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Nonstationary probability densities of system response of strongly nonlinear single-degree-of-freedom system subject to modulated white noise excitation*

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Abstract The nonstationary probability densities of system response of a single-degree-of-freedom system with lightly nonlinear damping and strongly nonlinear stiffness subject to modulated white noise excitation are studied. Using the stochastic averaging method based on the generalized harmonic functions, the averaged Fokker-Planck-Kolmogorov equation governing the nonstationary probability density of the amplitude is derived. The solution of the equation is approximated by the series expansion in terms of a set of properly selected basis functions with time-dependent coefficients. According to the Galerkin method, the time-dependent coefficients can be solved from a set of first-order linear differential equations. Then, the semi-analytical formulae of the nonstationary probability density of the amplitude response as well as the nonstationary probability density of the state response and the statistic moments of the amplitude response can be obtained. A van der Pol-Duffing oscillator subject to modulated white noise is given as an example to illustrate the proposed procedures. The effects of the system parameters, such as the linear damping coefficient and the nonlinear stiffness coefficient, on the system response are discussed.

Key words nonstationary probability density, modulated white noise, stochastic averaging method, Galerkin method

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1 Introduction

Stochastic perturbations such as ground motions, gusty winds, and sea waves are ubiquitous in nature, which are modeled as stationary or nonstationary random processes. In general, nonstationary random processes are more realistic models of the physical processes. Among them, the modulated random processes are extensively used to model the earthquake motions and atmospheric turbulence[1–6].

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Analysis of the response of systems subject to modulated random excitations has been conducted by many investigators[7–13]. Verdon[14] presented the expressions of the response correlations of a linear single-degree-of-freedom (SDOF) system under modulated white noise. To[15] obtained the expressions for the evolutionary cross-spectral density and spectral density of the responses of linear time-invariant multi-degree-of-freedom systems subject to exponentially decaying and uniformly modulated random excitations. Ahmadi[16], Iwan and Mason[17] evaluated the statistics of the response of nonlinear systems subject to modulated white noise excitation with the generalized method of equivalent linearization. Fang and Zhang[18] obtained the nonstationary mean square response of a time-invariant linear system subjected to uniformly amplitude modulated random excitations by the complex modal method. Nevertheless, only a few studies have been carried out about the nonstationary probability densities of the response of nonlinear systems under nonstationary excitations. Spanos[19] presented a method for analyzing the response of a class of weakly non-linear damping systems to a modulated random excitation and the approximate nonstationary probability density function of the response amplitude was obtained. Kougioumtzoglou and Spanos[20] presented a method based on the concept of stochastic averaging and equivalent linearization for determining the response of a lightly damped nonlinear SDOF oscillator to a evolutionary stochastic excitation. On the other hand, it is well known that the stochastic averaging method based on the generalized harmonic functions is a powerful tool and has been successfully applied to stochastic systems with strongly nonlinear stiffness[21–22]. Based on the stochastic averaging method using generalized harmonic function, the subject can be improved and this paper is written for this purpose.

In the present paper, the expressions of probability density of the state response and amplitude response, together with the statistic moments of the amplitude response, of an SDOF system with lightly nonlinear damping and strongly nonlinear stiffness subject to modulated white noise excitation are approximately obtained by using the stochastic averaging method based on the generalized harmonic functions and the Galerkin method. A van der Pol-Duffing oscillator subject to modulated white noise excitation is given as an example to illustrate the proposed procedures. The effects of the system parameters on system response are discussed as well.

2 Approximate nonstationary probability densities of system response

Consider a strongly nonlinear SDOF system subject to modulated white noise excitation. The equation of motion in the dimensionless form is

$$\ddot{X}(t) + \varepsilon c(X, \dot{X})\dot{X}(t) + g(X) = \varepsilon^{1/2} f(t) W(t), \tag{1}$$

where ε is a small positive parameter, $\varepsilon c(X, \dot{X})$ is the damping coefficient of the generalized displacement X and velocity $\dot{X}$, $g(X)$ is the nonlinear stiffness which is an odd function of the generalized displacement X, i.e., $g(-X) = -g(X)$, $W(t)$ is a Gaussian white noise with intensity $2D$, and $\varepsilon^{1/2} f(t) \geqslant 0$ is a deterministic modulating function. It should be noted that all the variables in the present paper are dimensionless and it will not further specified thereafter.

