Appl. Math. Mech. -Engl. Ed., **33**(2), 139–154 (2012)
DOI 10.1007/s10483-012-1539-7

Applied Mathematics and Mechanics (English Edition)

Transition sets of bifurcations of dynamical systems with two state variables with constraints*

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Abstract Bifurcation of periodic solutions widely exists in nonlinear dynamical systems. In this paper, categories of bifurcations of systems with two state variables with different types of constraints are discussed, where some new types of transition sets are added. Additionally, the bifurcation properties of two-dimensional systems without constraints are compared with the ones with constraints. The results obtained in this paper can be used by engineers for the choice of the structural parameters of the systems.

Key words bifurcation, constraint, singularity theory, nonlinear dynamics, two state variables

Chinese Library Classification O322, TH133
2010 Mathematics Subject Classification 70K50, 34A34

1 Introduction

With the developments of nonlinear science, it is more and more widely applied in the fields of engineering and technology. Many researchers have done a great deal of work in these fields, such as Bogoliubov, Mitropolsky, Nayfeh and Mook, Smale, Arnold, and Golubitsky. Chen and Langford introduced the Lyapunov-Schmidt (LS) method into the nonlinear bifurcation analysis[1] and formed a complete set of analytical methods to study the bifurcation of periodic solutions[2–3].

We often encounter such problems that the state variables are of constraints when studying on nonlinear systems, e.g., the amplitude of oscillation or the concentration of reactant cannot be negative. For the single-degree of freedom system in periodic solutions with constraints, Wu et al. did a lot of work[4–7], which developed the singularity theory and provided a theoretical basis for the actual engineering and technology.

For the multi-degree of freedom systems, the main resonance, internal resonance, and combination resonance of the systems are often studied. Their bifurcation equations are of multiple dimensions. And for the actual vibration systems, the constraints of the state variables should be considered. For example, there is coupling of bending and torsion vibrations of the rotor system. For the existence of radial clearance, the amplitude of bending vibration is limited, which is a double-sided constraint, and the amplitude of torsional vibration is greater than zero, which is a one-sided constraint. Hu and Li[8] made much research on the bifurcations of

* Received Aug. 19, 2011 / Revised Dec. 4, 2011
Project supported by the National Natural Science Foundation of China (No. 10632040)
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the systems with multiple bifurcation parameters and two state variables. Qin et al.[9] studied on the singularity analysis of a two-dimensional cable system under 1:1 internal resonance. Although there are some papers on study of the singularity theory of the systems with two state variables, the research of the bifurcations of dynamical system with two state variables with constraints is little. In engineering, the state variables are often constrained. Therefore, study on the bifurcations of dynamical systems with two state variables with constraints is necessary. To reveal all possible bifurcation patterns of such systems, a new method is developed based on the singularity theory[10] in this paper.

2 Categories of bifurcations with two state variables with constraints

2.1 Case Ⅰ: bifurcations of systems—one state variable with single-sided constraints and the other without constraints

The idea of singularity theory of systems with one state variable with constraints was presented by Wu et al.[6]. In this paper, it is developed into the dynamical systems with two state variables.

For simplicity, we use a general form of bifurcation equation, given by

$$g=\begin{cases} g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\end{cases} \tag{1}$$

where $\delta s \geqslant \beta\,(\delta=\pm 1)$, s and m are the state variables, λ is the bifurcation parameter, and α is the unfolding parameter.

In order to convert the constrained bifurcation problem into a bifurcation problem without constraints, we introduce the following transformation:

$$G(x,y,\lambda,\alpha)\triangleq g(s,m,\lambda,\alpha)=\begin{cases} g_1(\delta(x^2+\beta),m,\lambda,\alpha),\\ g_2(\delta(x^2+\beta),m,\lambda,\alpha).\end{cases} \tag{2}$$

First, note that the following relations which can be obtained from (1) and (2):

$$\begin{cases} G_{1x}=g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta x g_{1s}, \quad G_{1xx}=2\delta g_{1s}+4x^2 g_{1ss},\\ G_{2x}=g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta x g_{2s}, \quad G_{2xx}=2\delta g_{2s}+4x^2 g_{2ss}.\end{cases} \tag{3}$$

Substituting the above relations into the bifurcation set B[10] results in

$$B=B_1\cup B_2,$$

$$B_1=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\ g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0,\\ g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\ g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \delta s\geqslant\beta,\end{cases} \tag{4}$$

$$B_2=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\ g_1(\delta\beta,m,\lambda,\alpha)=0,\\ g_2(\delta\beta,m,\lambda,\alpha)=0.\end{cases} \tag{5}$$

Here, B_1 is the same as the bifurcation set of the corresponding nonconstrained bifurcation except the region defined by the constraint. While B_2 may be called the boundary induced bifurcation set, which particularly takes into account the boundary effect.

Similarly, we obtain the regular hysteresis sets as follows:

$$H = H_1 \cup H_2,$$

$$H_1 = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \dfrac{4(\delta s - \beta)\big(g_{1ss} + g_{1mm}\big(\frac{g_{1s}}{g_{1m}}\big)^2 - 2g_{1sm}\big(\frac{g_{1s}}{g_{1m}}\big)\big) + 2g_{1s}\delta}{4(\delta s - \beta)\big(g_{2ss} + g_{2mm}\big(\frac{g_{1s}}{g_{1m}}\big)^2 - 2g_{2sm}\big(\frac{g_{1s}}{g_{1m}}\big)\big) + 2g_{1s}\delta} = \dfrac{g_{1s}}{g_{2s}} = \dfrac{g_{1m}}{g_{2m}}, \\ \delta s > \beta, \end{cases} \tag{6}$$

$$H_2 = \begin{cases} g_{1m}g_{2s} - g_{2m}g_{1s} = 0, \\ g_1(\delta\beta, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta, m, \lambda, \alpha) = 0. \end{cases} \tag{7}$$

