-similar solutions. Takhar et al.[5] considered a similar problem in the presence of a magnetic field and solved the problem numerically using a difference-differential method. Later, Rajeswari and Nath[6] and Nazar et al.[7] extended the

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problem considered by Wang[4] to the unsteady flow. Recently, Abbas et al.[8] extended the work of Nazar et al.[7] to the unsteady magnetohydrodynamic (MHD) and heat transfer on a stretching surface in a rotating fluid.

The flow of a viscoelastic fluid has attracted the attention of many researchers. One of the significant contributions in this area was given by Rajagopal et al.[9], who investigated numerically the viscoelastic fluid flow over a stretching sheet and obtained similarity solutions without taking into consideration the heat transfer aspect. The analytical solution to this problem was given by Troy et al.[10]. Quite recently, Cortell[11] analyzed the effect of the viscous dissipation on the viscoelastic fluid flow and heat transfer over a stretching sheet. Moreover, Hayat et al.[12] studied the three-dimensional flow of a viscoelastic fluid over a stretching surface and obtained the analytical solutions by the HAM.

To date, only a few investigations have focused on the study of flow in a rotating viscoelastic fluid. Motivated by this fact, in the present paper, we study a steady boundary layer flow due to a stretching surface in a rotating viscoelastic fluid. By means of the appropriate similarity transformation, the governing partial differential equations are transformed into ordinary differential equations before being solved numerically by the Keller-box method, which is well described in the book by Cebeci and Bradshaw[13]. To validate the obtained numerical results, comparisons are made with those of Wang[4] and Nazar et al.[7] for the steady state case when the viscoelastic parameter is absent.

2 Problem formulation

Consider a steady, laminar, and incompressible fluid flow caused by a stretching sheet in a rotating viscoelastic fluid as shown in Fig. 1. It is assumed that the surface is stretched in the x-direction such that the x component of the velocity varies linearly along it, i.e., $u_{\mathrm{w}}(x) = ax$, where a is a positive constant. Due to the Coriolis force, the fluid motion is three-dimensional. Here, u, v, and w are the velocity components in the direction of the Cartesian axes x, y, and z.

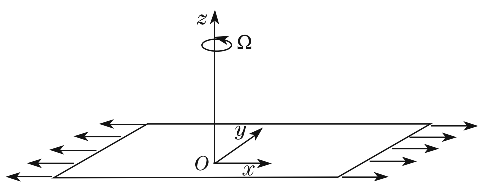

Fig. 1 Physical model and coordinate system

Under the usual boundary layer approximations, the flow is governed by the following equations (see Wang[4] and Hayat et al.[12]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v \\ = & \nu\frac{\partial^2 u}{\partial z^2} - k_0\Big(u\frac{\partial^3 u}{\partial x\partial^2 z} + w\frac{\partial^3 u}{\partial z^3} - \Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ & +2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\Big)\Big), \end{aligned} \tag{2}$$

$$
\begin{aligned}
& u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\Omega u \\
&=\nu\frac{\partial^2 v}{\partial z^2}-k_0\Big(v\frac{\partial^3 v}{\partial y\partial^2 z}+w\frac{\partial^3 v}{\partial z^3}-\Big(\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\
&\quad +2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\Big)\Big)
\end{aligned}
\tag{3}
$$

subjected to the following boundary conditions:

$$
\begin{cases}
u=u_{\mathrm{w}}(x)=ax, \quad v=0, \quad w=0 \quad \text{at} \quad z=0, \\
u\to 0, \quad v\to 0, \quad \dfrac{\partial u}{\partial z}\to 0, \quad \dfrac{\partial v}{\partial z}\to 0 \quad \text{as} \quad z\to\infty,
\end{cases}
\tag{4}
$$

where ν is the kinematic viscosity expressed as

$$
\nu=\frac{\mu}{\rho},
$$

μ is the dynamic viscosity, ρ is the fluid density, Ω is the angular velocity about the z-axis, and k_0 is the material fluid parameter. We look for the solution to Eqs. (1)–(3) of the form

$$
\begin{cases}
u=axf'(\eta), \\
v=axg(\eta), \\
w=-(a\nu)^{\frac{1}{2}}f(\eta), \\
\eta=\Big(\dfrac{a}{\nu}\Big)^{\frac{1}{2}}z,
\end{cases}
\tag{5}
$$

where primes denote differentiation with respect to η.