2.1 Application of stochastic averaging method based on generalized harmonic functions

Assuming that there exists a family of periodic solutions surrounding the origin of the phase plane, the following transformation can be introduced[22–23]:

$$\begin{cases} X(t) = A\cos\Theta(t), \\ \dot{X}(t) = -A\Lambda(A, \Theta)\sin\Theta(t), \end{cases} \tag{2}$$

where

$$\begin{cases} \Theta(t) = \Phi(t) + \Gamma(t), \\ \Lambda(A, \Theta) = \dfrac{\mathrm{d}\Phi}{\mathrm{d}t} = \sqrt{\dfrac{2(U(A) - U(A\cos\Theta))}{A^2 \sin^2\Theta}} = b_0(A) + \displaystyle\sum_{r=1}^{\infty} b_r(A)\cos r\Theta, \\ U(X) = \displaystyle\int_0^X g(u)\mathrm{d}u, \end{cases} \tag{3}$$

in which $A(t), \Theta(t)$, $\Phi(t)$, and $\Gamma(t)$ are stochastic processes. Substituting (2) into (1) and recalling the compatibility condition of (2), i.e., the derivative of the first equation of (2) with respect to t equal to the second equation of (2), system (1) can be written by the stochastic differential equations for the amplitude A and the phase Γ as follows:

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}t} = \varepsilon F_1(A, \Gamma) + \varepsilon^{1/2} Z_1(A, \Gamma, f) W(t), \\ \dfrac{\mathrm{d}\Gamma}{\mathrm{d}t} = \varepsilon F_2(A, \Gamma) + \varepsilon^{1/2} Z_2(A, \Gamma, f) W(t), \end{cases} \tag{4}$$

where

$$\begin{cases} F_1 = \dfrac{-\left(A\Lambda\sin\Theta\right)^2}{g(A)} \left(c(A\cos\Theta, -A\Lambda\sin\Theta)\right), \\ F_2 = \dfrac{-A\Lambda^2\cos\Theta\sin\Theta}{g(A)} \left(c(A\cos\Theta, -A\Lambda\sin\Theta)\right), \\ Z_1 = G_1 f, \quad Z_2 = G_2 f, \\ G_1 = \dfrac{-A\Lambda\sin\Theta}{g(A)}, \\ G_2 = \dfrac{-\Lambda\cos\Theta}{g(A)}. \end{cases} \tag{5}$$

It is assumed that $f(t)$ is a slowly varying function of t. Then, based on the Stratonovich-Khasminskii theorem[24], the slowly varying processes $A(t)$ and $\Gamma(t)$ converge weakly into a two-dimensional diffusion Markov process as $\varepsilon \to 0$, at a time interval $0 \leqslant t \leqslant T$, where $T \sim O(\varepsilon^{-1})$. The limit processes can be described by the averaged Itô stochastic differential equations. Because the averaged Itô equation for A is independent of Γ, $A(t)$ can be simplified as a one-dimensional diffusion process governed by

$$\mathrm{d}A = m(A, f)\mathrm{d}t + \sigma(A, f)\mathrm{d}B(t), \tag{6}$$

where

$$\begin{cases} m(A, f) = \varepsilon \left\langle F_1 \right\rangle_\Theta + \varepsilon D f^2 \left\langle \dfrac{\partial G_1}{\partial A} G_1 + \dfrac{\partial G_1}{\partial \Gamma} G_2 \right\rangle_\Theta, \\ \sigma^2(A, f) = \varepsilon f^2 \left\langle 2DG_1G_1 \right\rangle_\Theta, \end{cases} \tag{7}$$

in which $\langle\cdot\rangle_\Theta = \frac{1}{2\pi}\int_0^{2\pi} \langle\cdot\rangle \mathrm{d}\Theta$ denotes the deterministic averaging with respect to Θ, and A and f are treated as constants in the deterministic averaging.

The averaged Fokker-Planck-Kolmogorov (FPK) equation associated with (6) is

$$\frac{\partial p(A, t)}{\partial t} = -\frac{\partial}{\partial A}\left(m(A, f)p(A, t)\right) + \frac{1}{2}\frac{\partial^2}{\partial A^2}\left(\sigma^2(A, f)p(A, t)\right). \tag{8}$$

For simplification, it is assumed that the system (1) is initially at rest. Then, the initial condition of (8) is

$$p(A,0)=\hat{\delta}(A), \tag{9}$$

where $\hat{\delta}(A)$ denotes the one-sided Dirac delta function.