The regular double limit point sets can be derived respectively as

$$D = D_1 \cup D_2,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \delta s_i \geqslant \beta, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ i = 1, 2, \quad (s_1, m_1) \neq (s_2, m_2), \end{cases} \tag{8}$$

$$D_2 = \begin{cases} g_1(\delta\beta, m_1, \lambda, \alpha) = 0, \\ g_2(\delta\beta, m_1, \lambda, \alpha) = 0, \\ g_1(s, m_2, \lambda, \alpha) = 0, \\ g_2(s, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m_2, \lambda, \alpha} = 0, \\ \delta s > \beta. \end{cases} \tag{9}$$

2.2 Case II: bifurcations of systems—one state variable with double-sided constraints and the other without constraints

In this case, we consider the bifurcation equation with double-sided constraints,

$$g(s, m, \lambda, \alpha) = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \end{cases} \tag{10}$$

where $\beta_1 \leqslant s \leqslant \beta_2$.

Introduce the transformation as follows:

$$h(s, x) \triangleq (s - \beta_1)(s - \beta_2) + x^2 = 0. \tag{11}$$

In order to convert the constrained bifurcation problem into a bifurcation problem without constraints, we introduce the following transformation:

$$G(x, y, \lambda, \alpha) \triangleq g(s, m, \lambda, \alpha) = \begin{cases} g_1(X(x^2, \beta), m, \lambda, \alpha), \\ g_2(X(x^2, \beta), m, \lambda, \alpha), \end{cases} \tag{12}$$

where $X(x^2,\beta)=s$ is the solution to $h(s,x)=0$.

Note that the following relations which can be obtained from (11) and (12):

$$\begin{cases} G_{1x}=g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=\Big(-2x\Big/\Big(\dfrac{\partial h}{\partial s}\Big)\Big)g_{1s}, \\ G_{2x}=g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=\Big(-2x\Big/\Big(\dfrac{\partial h}{\partial s}\Big)\Big)g_{2s}, \\ G_{1xx}=g_{1s}\dfrac{\mathrm{d}^2s}{\mathrm{d}x^2}+g_{1ss}\Big(\dfrac{\mathrm{d}s}{\mathrm{d}x}\Big)^2 \\ \qquad =g_{1s}\Big(-2((2s-\beta_1-\beta_2)^2+4x^2)\Big/\Big(\dfrac{\partial h}{\partial s}\Big)^3\Big)+4x^2\Big/\Big(\dfrac{\partial h}{\partial s}\Big)^2 g_{1ss}, \\ G_{2xx}=g_{2s}\dfrac{\mathrm{d}^2s}{\mathrm{d}x^2}+g_{2ss}\Big(\dfrac{\mathrm{d}s}{\mathrm{d}x}\Big)^2 \\ \qquad =g_{2s}\Big(-2((2s-\beta_1-\beta_2)^2+4x^2)\Big/\Big(\dfrac{\partial h}{\partial s}\Big)^3\Big)+4x^2\Big/\Big(\dfrac{\partial h}{\partial s}\Big)^2 g_{2ss}. \end{cases} \tag{13}$$

Similar to Case I, we can obtain the transition sets.

The bifurcation sets are

$$B=B_1\cup B_2,$$

$$B_1=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0, \\ g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0, \\ g_{1s}g_{2m}-g_{2s}g_{1m}=0, \\ g_1(s,m,\lambda,\alpha)=0, \\ g_2(s,m,\lambda,\alpha)=0, \\ \beta_1<s<\beta_2, \end{cases} \tag{14}$$

$$B_2=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0, \\ g_1(\beta_i,m,\lambda,\alpha)=0, \\ g_2(\beta_i,m,\lambda,\alpha)=0, \\ i=1,2. \end{cases} \tag{15}$$

Following the same procedure, the hysteresis sets of the bifurcation systems are

$$H=H_1\cup H_2,$$

$$H_1=\begin{cases} g_1(s,m,\lambda,\alpha)=0, \\ g_2(s,m,\lambda,\alpha)=0, \\ \Big(g_{1s}(-2((2s-\beta_1-\beta_2)^2-4(s-\beta_1)(s-\beta_2)))-4(2_s-\beta_1-\beta_2) \\ \cdot(s-\beta_1)(s-\beta_2)\Big(g_{1ss}+g_{1sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)^2-2g_{1sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)\Big)\Big) \\ \Big/\Big(g_{2s}(-2((2s-\beta_1-\beta_2)^2-4(s-\beta_1)(s-\beta_2)))-4(2_s-\beta_1-\beta_2)(s-\beta_1) \\ (s-\beta_2)\Big(g_{2ss}+g_{2sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)^2-2g_{2sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)\Big)\Big)=\dfrac{g_{1s}}{g_{2s}}=\dfrac{g_{1m}}{g_{2m}}, \\ \beta_1<s<\beta_2, \end{cases} \tag{16}$$

$$H_2 = \begin{cases} g_1(\beta_i, m, \lambda, \alpha) = 0, \\ g_2(\beta_i, m, \lambda, \alpha) = 0, \\ g_{1m}g_{2s} - g_{2m}g_{1s} = 0, \\ i = 1, 2. \end{cases} \tag{17}$$