Substituting Eq. (5) into Eqs. (1) and (2), we can get the following ordinary differential equations:

$$
f'''+ff''-(f')^2+2\lambda g+K(ff^{(4)}-2f'f'''+(f'')^2)=0, \tag{6}
$$

$$
g''+fg'-f'g-2\lambda f'+K(fg'''-(f'')^2-2f'g'')=0, \tag{7}
$$

and the boundary conditions (2) become

$$
\begin{cases}
f(0)=0, \quad g(0)=0, \quad f'(0)=1, \\
f'(\infty)=0, \quad g(\infty)=0, \quad g'(\infty)=0, \quad f''(\infty)=0.
\end{cases}
\tag{8}
$$

Here, K is the dimensionless viscoelastic parameter, and λ is the dimensionless parameter signifying the relative importance of the rotation rate to the stretching rate, which are defined as follows:

$$\begin{cases} K = \dfrac{k_0 a}{\nu}, \\ \lambda = \dfrac{\Omega}{a}. \end{cases}$$

The physical quantities of interest are the skin friction coefficients along the x- and y-directions, i.e., $C_{\mathrm{f}x}$ and $C_{\mathrm{f}y}$, which are defined as

$$\begin{cases} C_{\mathrm{f}x} = \dfrac{\tau_{\mathrm{w}x}}{\rho u_{\mathrm{w}}^2}, \\ C_{\mathrm{f}y} = \dfrac{\tau_{\mathrm{w}y}}{\rho u_{\mathrm{w}}^2}, \end{cases} \tag{9}$$

where $\tau_{\mathrm{w}x}$ is the surface shear stress along the x-direction, and $\tau_{\mathrm{w}y}$ is the surface shear stress along the y-direction, which are defined as

$$\begin{cases} \tau_{\mathrm{w}x} = \mu\Big(\dfrac{\partial u}{\partial z}\Big)_{\mathrm{w}}, \\ \tau_{\mathrm{w}y} = \mu\Big(\dfrac{\partial v}{\partial z}\Big)_{\mathrm{w}}. \end{cases} \tag{10}$$

Using the variables in Eq. (5), we can obtain

$$\begin{cases} (Re_x)^{\frac{1}{2}} C_{\mathrm{f}x} = f''(0), \\ (Re_x)^{\frac{1}{2}} C_{\mathrm{f}y} = g'(0), \end{cases} \tag{11}$$

where Re_x is the local Reynolds number,

$$Re_x = \frac{u_{\mathrm{w}} x}{\nu}.$$

3 Results and discussion

The nonlinear ordinary differential equations (4) and (5) subjected to the boundary conditions (6) were solved numerically using a finite-difference scheme which is known as the Keller-box method[13–14]. The results presented in a table and several graphs illustrate the influence of the dimensionless viscoelastic parameter K and the ratio of the rotation rate to the stretching rate λ on the skin friction coefficient and the similarity velocity profiles both along the x- and y-directions. In order to validate our findings, the case when $K = 0$ (the regular fluid) is also considered, and the results are compared with those reported by Wang[4] and Nazar et al.[7] as presented in Table 1 for different values of the skin-friction coefficient in the x- and y-directions, $f''(0)$ and $g'(0)$, respectively. It is found that the results show favorable agreement.