2.2 Approximate solution of FPK equation

In this subsection, the general Galerkin method is applied to solving the averaged FPK equation by selecting as basis functions the eigenfunctions which pertain to the amplitude of the corresponding linear system. This method does not require any perturbation analysis and is valid for large values of the nonlinearity parameter. Furthermore, the selected basis functions possess notable properties that have substantial computational advantages[25–26]. The detailed procedure is presented as follows. First, the nonstationary probability density of the amplitude $p(A,t)$ of (8) with the initial condition (9) can be approximated by

$$p(A,t)=\sum_{r=0}^{\infty}s_r(t)J_r(A), \tag{10}$$

where $s_r(t)$ are functions of time to be determined, and the basis functions $J_r(A)$ are chosen as

$$\begin{cases} J_r(A)=\dfrac{1}{r!}\dfrac{A}{\sigma_s^2}\exp\Big(-\dfrac{A^2}{2\sigma_s^2}\Big)L_r\Big(\dfrac{A^2}{2\sigma_s^2}\Big),\\ \sigma_s^2=\dfrac{D}{|\varepsilon c(0,0)|\left.\frac{\mathrm{d}g}{\mathrm{d}X}\right|_{X=0}},\\ r=0,1,\cdots, \end{cases} \tag{11}$$

in which $L_r(\cdot)$ is the Laguerre polynomial of order r. An analogous series expansion of the solution of the FPK equation has been proposed in [19].

By using (10) and the properties of Laguerre polynomials, the initial condition (9) can be rewritten as

$$s_r(0)=1,\quad r=0,1,2,\cdots. \tag{12}$$

Substituting (10) into (8) yields the following residual error:

$$\begin{aligned} R=\sum_{r=0}^{\infty}\dot{s}_r(t)J_r(A)-\Big(\sum_{r=0}^{\infty}s_r(t)\Big(-\frac{\mathrm{d}}{\mathrm{d}A}(m_1(A)J_r(A))\\ +f^2\Big(-\frac{\mathrm{d}}{\mathrm{d}A}\big(m_2(A)J_r(A)\big)+\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}A^2}\big(b(A)J_r(A)\big)\Big)\Big)\Big), \end{aligned} \tag{13}$$

where

$$\begin{cases} m_1(A)=\varepsilon\left\langle F_1\right\rangle_{\Theta},\\ m_2(A)=\varepsilon D\left\langle\dfrac{\partial G_1}{\partial A}G_1+\dfrac{\partial G_1}{\partial\Gamma}G_2\right\rangle_{\Theta},\\ b(A)=\varepsilon\left\langle 2DG_1G_1\right\rangle_{\Theta}. \end{cases} \tag{14}$$

In numerical calculation, the series in (10) will be truncated. Herein $N+1$ terms are adopted. According to the Galerkin method, the $N+1$ unknown functions $s_r(t)$ can be evaluated by vanishing the projection of the residual error R on a proper set of independent functions. By selecting $\frac{J_r(A)}{J_0(A)}$ as the weighting function, this condition can be expressed as

$$\int_0^\infty \frac{J_k(A)}{J_0(A)} R\,\mathrm{d}A = 0, \quad k = 0, 1, 2, \cdots, N. \tag{15}$$

Substituting (13) into (15) and rearranging the resulting equations with the properties of Laguerre polynomials yield the following set of first-order linear differential equations:

$$\dot{\boldsymbol{s}} = \big(\boldsymbol{B} + f^2(t)\boldsymbol{C}\big)\boldsymbol{s}(t), \tag{16}$$

where $\boldsymbol{s} = [s_0(t), s_1(t), \cdots, s_N(t)]^{\mathrm{T}}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices of order $(N+1)\times(N+1)$, and their elements are

$$\begin{cases} B_{kr} = \displaystyle\int_0^\infty kJ_r(A)\Big(\frac{2m_1(A)}{A}\Big)\Big(LL_k\Big(\frac{A^2}{2\sigma_s^2}\Big) - LL_{k-1}\Big(\frac{A^2}{2\sigma_s^2}\Big)\Big)\mathrm{d}A \\ \qquad - \Big(m_1(A)J_r(A)LL_k\Big(\dfrac{A^2}{2\sigma_s^2}\Big)\Big)\Big\|_0^\infty, \\ C_{kr} = \displaystyle\int_0^\infty kJ_r(A)\Big(\Big(\frac{2m_2(A)}{A} - \frac{b(A)}{A^2}\Big)LL_k\Big(\frac{A^2}{2\sigma_s^2}\Big) - \Big(\frac{2m_2(A)}{A} - \frac{b(A)}{A^2} \\ \qquad + \dfrac{1}{\sigma_s^2}b(A)\Big)LL_{k-1}\Big(\dfrac{A^2}{2\sigma_s^2}\Big)\Big)\mathrm{d}A + \Big(\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}A}\big(b(A)J_r(A)\big)LL_k\Big(\dfrac{A^2}{2\sigma_s^2}\Big) \\ \qquad - \dfrac{kb(A)J_r(A)}{A}\Big(LL_k\Big(\dfrac{A^2}{2\sigma_s^2}\Big) - LL_{k-1}\Big(\dfrac{A^2}{2\sigma_s^2}\Big)\Big) \\ \qquad - m_2(A)J_r(A)LL_k\Big(\dfrac{A^2}{2\sigma_s^2}\Big)\Big)\Big\|_0^\infty, \\ LL_k(\cdot) = \dfrac{1}{k!}L_k(\cdot). \end{cases} \tag{17}$$