The double limit point sets are described by the following equations:

$$D = D_1 \cup D_2 \cup D_3,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ \beta_1 < s < \beta_2, \\ i = 1, 2, \ (s_1, m_1) \neq (s_2, m_2), \end{cases} \tag{18}$$

$$D_2 = \begin{cases} g_1(\beta_1, m_1, \lambda, \alpha) = 0, \\ g_2(\beta_1, m_1, \lambda, \alpha) = 0, \\ g_1(\beta_2, m_2, \lambda, \alpha) = 0, \\ g_2(\beta_2, m_2, \lambda, \alpha) = 0, \end{cases} \tag{19}$$

$$D_3 = \begin{cases} g_1(\beta_i, m_i, \lambda, \alpha) = 0, \\ g_2(\beta_i, m_i, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m, \lambda, \alpha} = 0, \\ \beta_1 < s < \beta_2, \\ i = 1, 2. \end{cases} \tag{20}$$

2.3 Case Ⅲ: bifurcations of systems—one state variable with piecewise constraints and the other without constraints

There is a bifurcation equation with piecewise constraints as follows:

$$G(x, y, \lambda, \alpha) = \begin{cases} g_1(s, m, \lambda, \alpha), \\ g_2(s, m, \lambda, \alpha), \end{cases}$$

$$G(x, y, \lambda, \alpha) = \begin{cases} g_{11}(s, m, \lambda, \alpha), & g_{21}(s, m, \lambda, \alpha), & s \leqslant \beta_1, \\ g_{12}(s, m, \lambda, \alpha), & g_{22}(s, m, \lambda, \alpha), & \beta_1 < s < \beta_2, \\ g_{13}(s, m, \lambda, \alpha), & g_{23}(s, m, \lambda, \alpha), & s \geqslant \beta_2. \end{cases} \tag{21}$$

We can use the results obtained in the previous two subsections. For $s \leqslant \beta_1$ or $s \geqslant \beta_2$, the singularity theory of two state variable systems with one-sided constrains can be used. And for $\beta_1 < s < \beta_2$, the singularity theory of two state variable systems with two-sided constrains can be used.

2.4 Case Ⅳ: bifurcations of systems—two state variables both with single-sided constraints

In this subsection, we consider the bifurcation systems with single-sided constraints:

$$g(s, m, \lambda, \alpha) = 0, \quad \begin{cases} \delta_1 s = x^2 + \beta_1, & x^2 \geqslant 0, & \delta_1 = \pm 1, \\ \delta_2 m = y^2 + \beta_2, & y^2 \geqslant 0, & \delta_2 = \pm 1. \end{cases} \tag{22}$$

Introduce the transformation,

$$G(x, y, \lambda, \alpha) \triangleq g(s, m, \lambda, \alpha) = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0. \end{cases} \tag{23}$$

We note that the following relations can be obtained from (23):

$$\begin{cases} G_{1x} = g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = 2\delta_1 x g_{1s}, \quad G_{1y} = g_{1m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = 2\delta_2 y g_{1m}, \\ G_{2x} = g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = 2\delta_1 x g_{2s}, \quad G_{2y} = g_{2m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = 2\delta_2 y g_{2m}, \\ G_{1xx} = 2\delta_1 g_{1s} + 4x^2 g_{1ss}, \quad G_{1yy} = 2\delta_2 g_{1m} + 4y^2 g_{1mm}, \\ G_{2xx} = 2\delta_1 g_{2s} + 4x^2 g_{2ss}, \quad G_{2yy} = 2\delta_2 g_{2m} + 4y^2 g_{2mm}. \end{cases} \tag{24}$$

Similarly, the transition sets can be obtained.

The bifurcation sets are

$$B = B_1 \cup B_2 \cup B_3 \cup B_4,$$

$$B_1 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \delta_1 s \geqslant \beta_1, \quad \delta_2 m \geqslant \beta_2, \end{cases} \tag{25}$$

$$B_2 = \begin{cases} g_1(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \end{cases} \tag{26}$$

$$B_3 = \begin{cases} g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_1(s, \delta\beta_2, \lambda, \alpha) = 0, \\ g_2(s, \delta\beta_2, \lambda, \alpha) = 0, \\ \delta_1 s > \beta_1, \end{cases} \tag{27}$$

$$B_4 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_1(\delta\beta_1, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta_1, m, \lambda, \alpha) = 0, \\ \delta_2 m > \beta_2. \end{cases} \tag{28}$$

Here, B_1 is the same as the bifurcation set of the associated nonconstrained bifurcation except the region defined by the constraint. While B_2, B_3, and B_4 may be called the boundary induced bifurcation sets which particularly take into account the boundary effect.

Following the same procedure, one can find the hysteresis sets of the bifurcation system,

$$H = H_1 \cup H_2 \cup H_3 \cup H_4,$$

$$H_1 = \begin{cases} g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \Big(2\delta_1 g_{1s} + 4(\delta_1 s - \beta_1)g_{1ss} + (2\delta_2 g_{1m} + 4(\delta_2 m - \beta_2)g_{1mm})\Big(\dfrac{\delta_2 m - \beta_2}{\delta_1 s - \beta_1}\Big)\Big(\dfrac{g_{1s}}{g_{1m}}\Big)^2 \\ \quad - 8(\delta_1 s - \beta_1)g_{1sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)\Big) \Big/ \Big(2\delta_1 g_{2s} + 4(\delta_1 s - \beta_1)g_{2ss} + (2\delta_2 g_{2m} + 4(\delta_2 m - \beta_2) \\ \quad \cdot g_{2mm})\Big(\dfrac{\delta_2 m - \beta_2}{\delta_1 s - \beta_1}\Big)\Big(\dfrac{g_{1s}}{g_{1m}}\Big)^2 - 8(\delta_1 s - \beta_1)g_{2sm}\Big(\dfrac{g_{1s}}{g_{1m}}\Big)\Big) = \dfrac{g_{1s}}{g_{2s}} = \dfrac{g_{1m}}{g_{2m}}, \\ \delta_1 s > \beta_1, \quad \delta_2 m > \beta_2, \end{cases} \tag{29}$$

$$H_2 = \begin{cases} g_1(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \end{cases} \tag{30}$$

$$H_3 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s, \delta\beta_2, \lambda, \alpha) = 0, \\ g_2(s, \delta\beta_2, \lambda, \alpha) = 0, \\ \delta_1 s > \beta_1, \end{cases} \tag{31}$$

$$H_4 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(\delta\beta_1, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta_1, m, \lambda, \alpha) = 0, \\ \delta_2 m > \beta_2. \end{cases} \tag{32}$$

