Appl. Math. Mech. -Engl. Ed. **31**(7), 929–936 (2010)
DOI 10.1007/s10483-010-1327-z

Applied Mathematics and Mechanics (English Edition)

A new auxiliary equation method for finding travelling wave solutions to KdV equation*

Jing PANG (庞 晶)[1], Chun-quan BIAN (边春泉)[1], Lu CHAO (朝 鲁)[2]

(1. College of Science, Inner Mongolia University of Technology, Hohhot 010051, P. R. China;
2. Department of Mathematics, Shanghai Maritime University, Shanghai 200315, P. R. China)

(Communicated by Xing-ming GUO)

Abstract In this paper, a new auxiliary equation method is used to find exact travelling wave solutions to the (1+1)-dimensional KdV equation. Some exact travelling wave solutions with parameters have been obtained, which cover the existing solutions. Compared to other methods, the presented method is more direct, more concise, more effective, and easier for calculations. In addition, it can be used to solve other nonlinear evolution equations in mathematical physics.

Key words auxiliary equation method, travelling wave solution, KdV equation, homogeneous balance method

Chinese Library Classification O175.29, 0241.82
2000 Mathematics Subject Classification 37K10, 37L05

1 Introduction

The research area of nonlinear evolution equations has been very active for the past few decades. There are various kinds of nonlinear evolution equations in the fields of physical and mathematical sciences. Much effort has been made to construct the exact solutions to nonlinear equations because of their important roles in the study of nonlinear physical phenomena. In recent years, the powerful and efficient methods for finding the analytic solutions to nonlinear equations have attracted a lot of interest of a diverse group of scientists, such as the homogeneous balance method[1–3], the tanh-function method[1–2,4–5], the tanh-coth function method[1–2], the Painleve expansion method[2–3,6], the auxiliary equation method[7–9], the Jacobi elliptic function method[1–2,6,10], the sine-cosine function method[1–2,11], and the exp-function method[2,12]. These methods have been used to solve nonlinear dispersive and dissipative problems.

In this paper, we give a new auxiliary equation method for finding the exact travelling wave solutions to the (1+1)-dimensional KdV equation. Some exact solutions with parameters have been obtained successfully, which include existing solutions. The results show that this

* Received Sept. 17, 2009 / Revised May 5, 2010
Project supported by the National Natural Science Foundation of China (No. 10461005), the Ph. D. Programs Foundation of Ministry of Education of China (No. 20070128001), and the High Education Science Research Program of Inner Mongolia (No. NJZY08057)
Corresponding author Jing PANG, Professor, E-mail: pang_j@imut.edu.cn, bianchunquan2005@163.com

method not only is effective and straightforward, but also can be used for many other nonlinear evolution equations in mathematical physics. Furthermore, this method includes the methods of [13–17], and it is better than those methods. It is also a powerful mathematical tool to obtain the exact travelling wave solutions for some high-dimensional or high-order nonlinear evolution equations.

2 Description of the method

In this section, we describe the method of finding the travelling wave solutions to nonlinear evolution equations as follows. First, a given nonlinear partial differential equation has the form

$$p(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{1}$$

Our method mainly consists of four steps:

Step 1 Take the complex solutions of (1) in the form

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{2}$$

where v is a real constant. Under the transformation (2), (1) becomes an ordinary differential equation

$$Q(u, u', u'', \cdots) = 0. \tag{3}$$

Step 2 Take the solutions of (3) in the more general form

$$u(\xi) = a_0 + \sum_{i=1}^{m} \Big(a_i \Big(\frac{G(\xi)}{G'(\xi)}\Big)^{i} + b_i \Big(\frac{G(\xi)}{G'(\xi)}\Big)^{-i} \Big), \tag{4}$$

where a_m and b_m are not zero at the same time, and a_0, a_i, and b_i $(i = 1, 2, \cdots, m)$ are constants to be determined later. The integer m in (4) can be determined by balancing the highest order nonlinear terms and the highest order linear terms of $u(\xi)$ in (3). $G = G(\xi)$ satisfies the second-order linear ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where λ and μ are constants to be determined later.