(16) can be numerically solved with the initial condition (12). Then, the approximate solution $p(A,t)$ can be obtained. The jth order moment of the amplitude A can be simply expressed as

$$m_j(t) = E[A^j] = \sum_{r=0}^{N} s_r(t) I_{j,r}, \tag{18}$$

where $I_{j,r} = \int_0^\infty A^j J_r(A)\mathrm{d}A$, which have the following properties[26]:

$$\begin{cases} I_{n,r+1} = \dfrac{r - n/2}{r+1} I_{n,r}, \\ I_{k,r} = 0, \quad r > k/2, \end{cases} \tag{19}$$

where k is an even number. The joint nonstationary probability density of the generalized displacement and velocity can be derived from the nonstationary probability density of the amplitude as well,

$$p(X, \dot{X}, t) = \frac{p(A,t)\omega(A)}{2\pi g(A)}\bigg|_{A=U^{-1}(\dot{X}^2/2+U(X))}, \tag{20}$$

where $U^{-1}(\cdot)$ is the inverse function of $U(\cdot)$, and $\omega(A) = 2\pi\Big/\int_0^{2\pi} \Lambda^{-1}(A,\Theta)\mathrm{d}\Theta$. The nonstationary marginal probability density of the generalized displacement $p(X,t)$ can be obtained by integrating (20) over $\dot{X}$.

3 Example

Consider a van der Pol-Duffing oscillator under modulated white noise excitation. The equation of motion of the system is governed by

$$\ddot{X}(t) + (c_1 + c_2 X^2)\dot{X}(t) + \omega^2 X + \alpha X^3 = f(t)W(t), \tag{21}$$

where c_1 and c_2 represent linear and nonlinear damping coefficients, respectively, ω is the frequency of the linearized system, α is the coefficient of nonlinear stiffness, $W(t)$ is a Gaussian white noise with intensity $2D$, and $f(t) = k_1(\exp(-d_1 t) - \exp(-d_2 t))$ $(d_1 < d_2)$ is a deterministic modulating function.

Based on the procedures in Section 2, the averaged Itô equation for $A(t)$ associated with the system (21) is of the form of (6) with the averaged drift and diffusion coefficients as follows:

$$\begin{cases} m(A) = -\dfrac{c_1 A\left(\omega^2 + 5\alpha A^2/8\right)}{2\left(\omega^2 + \alpha A^2\right)} - \dfrac{c_2 A^3\left(\omega^2 + 3\alpha A^2/4\right)}{8\left(\omega^2 + \alpha A^2\right)} + Df^2 \dfrac{8\omega^4 + 3\omega^2\alpha A^2 + \alpha^2 A^4}{16A\left(\omega^2 + \alpha A^2\right)^3}, \\ \sigma^2(A) = Df^2 \dfrac{\omega^2 + 5\alpha A^2/8}{\left(\omega^2 + \alpha A^2\right)^2}. \end{cases} \tag{22}$$

The approximate expression of $p(A, t)$ has the form of (10), where $\sigma_s^2 = D/\left(|c_1|\,\omega^2\right)$ and $s_r(t)$ satisfy the set of first-order linear differential equations in the form of (16). In this example, the coefficient matrices in (17) can be numerically calculated by Gauss-Lagurre quadrature. If $\alpha = 0$, the coefficient matrices can be further reduced to

$$\begin{cases} B_{kr} = -c_1 k\left(\delta_{rk} - \delta_{r,k-1}\right) - \dfrac{c_2 k \sigma_s^2}{2}((3k+1)\delta_{rk} - (3k-1)\delta_{r,k-1} \\ \qquad + (k-1)\delta_{r,k-2} - (k+1)\delta_{r,k+1}), \\ C_{kr} = -\dfrac{D}{\omega^2\sigma_s^2} k\delta_{r,k-1}. \end{cases} \tag{23}$$

Applying (18), the first- and second-order statistical moments of A are

$$\begin{cases} m_1(t) = E[A] = \displaystyle\sum_{r=0}^{N} s_r(t) I_{1,r}, \\ m_2(t) = E[A^2] = 2\sigma_s^2\left(1 - s_1(t)\right). \end{cases} \tag{24}$$