The double limit point sets are described by the following equations:

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ i = 1, 2, \quad \delta_1 s \geqslant \beta_1, \quad \delta_2 m \geqslant \beta_2, \\ (s_1, m_1) \neq (s_2, m_2), \end{cases} \tag{33}$$

$$D_2 = \begin{cases} g_1(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m, \lambda, \alpha} = 0, \\ \delta_1 s > \beta_1, \quad \delta_2 m > \beta_2, \end{cases} \tag{34}$$

$$D_3 = \begin{cases} g_1(s_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(s_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_1(s_2, m, \lambda, \alpha) = 0, \\ g_2(s_2, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m, \lambda, \alpha} = 0, \\ \delta_1 s_i > \beta_1, \quad \delta_2 m > \beta_2, \\ i = 1, 2, \end{cases} \tag{35}$$

$$D_4 = \begin{cases} g_1(\delta_1\beta_1, m_1, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, m_1, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \delta_1 s_2 > \beta_1, \quad \delta_2 m_i > \beta_2, \\ i = 1, 2. \end{cases} \tag{36}$$

2.5 Case V: bifurcations of systems—two state variables both with double-sided constraints

In this subsection, we consider two state variables both with double-sided constraints,

$$\begin{cases} G(x, y, \lambda, \alpha) \triangleq g(s, m, \lambda, \alpha) = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, & \beta_1 \leqslant s \leqslant \beta_2, \\ g_2(s, m, \lambda, \alpha) = 0, & \beta_3 \leqslant m \leqslant \beta_4, \end{cases} \\ h_1(s, x) \triangleq (s - \beta_1)(s - \beta_2) + x^2 = 0, \\ h_2(s, x) \triangleq (m - \beta_3)(m - \beta_4) + y^2 = 0. \end{cases} \tag{37}$$

Note that the following relations which can be obtained from (37):

$$\begin{cases} G_{1x} = g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = \Big(-2x\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)\Big)g_{1s}, \\ G_{2x} = g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = \Big(-2x\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)\Big)g_{2s}, \\ G_{1xx} = g_{1s}\dfrac{\mathrm{d}^2 s}{\mathrm{d}x^2} + g_{1ss}\Big(\dfrac{\mathrm{d}s}{\mathrm{d}x}\Big)^2 \\ \quad = g_{1s}\Big(-2((2s-\beta_1-\beta_2)^2+4x^2)\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)^3\Big) + \Big(4x^2\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)^2\Big)g_{1ss}, \\ G_{2xx} = g_{2s}\dfrac{\mathrm{d}^2 s}{\mathrm{d}x^2} + g_{2ss}\Big(\dfrac{\mathrm{d}s}{\mathrm{d}x}\Big)^2 \\ \quad = g_{2s}\Big(-2((2s-\beta_1-\beta_2)^2+4x^2)\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)^3\Big) + \Big(4x^2\Big/\Big(\dfrac{\partial h_1}{\partial s}\Big)^2\Big)g_{2ss}, \end{cases} \tag{38}$$

$$\begin{cases} G_{1y} = g_{1m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = \Big(-2y\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)\Big)g_{1m}, \\ G_{2y} = g_{2m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = \Big(-2y\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)\Big)g_{2m}, \\ G_{1yy} = g_{1m}\dfrac{\mathrm{d}^2 m}{\mathrm{d}y^2} + g_{1mm}\Big(\dfrac{\mathrm{d}m}{\mathrm{d}y}\Big)^2 \\ \quad = g_{1m}\Big(-2((2m-\beta_3-\beta_4)^2+4y^2)\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)^3\Big) + \Big(4y^2\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)^2\Big)g_{1mm}, \\ G_{2yy} = g_{2s}\dfrac{\mathrm{d}^2 m}{\mathrm{d}y^2} + g_{2ss}\Big(\dfrac{\mathrm{d}m}{\mathrm{d}y}\Big)^2 \\ \quad = g_{2m}\Big(-2((2m-\beta_3-\beta_4)^2+4y^2)\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)^3\Big) + \Big(4y^2\Big/\Big(\dfrac{\partial h_2}{\partial m}\Big)^2\Big)g_{2mm}. \end{cases} \tag{39}$$

The bifurcation sets are

$$B = B_1 \cup B_2 \cup B_3 \cup B_4,$$

$$B_1 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \beta_1 \leqslant s \leqslant \beta_2, \\ \beta_3 \leqslant m \leqslant \beta_4, \end{cases} \tag{40}$$

$$B_2 = \begin{cases} g_1(\beta_i,\beta_j,\lambda,\alpha) = 0, \\ g_2(\beta_i,\beta_j,\lambda,\alpha) = 0, \\ i = 1,2, \\ j = 3,4, \end{cases} \tag{41}$$

$$B_3 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_1(\beta_i, m, \lambda, \alpha) = 0, \\ g_2(\beta_i, m, \lambda, \alpha) = 0, \\ \beta_3 < m < \beta_4, \\ i = 1, 2, \end{cases} \tag{42}$$

$$B_4 = \begin{cases} g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_1(s, \beta_j, \lambda, \alpha) = 0, \\ g_2(s, \beta_j, \lambda, \alpha) = 0, \\ \beta_1 < s < \beta_2, \\ j = 3, 4. \end{cases} \tag{43}$$