The explicit expressions for the general solution of (5) are as follows:

$$\begin{cases} \text{when } \lambda^2 - 4\mu > 0, \quad G(\xi) = c_1 \exp\Big(\dfrac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}\xi\Big) + c_2 \exp\Big(\dfrac{-\lambda - \sqrt{\lambda^2 - 4\mu}}{2}\xi\Big); \\ \text{when } \lambda^2 - 4\mu = 0, \quad G(\xi) = (c_1 + c_2\xi) \exp\Big(-\dfrac{\lambda}{2}\xi\Big); \\ \text{when } \lambda^2 - 4\mu < 0, \quad G(\xi) = \exp\Big(-\dfrac{\lambda}{2}\xi\Big)\Big(c_1 \cos\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi + c_2 \sin\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi\Big). \end{cases}$$

Remark 1 Let $a_i = 0$, $i = 1, 2, \cdots, m$. Equation (4) changes to

$$u(\xi) = a_0 + \sum_{i=1}^{m} b_i \Big(\frac{G(\xi)}{G'(\xi)}\Big)^{-i}. \tag{6}$$

The form of (6) has been used in [3–5, 7]. If we set $b_i = 0$ $(i = 1, 2, \cdots, m,)$, (4) changes to

$$u(\xi) = a_0 + \sum_{i=1}^{m} a_i \Big(\frac{G(\xi)}{G'(\xi)}\Big)^{i}. \tag{7}$$

Step 3 Substitute (4) into (3) and collect all terms with the same order of $\frac{G}{G'}$ together. The left-hand side of (3) is converted into a polynomial in $\frac{G}{G'}$. Then, let each coefficient of this polynomial to be zero to derive a set of over-determined partial differential equations for a_0, a_i, $b_i\,(i=1,2,\cdots,m)$, λ, μ, and v.

Step 4 Solve the algebraic equations obtained in Step 3 with the aid of a computer algebra system (such as Mathematica or Maple) to determine these constants. Moreover, the solutions of (5) are well known. Then, substituting a_0, a_i, b_i $(i=1,2,\cdots,m)$, v, and the solutions of (5) into (4), we can obtain the exact travelling wave solutions of (1).

3 Applications of the method

Consider the KdV equation[13]

$$u_t+uu_x+\delta u_{xxx}=0. \tag{8}$$

Choose the travelling wave transformation (2). Substituting (2) into (8), integrating it with respect to ξ once, and letting the integrating constant to be zero, we have

$$\delta u''+\frac{1}{2}u^2-vu=0. \tag{9}$$

According to Step 2, we get $m=2$. Therefore, we can write the solution of (9) in the form

$$u(\xi)=a_0+a_1\frac{G}{G'}+a_2\Big(\frac{G}{G'}\Big)^2+b_1\Big(\frac{G}{G'}\Big)^{-1}+b_2\Big(\frac{G}{G'}\Big)^{-2}, \tag{10}$$

where a_2 and b_2 are not zero at the same time. By using (5) and from (10), we have

$$\begin{aligned}u''=&(\lambda a_1+2a_2+\lambda\mu b_1+2\mu^2b_2)+(\lambda^2a_1+6\lambda a_2+2\mu a_1)\frac{G}{G'}\\&+(8\mu a_2+3\lambda\mu a_1+4\lambda^2a_2)\Big(\frac{G}{G'}\Big)^2+(2\mu^2a_1+10\lambda\mu a_2)\Big(\frac{G}{G'}\Big)^3+6\mu^2a_2\Big(\frac{G}{G'}\Big)^4\\&+(2\mu b_1+6\lambda\mu b_2+\lambda^2b_1)\Big(\frac{G}{G'}\Big)^{-1}+(3\lambda b_1+4\lambda^2b_2+8\mu b_2)\Big(\frac{G}{G'}\Big)^{-2}\\&+(2b_1+10\lambda b_2)\Big(\frac{G}{G'}\Big)^{-3}+6b_2\Big(\frac{G}{G'}\Big)^{-4}.\end{aligned} \tag{11}$$