Table 1 Values of $(Re_x)^{\frac{1}{2}}C_{fx}$ and $(Re_x)^{\frac{1}{2}}C_{fy}$ for different values of λ when $K = 0$

λ	$(Re_x)^{\frac{1}{2}}C_{fx}$			$(Re_x)^{\frac{1}{2}}C_{fy}$		
	Wang[4]	Nazar et al.[7]	Present results	Wang[4]	Nazar et al.[7]	Present results
0.0	–1.000 0	–1.000 0	–1.000 0	0.000 0	0.000 0	0.000 0
0.2	–	–	–1.033 1	–	–	–0.238 5
0.4	–	–	–1.100 9	–	–	–0.431 0
0.5	–1.138 4	–1.138 4	–1.138 4	–0.512 8	–0.512 8	–0.512 8
0.6	–	–	–1.176 4	–	–	–0.587 4
0.8	–	–	–1.251 8	–	–	–0.720 4
1.0	–1.325 0	–1.325 0	–1.325 0	–0.837 1	–0.837 1	–0.837 1
1.2	–	–	–1.395 6	–	–	–0.942 0
1.4	–	–	–1.463 4	–	–	–1.037 9
1.6	–	–	–1.528 7	–	–	–1.126 5
1.8	–	–	–1.591 6	–	–	–1.209 3
2.0	–1.652 3	–1.652 3	–1.652 3	–1.287 3	–1.287 3	–1.287 3
3.0	–	–	–1.928 9	–	–	–1.624 8
4.0	–	–	–2.171 6	–	–	–1.905 4
5.0	–	–	–2.390 1	–	–	–2.150 6

Figures 2 and 3 depict the effects of the viscoelastic parameter K on the similarity velocity profiles $f'(\eta)$ and $g(\eta)$ in the x- and y-directions, respectively. The velocity profiles for different values of the viscoelastic parameter K, in both directions, satisfy the far field boundary condition asymptotically, and thus support the validity of the obtained numerical results. In Fig. 2, we note that the velocity increases as the parameter K increases. This is due to the fact that the viscoelastic fluid accelerates the fluid motion, which increases the boundary layer thickness, and thus increases the velocities in both the x- and y-directions as presented in Figs. 2 and 3. As presented in Fig. 2, for large values of K, the velocities decay monotonically exponentially, while for zero value of K, the decay is oscillatory.

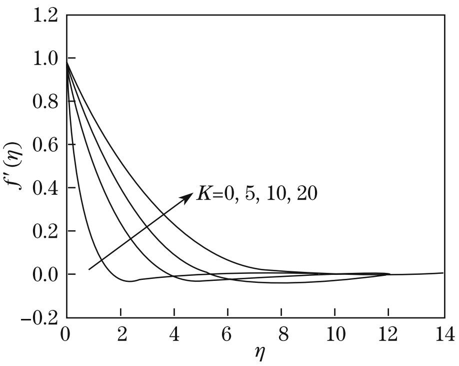

Fig. 2 Similarity velocity profiles in x-direction, $f'(\eta)$, for different values of K when $\lambda = 1$

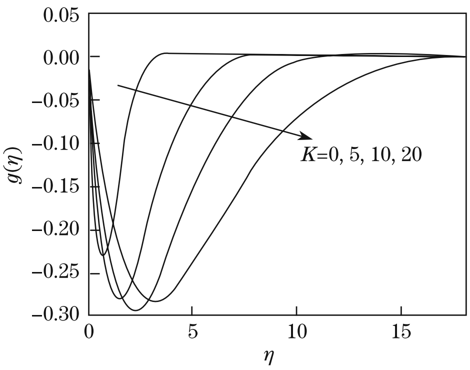

Fig. 3 Similarity velocity profiles in y-direction, $g(\eta)$, for different values of K when λ=1

The effects of the rotation parameter λ on the similarity velocity profiles in the x- and y-directions, $f'(\eta)$ and $g(\eta)$, are presented in Figs. 4 and 5, respectively. The velocity profiles in both directions satisfy the boundary conditions and are represented by the smooth curve for different rotation parameter λ. Figure 4 indicates that the similarity velocity in the x-direction, $f'(\eta)$, decreases with the increase in the rotation parameter λ. On the physical aspect, the rotation parameter λ decreases the flow motion of the entrained fluid, as denoted in the decrease in $f'(\eta)$. For small values of λ, the velocities decay oscillatory, while for large

λ, the decay is monotonically exponentially. The similar observation was reported by Wang[4] and Nazar et al.[7].