The hysteresis sets are

$$H = H_1 \cup H_2 \cup H_3 \cup H_4,$$

$$H_1 = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \dfrac{G_{1xx}\left(\frac{G_{1y}}{G_{1x}}\right)^2 + G_{1yy} + 2G_{1xy}\left(-\frac{G_{1y}}{G_{1x}}\right)}{G_{2xx}\left(\frac{G_{1y}}{G_{1x}}\right)^2 + G_{2yy} + 2G_{2xy}\left(-\frac{G_{1y}}{G_{1x}}\right)} = \dfrac{G_{1x}}{G_{2x}} = \dfrac{G_{1y}}{G_{2y}}, \\ \dfrac{G_{1y}}{G_{1x}} = \dfrac{\left(-2y/\left(\dfrac{\partial h_2}{\partial m}\right)\right)g_{1m}}{\left(-2x/\left(\dfrac{\partial h_1}{\partial s}\right)\right)g_{1s}}, \\ \dfrac{\partial h_2}{\partial m} = 2m - \beta_3 - \beta_4, \\ \dfrac{\partial h_1}{\partial s} = 2s - \beta_1 - \beta_2, \\ \beta_1 \leqslant s \leqslant \beta_2, \\ \beta_3 \leqslant m \leqslant \beta_4, \end{cases} \tag{44}$$

$$H_2 = \begin{cases} g_1(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ g_2(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ i = 1, 2, \\ j = 3, 4, \end{cases} \tag{45}$$

$$H_3 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s, \beta_j, \lambda, \alpha) = 0, \\ g_2(s, \beta_j, \lambda, \alpha) = 0, \\ \beta_1 < s < \beta_2, \\ j = 3, 4, \end{cases} \tag{46}$$

$$H_4 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(\beta_i, m, \lambda, \alpha) = 0, \\ g_2(\beta_i, m, \lambda, \alpha) = 0, \\ \beta_3 < m < \beta_4, \\ i = 1, 2. \end{cases} \tag{47}$$

The double limit point sets are

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ i = 1, 2, \quad \beta_1 \leqslant s \leqslant \beta_2, \\ \beta_3 \leqslant m \leqslant \beta_4, \\ (s_1, m_1) \neq (s_2, m_2), \end{cases} \tag{48}$$

$$D_2 = \begin{cases} g_1(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ g_2(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m, \lambda, \alpha} = 0, \\ i = 1, 2, \\ j = 3, 4, \end{cases} \tag{49}$$

$$D_3 = \begin{cases} g_1(s_1, \beta_j, \lambda, \alpha) = 0, \\ g_2(s_1, \beta_j, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \beta_1 < s < \beta_2, \\ \beta_3 < m_2 < \beta_4, \\ j = 3, 4, \end{cases} \tag{50}$$

$$D_4 = \begin{cases} g_1(\beta_i, m_1, \lambda, \alpha) = 0, \\ g_2(\beta_i, m_1, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \beta_1 < s_2 < \beta_2, \\ \beta_3 < m < \beta_4, \\ i = 1, 2. \end{cases} \tag{51}$$

2.6 Case Ⅵ: bifurcations of systems—one state variable with single-sided constraints and the other with double-sided constraints

In this subsection, we consider two state variables, where one with single-sided constraints and the other with double-sided constraints,

$$G(x, y, \lambda, \alpha) \triangleq g(s, m, \lambda, \alpha) = \begin{cases} \delta s = x^2 + \beta_0, \quad (x^2 > 0, \delta = \pm 1), \\ \beta_1 \leqslant m \leqslant \beta_2, \quad h(m, x) \triangleq (m - \beta_1)(m - \beta_2) + y^2 = 0. \end{cases} \tag{52}$$

The bifurcation sets are

$$B = B_1 \cup B_2 \cup B_3 \cup B_4,$$

$$B_1 = \begin{cases} g_{1m} g_{2\lambda} - g_{2m} g_{1\lambda} = 0, \\ g_{1s} g_{2\lambda} - g_{2s} g_{1\lambda} = 0, \\ g_{1s} g_{2m} - g_{2s} g_{1m} = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \delta s \geqslant \beta_0, \\ \beta_1 \leqslant m \leqslant \beta_2, \end{cases} \tag{53}$$

$$B_2 = \begin{cases} g_1(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ i = 1, 2, \end{cases} \tag{54}$$

$$B_3 = \begin{cases} g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_1(s, \beta_i, \lambda, \alpha) = 0, \\ g_2(s, \beta_i, \lambda, \alpha) = 0, \\ \delta s \geqslant \beta_0, \\ i = 1, 2, \end{cases} \tag{55}$$

$$B_4 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_1(\delta\beta_0, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, m, \lambda, \alpha) = 0, \\ \beta_1 < m < \beta_2. \end{cases} \tag{56}$$

The hysteresis sets are

$$H = H_1 \cup H_2 \cup H_3 \cup H_4,$$

$$H_1 = \begin{cases} g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \dfrac{G_{1xx}\big(\frac{G_{1y}}{G_{1x}}\big)^2 + G_{1yy} + 2G_{1xy}\big(-\frac{G_{1y}}{G_{1x}}\big)}{G_{2xx}\big(\frac{G_{1y}}{G_{1x}}\big)^2 + G_{2yy} + 2G_{2xy}\big(-\frac{G_{1y}}{G_{1x}}\big)} = \dfrac{G_{1x}}{G_{2x}} = \dfrac{G_{1y}}{G_{2y}}, \\ \dfrac{G_{1y}}{G_{1x}} = \dfrac{\big(-2y/\big(\frac{\partial h_2}{\partial m}\big)\big)g_{1m}}{2xg_{1s}\delta}, \\ \dfrac{\partial h_2}{\partial m} = 2m - \beta_1 - \beta_2, \\ \delta s \geqslant \beta_0, \\ \beta_1 \leqslant m \leqslant \beta_2, \end{cases} \tag{57}$$

$$H_2 = \begin{cases} g_1(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ i = 1, 2, \end{cases} \tag{58}$$

$$H_3 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s, \beta_i, \lambda, \alpha) = 0, \\ g_2(s, \beta_i, \lambda, \alpha) = 0, \\ \delta s > \beta_0, \\ i = 1, 2, \end{cases} \tag{59}$$

$$H_4 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(\delta\beta_0, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, m, \lambda, \alpha) = 0, \\ \beta_1 < m < \beta_2. \end{cases} \tag{60}$$