According to (20), the joint nonstationary probability density of the generalized displacement and velocity can be obtained as

$$p(X, \dot{X}, t) = \left.\frac{p(A, t)\omega(A)}{2\pi\left(\omega^2 A + \alpha A^3\right)}\right|_{A = U^{-1}(\dot{X}^2/2 + U(X))}, \tag{25}$$

where

$$\omega(A) = \frac{\pi\sqrt{\omega^2 + \alpha A^2}}{2\text{Elliptick}\left(\sqrt{\frac{\alpha A^2}{\omega^2 + 2\alpha A^2}}\right)}, \quad U(X) = \frac{1}{2}\omega^2 X^2 + \frac{1}{4}\alpha X^4.$$

In numerical calculation, the frequency of the linearized system and parameters on the modulated white noise are set as: $\omega = 1, c_2 = 0.08, k_1 = 3.06, d_1 = 0.04\omega, d_2 = 0.1\omega$, and $D = 0.02$. The nonstationary probability densities of the amplitude $p(A, t)$ at different time instants with different linear damping coefficients and different nonlinear stiffness coefficients are shown in Fig. 1. Under corresponding conditions, the nonstationary marginal probability densities of the generalized displacement $p(X,t)$ and the time evolutions of the first- and second-order statistical moments of the amplitude are shown in Figs. 2–3. Monte Carlo simulation is implemented to validate the results, where the time step is 0.01, the number of samples is 100 000. In Figs. 1–2, the selections of number N vary for different time instants. It can be seen from the insert in Fig.1(a) that negative values of $p(A,70)$ with $N=30$ appear. Therefore, to be concise and with reasonable precision, $N=30$ when $t=10,15,20,30,40,50$ while $N=110$ when $t=70$. It is found from Figs. 1–2 that the larger number N is required if the nonstationary