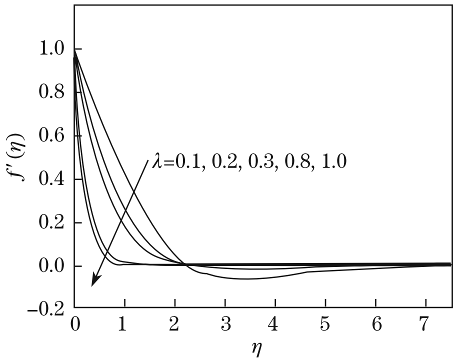

Fig. 4 Similarity velocity profiles in x-direction, $f'(\eta)$, for different values of λ when $K = 1$

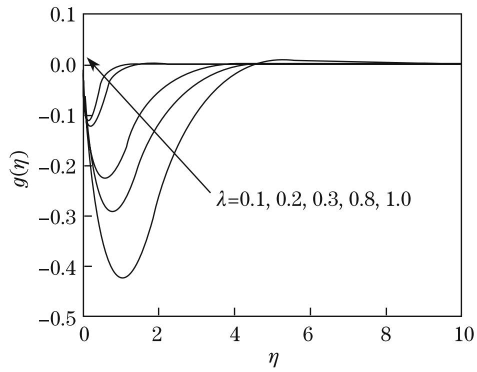

Fig. 5 Similarity velocity profiles in y-direction, $g(\eta)$, for different values of λ when K=1

The typical results for the variations of the skin friction coefficients in the x- and y-directions, $(Re_x)^{\frac{1}{2}}C_{fx}$ and $(Re_x)^{\frac{1}{2}}C_{fy}$, are illustrated in Figs. 6 and 7 against λ for some particular values of K. These figures show that the values of $(Re_x)^{\frac{1}{2}}C_{fx}$ and $(Re_x)^{\frac{1}{2}}C_{fy}$ increase as K increases. It is also observed from Figs. 6 and 7 that the values of $|(Re_x)^{\frac{1}{2}}C_{fx}|$ and $|(Re_x)^{\frac{1}{2}}C_{fy}|$ increase as the rotation parameter increases, suggesting that the rotation parameter affects the skin friction coefficient in both directions. It can be seen that there are large increases in the magnitudes of $(Re_x)^{\frac{1}{2}}C_{fx}$ and $(Re_x)^{\frac{1}{2}}C_{fy}$ for a regular fluid ($K = 0$). Moreover, for the case of the viscoelastic fluid ($K > 0$), the figure shows a slight increase in the magnitude of $(Re_x)^{\frac{1}{2}}C_{fx}$ within a certain range of λ. Figure 7 shows that the magnitude of $(Re_x)^{\frac{1}{2}}C_{fy}$ increases dramatically for $K = 0$ (the regular fluid) which leads to more steep profiles. In contrast, for the viscoelastic fluid, the variations of the skin friction coefficient in the y-direction increase steadily. This is known from the fact that the fluid motion is accelerated due to the increase in K (see Figs. 2 and 3). Therefore, the inclusion of the viscoelastic parameter increases the velocity and the boundary layer thickness, which results in the decrease in the velocity gradient at the surface and the increase in the skin friction coefficient in both directions as shown in Figs. 6 and 7.