The double limit point sets are

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ \delta s \geqslant \beta_0, \\ \beta_1 \leqslant m \leqslant \beta_2, \\ i = 1, 2, \end{cases} \tag{61}$$

$$D_2 = \begin{cases} g_1(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, \beta_i, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s,m,\lambda,\alpha} = 0, \\ \delta s > \beta_0, \\ \beta_1 < m < \beta_2, \\ i = 1, 2, \end{cases} \tag{62}$$

$$D_3 = \begin{cases} g_1(s_1, \beta_i, \lambda, \alpha) = 0, \\ g_2(s_1, \beta_i, \lambda, \alpha) = 0, \\ g_1(s_2, m, \lambda, \alpha) = 0, \\ g_2(s_2, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2,m,\lambda,\alpha} = 0, \\ \delta s > \beta_0, \\ \beta_1 < m < \beta_2, \\ i = 1, 2, \end{cases} \tag{63}$$

$$D_4 = \begin{cases} g_1(\delta\beta_0, m_1, \lambda, \alpha) = 0, \\ g_2(\delta\beta_0, m_1, \lambda, \alpha) = 0, \\ g_1(s, m_2, \lambda, \alpha) = 0, \\ g_2(s, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s,m_2,\lambda,\alpha} = 0, \\ \delta s > \beta_0, \\ \beta_1 < m < \beta_2. \end{cases} \tag{64}$$

2.7 Case Ⅶ: bifurcations of systems—two state variables both with piecewise constraints

We can use the results obtained in the previous subsections, for one-sided, double-sided and one-double-sided constraints. Bifurcation equations with two state variables with piecewise constraints are as follows:

$$G(x, y, \lambda, \alpha) = \begin{cases} g_1(s, m, \lambda, \alpha), \\ g_2(s, m, \lambda, \alpha), \end{cases} \tag{65}$$

$$G(x, y, \lambda, \alpha) = \begin{cases} g_{11}(s, m, \lambda, \alpha), & g_{21}(s, m, \lambda, \alpha), & s \leqslant \beta_1, & m \leqslant \gamma_1 \\ g_{12}(s, m, \lambda, \alpha), & g_{22}(s, m, \lambda, \alpha), & \beta_1 < s < \beta_2, & \gamma_1 < m < \gamma_2, \\ g_{13}(s, m, \lambda, \alpha), & g_{23}(s, m, \lambda, \alpha), & \beta_1 < s < \beta_2, & m > \gamma_2. \end{cases} \tag{66}$$

For $s \leqslant \beta_1$ and $m \leqslant \gamma_1$, the singularity theory of two state variables with one-sided constrains can be used. For $\beta_1 < s < \beta_2$ and $\gamma_1 < m < \gamma_2$, the singularity theory of two state variables with two-sided constrains can be used. And for $\beta_1 < s < \beta_2$ and $m > \gamma_2$, the singularity theory of two state variables with one-double-sided constrains can be used.

3 Singularity analysis of two-dimensional bifurcation systems

3.1 Singularity analysis without constraints

The following two-dimensional bifurcation equations are considered[11]:

$$\begin{cases} g_1 = s^2 - m^2 - \lambda + \alpha m, \\ g_2 = m^2 - \lambda + \beta s, \end{cases} \tag{67}$$

where s and m are the state variables, λ is the bifurcation parameter, and α and β are the unfolding parameters.

The transition sets of the bifurcation system (67) can be obtained as

$$B = \left\{\frac{\alpha^2}{2} - \beta^2 = 0\right\}, \tag{68}$$

$$\begin{cases} H_1 = \{\beta(2\alpha{+}3\beta + 3(\beta(4\alpha+\beta))^{\frac{1}{2}}) = 0\}, \\ H_2 = \{\beta(2\alpha{+}3\beta - 3(\beta(4\alpha+\beta))^{\frac{1}{2}}) = 0\}, \end{cases} \tag{69}$$

which are shown in Fig. 1. And the bifurcation diagrams in different persistent regions are shown in Table 1.