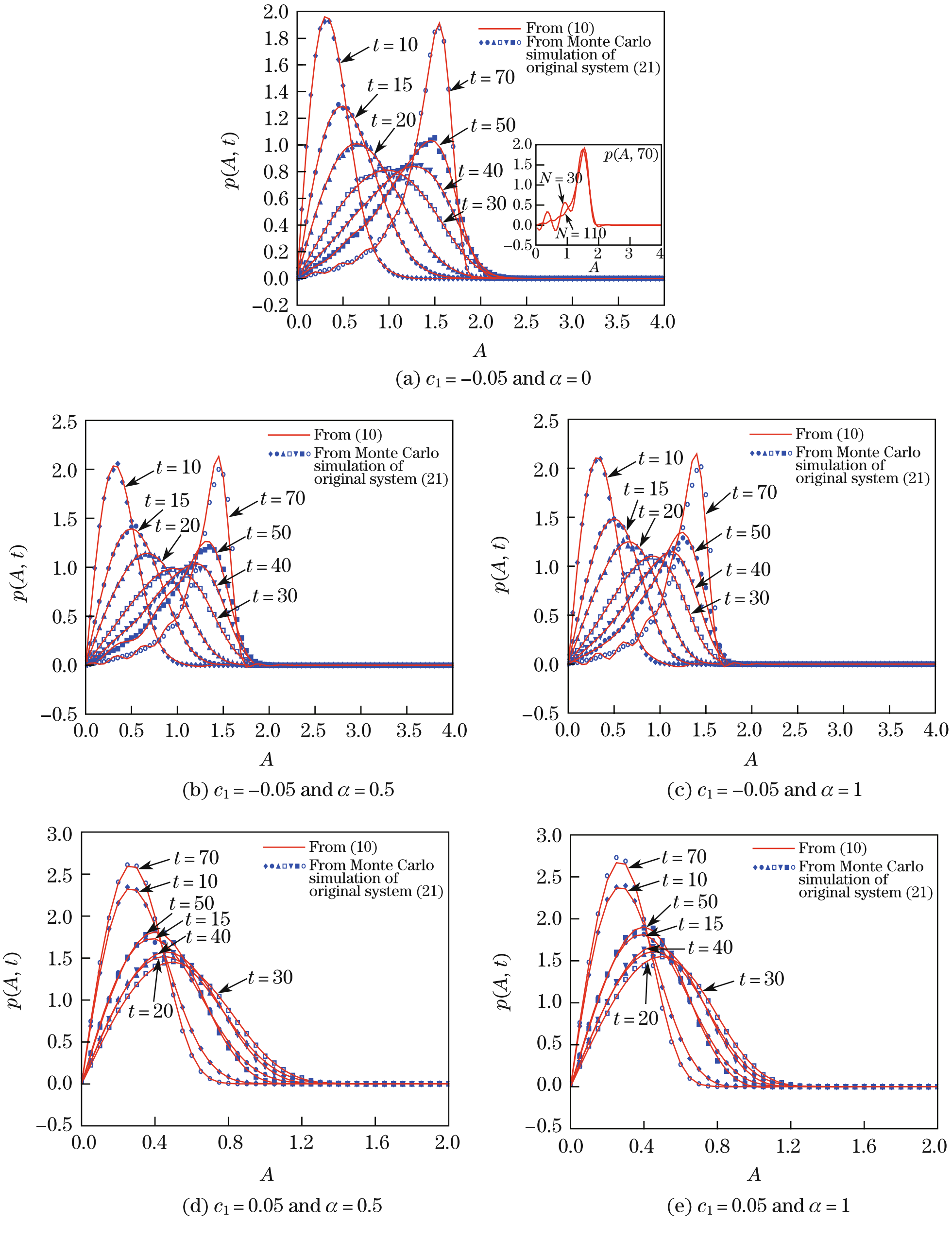


Fig. 1 Nonstationary probability densities of amplitude $p(A,t)$

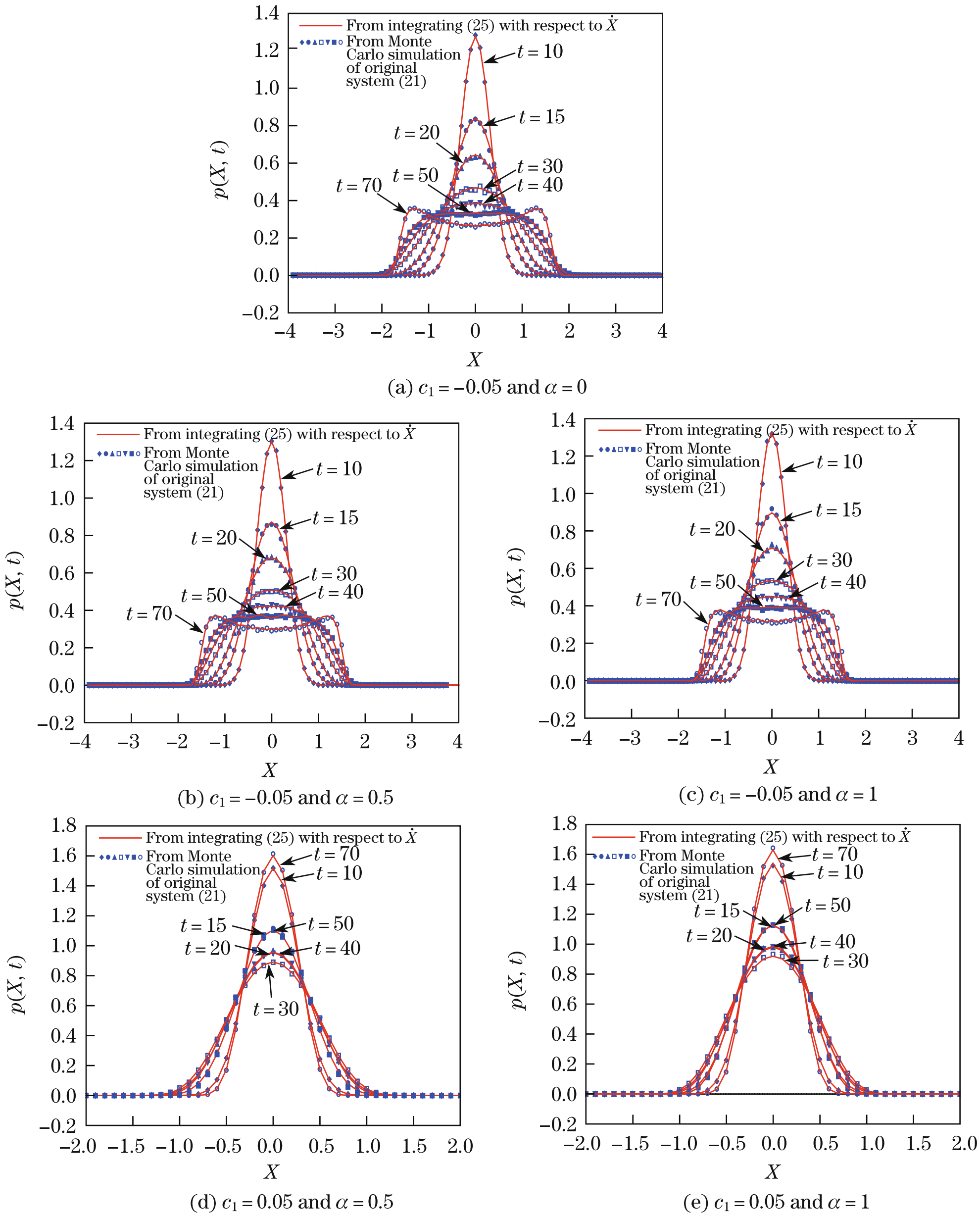
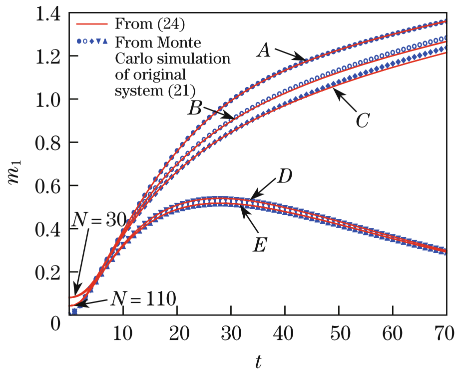
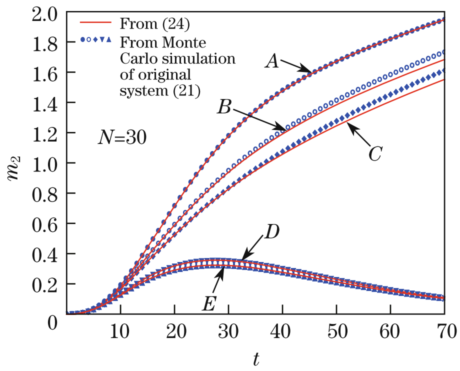