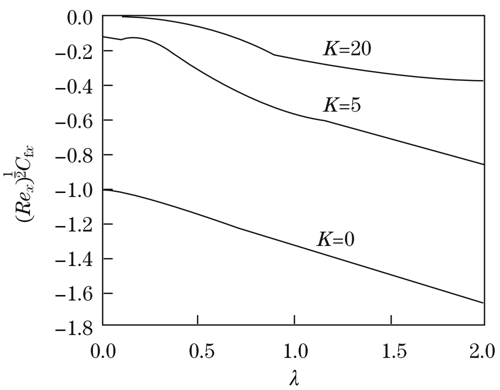

Fig. 6 Variation with λ of skin friction coefficient along x-direction, $(Re_x)^{\frac{1}{2}}C_{fx}$, for different values of K

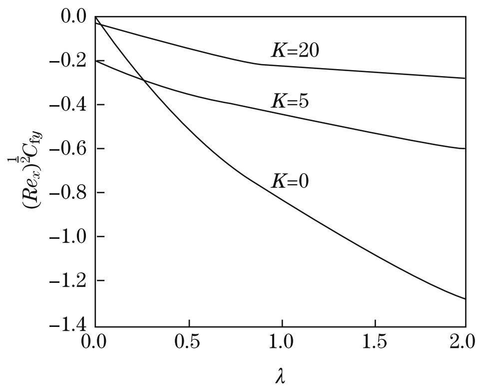

Fig. 7 Variation with λ of skin friction coefficient along y-direction, $(Re_x)^{\frac{1}{2}}C_{fy}$, for different values of K

The variations of $(Re_x)^{\frac{1}{2}}C_{fx}$ and $(Re_x)^{\frac{1}{2}}C_{fy}$ with the viscoelastic parameter K for some values of the rotation parameter λ are depicted in Figs. 8 and 9. In general, the obtained results point out that the values of $|(Re_x)^{\frac{1}{2}}C_{fx}|$ and $|(Re_x)^{\frac{1}{2}}C_{fy}|$ increase as the rotation parameter increases. This behavior is in agreement with the steady state flow considered by Wang[4] and Nazar et al.[7].

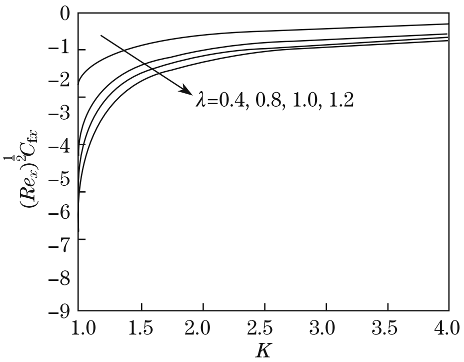

Fig. 8 Variation with K of skin friction coefficient along x-direction, $(Re_x)^{\frac{1}{2}}C_{fx}$, for different values of λ

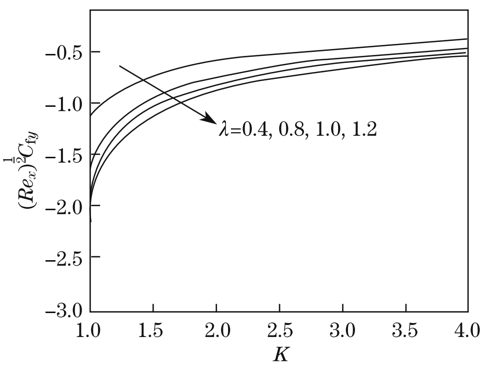

Fig. 9 Variation with K of skin friction coefficient along y-direction, $(Re_x)^{\frac{1}{2}}C_{fy}$, for different values of λ

4 Conclusions

The steady laminar boundary layer flow over a stretching surface in a rotating viscoelastic fluid is investigated. The governing partial differential equations are solved numerically using the well-known implicit finite difference scheme known as the Keller-box method. Numerical results for some values of the viscoelastic parameter K and the rotation parameter λ are tabulated and graphically presented. The values of the skin friction coefficients are compared with the results available in the literature for $K = 0$ (the regular fluid) in order to validate the obtained numerical results, which show good agreement. The skin friction coefficients and the similarity velocity profiles in both directions are notably affected by the velocity distributions of the stretching surface in the x-direction, the viscoelastic parameter, and the rotation parameter. The skin friction coefficients in both directions increase as the viscoelastic parameter K and the rotation parameter λ increase. The similarity velocity in the x-direction increases as K and λ increase. However, the opposite behaviors are observed for the velocity in the y-direction.

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