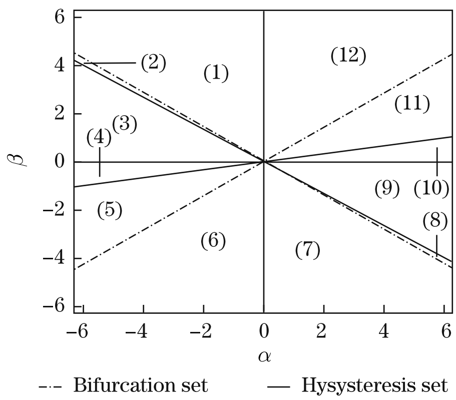
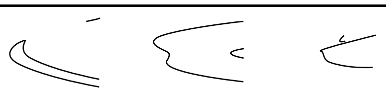
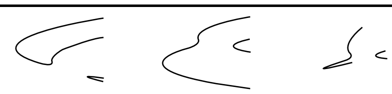
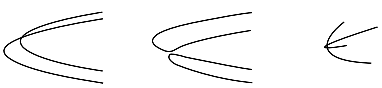
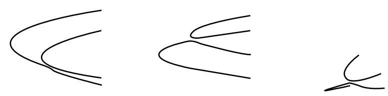
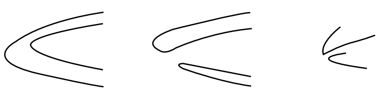
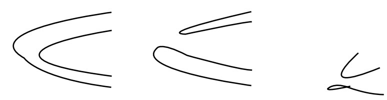
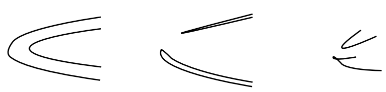
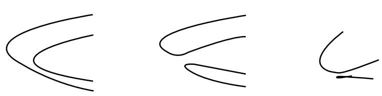
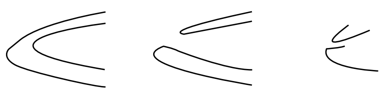
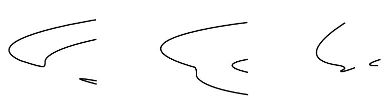
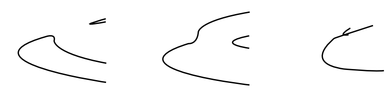


Fig. 1 Transition sets of system (67)

Table 1 Bifurcation diagrams in different persistent regions of system (67)

Persistent regions	λ-s curve	λ-m curve	λ-s-m curve	Persistent regions	λ-s curve	λ-m curve	λ-s-m curve
	s, λ	m, λ	m, s, λ		s, λ	m, λ	m, s, λ
(1)				(7)			
(2)				(8)			
(3)				(9)			
(4)				(10)			
(5)				(11)			
(6)				(12)			

Form Fig. 1, we can conclude that for the case of nonconstraints, the whole parametric plane is divided into 12 different persistent regions according to the singularity theory. From Table 1, we can see that there are 12 kinds of bifurcation patterns. For example, in region (2) the bifurcation phenomenon occurs. And in regions (1) and (6), there are hysteresis phenomena. All these bifurcation patterns can provide guidance to the dynamical analysis and design of the system.

3.2 Singularity analysis of system with constraints—one state variable with single-sided constraints and the other with double-sided constraints

The constrained bifurcation equations are

$$\begin{cases} g_1 = s^2 - m^2 - \lambda + \alpha m, \\ g_2 = m^2 - \lambda + \beta s, \\ s \geqslant 1, \\ 1 \leqslant m \leqslant 6. \end{cases} \tag{70}$$

Let

$$\begin{cases} s = x^2 + 1, \\ h = (m-1)(m-6) + y^2. \end{cases} \tag{71}$$

The transition sets of the bifurcation equations with two state variables, where one is single-sided constrained and the other is double-sided constrained, are obtained as the following equation, which are shown in Fig. 2.

$$\begin{cases} B_1 = \left\{ \dfrac{\alpha^2}{2} - \beta^2 = 0 \right\}, \quad B_{21} = \{\alpha - \beta - 1 = 0\}, \\ B_{22} = \{6\alpha - \beta - 71 = 0\}, \quad B_{31} = \left\{ -\dfrac{\beta^2}{4} + \alpha - 2 = 0 \right\}, \\ B_{32} = \left\{ -\dfrac{\beta^2}{4} + 6\alpha - 72 = 0 \right\}, \quad B_4 = \left\{ \dfrac{\alpha^2}{8} - \beta + 1 = 0 \right\}. \end{cases} \tag{72}$$

$$\begin{cases} H_1 = \left\{ \dfrac{\alpha^2 \beta}{(\beta+2)^2} - \beta + 1 = 0 \right\}, \quad H_{21} = \{\alpha - \beta - 1 = 0\}, \\ H_{22} = \{6\alpha - \beta - 71 = 0\}, \quad H_3 = \left\{ (\alpha - 2)(\alpha\beta^2 - 6\beta^2 + 16) = 0 \right\}, \\ H_4 = \left\{ -\dfrac{\alpha^2 \beta}{(\beta+2)^2} + \beta - 1 = 0 \right\}. \end{cases} \tag{73}$$