Fig. 2 Marginal probability density of generalized displacement $p(X,t)$

probability density of amplitude response closes to delta distribution. And it can also be observed in Fig. 3. It is known from the context that the nonstationary probability density is close to delta distribution when t is small. Thus, for $N = 30$, it can be shown from Fig. 3(a) that there is discrepancy when t is small. However, this discrepancy can be reduced by increasing N. All in all, from Figs. 1–3, it can be clearly seen that the results obtained from the proposed procedure agree well with those from Monte Carlo simulation of the original system (21). A larger value of N will lead to more accurate results. With a similar level of accuracy, the larger nonlinearity and the more complex behavior of the system result in a larger N. Moreover, under the present type of the modulated white noise, the peak values of $p(A,t)$ decrease with time at the beginning, and then increase again because the amplitude of modulation increases with time at first and then decreases after a critical value. However, $p(X,t)$ has this behavior only when the coefficient c_1 is positive. Also observed is that the effect of system parameters, especially the linear damping coefficient, on system response is significant. Figure 3 further shows that the first- and second-order statistical moments of the amplitude decrease as nonlinear stiffness coefficient α increases.



(a) First-order statistical moments of the amplitude
A: $c_1 = -0.05$, $\alpha = 0$; B: $c_1 = -0.05$, $\alpha = 0.5$;
C: $c_1 = -0.05$, $\alpha = 1$; D: $c_1 = 0.05$, $\alpha = 0.5$;
E: $c_1 = 0.05$, $\alpha = 1$



(b) Second-order statistical moments of the amplitude
A: $c_1 = -0.05$, $\alpha = 0$; B: $c_1 = -0.05$, $\alpha = 0.5$;
C: $c_1 = -0.05$, $\alpha = 1$; D: $c_1 = 0.05$, $\alpha = 0.5$;
E: $c_1 = 0.05$, $\alpha = 1$

Fig. 3 Time evolutions of first- and second-order statistical moments of amplitude with different system parameter values

4 Conclusions

In this paper, the nonstationary probability densities of the state and amplitude response of an SDOF system with lightly nonlinear damping and strongly nonlinear stiffness subject to modulated white noise excitation are investigated. By using the stochastic averaging method based on the generalized harmonic functions, the averaged FPK equation of the nonstationary probability density of the amplitude is derived. The nonstationary probability density of the system is approximated by the series expansion in terms of Laguerre polynomials type basis functions. Applying the Galerkin method, the nonstationary probability densities and the statistic moments of the amplitude response and the nonstationary probability densities of the state response are obtained. The proposed procedure is applied to a van der Pol-Duffing oscillator subject to modulated white noise excitation. The results obtained from the proposed procedures agree well with those from the Monte Carlo simulation of the original system. The effects of system parameters such as linear damping coefficient and nonlinear stiffness coefficient on the system response are discussed.

References

[1] Wirsching, P. H. and Yao, J. T. P. Monte Carlo study of seismic structural safety. *ASCE Journal of the Structural Division*, **97**, 1497–1519 (1971)

[2] Howell, L. J. and Lin, Y. K. Response of flight vehicles to nonstationary atmospheric turbulence. *AIAA Journal*, **9**, 2201–2207 (1971)

[3] Yang, Y. N. Nonstationary envelope process and first excursion probability. *Journal of Structural Mechanics*, **1**, 231–248 (1972)

[4] Fujimori, Y. and Lin, Y. K. Analysis of airplane response to nonstationary turbulence including wing bending flexibility. *AIAA Journal*, **11**, 334–339 (1973)

[5] Zhang, Z. C., Lin, J. H., Zhang, Y. H., Zhao, Y., Howson, W. P., and Williams, F. W. Nonstationary random vibration analysis for train-bridge systems subjected to horizontal earthquakes. *Engineering Structures*, **32**, 3571–3582 (2010)

[6] Abbas, A. M. and Manohar, C. S. Reliability-based vector nonstationary random critical earthquake excitations for parametrically excited systems. *Structural Safety*, **29**, 32–48 (2007)

[7] Caughey, T. K. and Stumpf, H. F. Transient response of a dynamic system under random excitation. *Journal of Applied Mechanics-Transactions of the ASME*, **28**, 563–566 (1961)

[8] Corotis, R. B. and Marshall, T. A. Oscillator response to modulated random-excitation. *Journal of the Engineering Mechanics Division-ASCE*, **103**, 501–513 (1977)

[9] Iwan, W. D. and Hou, Z. K. Explicit solutions for the response of simple systems subjected to nonstationary random-excitation. *Structural Safety*, **6**, 77–86 (1989)

[10] Fu, G. Seismic response statistics of SDOF system to exponentially modulated coloured input: an explicit solution. *Earthquake Engineering & Structural Dynamics*, **24**, 1355–1370 (1995)

[11] Michaelov, G., Sarkani, S., and Lutes, L. D. Spectral characteristics of nonstationary random processes response of a simple oscillator. *Structural Safety*, **21**, 245–267 (1999)

[12] Jangid, R. S. Stochastic response of building frames isolated by lead-rubber bearings. *Structural Control & Health Monitoring*, **17**, 1–22 (2010)

[13] Allam, S. M. and Datta, T. K. Seismic response of a cable-stayed bridge deck under multi-component non-stationary random ground motion. *Earthquake Engineering & Structural Dynamics*, **33**, 375–393 (2004)

[14] Verdon, J. M. Response of a single-degree-of-freedom system to modulated white noise. *Journal of Applied Mechanics-Transactions of the ASME*, **40**, 296–297 (1973)

[15] To, C. W. S. Non-stationary random responses of a multi-degree-of-freedom system by the theory of evolutionary spectra. *Journal of Sound and Vibration*, **83**, 273–291 (1982)

[16] Ahmadi, G. Mean square response of a Duffing oscillator to a modulated white noise excitation by the generalized method of equivalent linearization. *Journal of Sound and Vibration*, **71**, 9–15 (1980)

[17] Iwan, W. D. and Mason, A. B. Equivalent linearization for systems subjected to non-stationary random excitation. *International Journal of Non-Linear Mechanics*, **15**, 71–82 (1980)

[18] Fang, T. and Zhang, T. S. Non-stationary mean-square response due to uniformly amplitude modulated random excitations. *Journal of Sound and Vibration*, **182**, 369–379 (1995)

[19] Spanos, P. D. A method for analysis of non-linear vibrations caused by modulated random-excitation. *International Journal of Non-Linear Mechanics*, **16**, 1–11 (1981)

[20] Kougioumtzoglou, I. A. and Spanos, P. D. An approximate approach for nonlinear system response determination under evolutionary stochastic excitation. *Current Science*, **97**, 1203–1211 (2009)

[21] Zhu, W. Q. Nonlinear stochastic dynamics and control in Hamiltonian formulation. *Applied Mechanics Reviews*, **59**, 230–248 (2006)

[22] Zhu, W. Q., Huang, Z. L., and Suzuki, Y. Response and stability of strongly non-linear oscillators under wide-band random excitation. *International Journal of Non-Linear Mechanics*, **36**, 1235–1250 (2001)

[23] Xu, Z. and Chung, Y. K. Averaging method using generalized harmonic functions for strongly non-linear oscillators. *Journal of Sound and Vibration*, **174**, 563–576 (1994)

[24] Khasminskii, R. Z. On the averaging principle for Itô stochastic differential equations (in Russian). *Kibernetika*, **4**, 260–279 (1968)

[25] Spanos, P. D. and Iwan, W. D. Computational aspects of random vibration analysis. *Journal of the Engineering Mechanics Division-ASCE*, **104**, 1403–1415 (1978)

[26] Spanos, P. D., Sofi, A., and di Paola, M. Nonstationary response envelope probability densities of nonlinear oscillators. *Journal of Applied Mechanics-Transactions of the ASME*, **74**, 315–324 (2007)