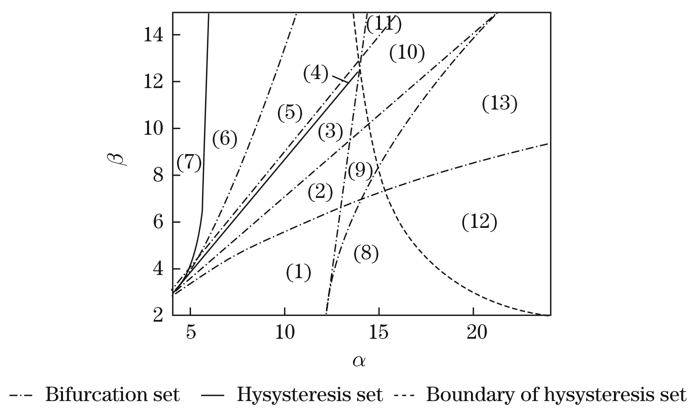
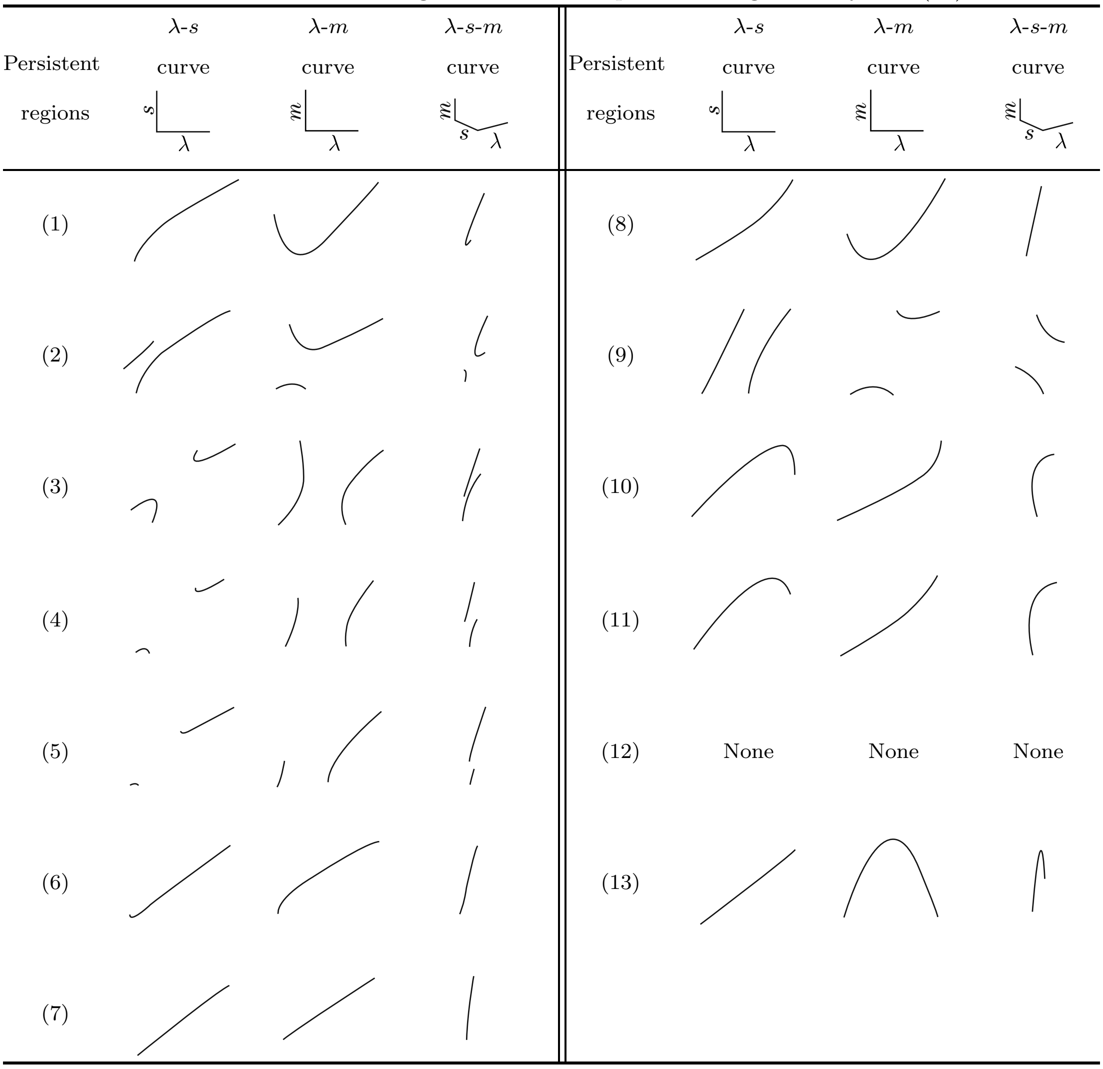


Fig. 2 Transition sets of system (70)

Table 2 Bifurcation diagrams in different persistent regions of system (70)



Form Fig. 2, we can conclude that for the case of constraints, the parametric plane is divided into 13 different persistent regions. From Table 2, we can see that there are 13 kinds of bifurcation patterns. In region (12), there is no solution. In regions (1), (6), (7), and (11), there is only one solution for different bifurcation parameters. And in region (2), there are two solution branches for some bifurcation parameters.

The bifurcation properties of two-dimensional systems with constraints are compared with the systems without constraints. When $24 \geqslant \alpha \geqslant 4$ and $\beta \geqslant 2$, there are 3 bifurcation patterns for the system without constrains. However, there are 13 bifurcation patterns for the system with constrains. Here, 10 bifurcation patterns are aroused by constraints. Moreover, the amplitude of the system is limited in some ranges by the constraints. All these results can provide a theoretical basis for the parametric design of engineering.

4 Conclusions

In this paper, the singularity theory of systems with two state variables with constraints has been developed. The transition sets of the two-dimensional systems with constraints are calculated. Different kinds of bifurcations with two state variables with constraints are obtained. The bifurcation set B_1 and the double limit point set D_1 are the same as the ones of the associated nonconstrain in the region defined by the constraint. The hysteresis set H_1 is different with the ones of the associated nonconstrain in the region defined by the constraint. Moreover, $B_2, B_3, B_4, H_2, H_3, H_4, D_2, D_3$, and D_4 are aroused by the boundary conditions. Therefore, they may be called boundary induced bifurcation sets.

Bifurcation properties of the two-dimensional bifurcation systems are analyzed. For the case of nonconstraints the whole parametric plane is divided into 12 different persistent regions. In region (2), the bifurcation phenomenon occurs. And in region (1) and (6), there are hysteresis phenomena, which maybe arouse destroy of the structure. For the case of constraints there are 13 kinds of bifurcation patterns. In region (12), there is no solution. In regions (1), (6), (7), and (11), there is only one solution for different bifurcation parameter. And in region (2), there are two solutions for some bifurcation parameters.

The bifurcation properties of the two-dimensional constraints systems are compared with the systems without constraints. When $24 \geqslant \alpha \geqslant 4$ and $\beta \geqslant 2$, there are 3 bifurcation patterns for the system without constrains. However, there are 13 bifurcation patterns for the system with constrains. 10 bifurcation patterns are aroused by constrains. Moreover, the amplitude of the system is limited in some ranges by the constraints.

All these bifurcation patterns can provide guidance to the dynamical analysis and design of the system.

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