Substituting (10) and (11) into (9), collecting the coefficients of $(\frac{G}{G'})^i$ $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4)$, and letting it be zero, we obtain the system

$$\begin{aligned}&\frac{a_2^2}{2}+6\delta\mu^2a_2=0,\quad a_0b_1+a_1b_2-vb_1=0,\\&\frac{b_2^2}{2}+6\delta b_2=0,\quad a_1a_2+10\lambda\mu\delta a_2+2\delta\mu^2a_1=0,\\&\frac{b_1^2}{2}+a_0b_2-vb_2+3\lambda\delta b_1+4\lambda^2\delta b_2+8\mu\delta b_2=0,\quad b_1b_2+10\lambda\delta b_2+2\delta b_1=0,\\&\frac{a_0^2}{2}+a_1b_1+a_2b_2-va_0+2\delta a_2+\lambda\delta a_1+\lambda\mu\delta b_1+2\mu^2\delta b_2=0,\\&\frac{a_1^2}{2}+a_0a_2-va_2+4\lambda^2\delta a_2+8\mu\delta a_2+3\lambda\mu\delta a_1=0,\\&a_0a_1+a_2b_1-va_1+6\lambda\delta a_2+\lambda^2\delta a_1-\lambda^2\delta b_1+2\mu\delta a_1-2\delta\mu b_1-6\lambda\mu\delta b_2=0.\end{aligned}$$

Solving this system by Mathematica, we obtain

$$a_0=-12\delta\mu,\quad a_1=-12\lambda\mu\delta,\quad a_2=-12\delta\mu^2,\quad b_1=b_2=0,\quad v=\delta(\lambda^2-4\mu), \tag{12}$$

$$a_0=-2(\lambda^2\delta+2\delta\mu),\quad a_1=-12\lambda\mu\delta,\quad a_2=-12\delta\mu^2,\quad b_1=b_2=0,\quad v=-\delta(\lambda^2-4\mu), \tag{13}$$

$$a_0=-24\delta\mu,\quad a_1=0,\quad a_2=-12\delta\mu^2,\quad b_1=0,\quad b_2=-12\delta,\quad v=-16\delta\mu,\quad \lambda=0, \tag{14}$$

$$a_0=8\delta\mu,\quad a_1=0,\quad a_2=-12\delta\mu^2,\quad b_1=0,\quad b_2=-12\delta,\quad v=16\delta\mu,\quad \lambda=0, \tag{15}$$

$$a_0=-12\delta\mu,\quad a_1=a_2=0,\quad b_1=0,\quad b_2=-12\delta,\quad v=-4\delta\mu,\quad \lambda=0, \tag{16}$$

$$a_0=-4\delta\mu,\quad a_1=a_2=0,\quad b_1=0,\quad b_2=-12\delta,\quad v=4\delta\mu,\quad \lambda=0, \tag{17}$$

$$a_0=-2\delta(\lambda^2+2\mu),\quad a_1=a_2=0,\quad b_1=-12\lambda\delta,\quad b_2=-12\delta,\quad v=-2\delta(\lambda^2+2\mu), \tag{18}$$

$$a_0=-12\delta\mu,\quad a_1=a_2=0,\quad b_1=-12\lambda\delta,\quad b_2=-12\delta,\quad v=-12\delta\mu, \tag{19}$$

where λ, μ, and δ are arbitrary constants. By using (12)–(19), (10) can be written as

$$u(\xi)=-12\delta\mu-12\lambda\mu\delta\frac{G}{G'}-12\delta\mu^2\Big(\frac{G}{G'}\Big)^2,\quad \xi=x-\delta(\lambda^2-4\mu)t, \tag{20}$$

$$u(\xi)=-2(\lambda^2\delta+2\delta\mu)-12\lambda\mu\delta\frac{G}{G'}-12\delta\mu^2\Big(\frac{G}{G'}\Big)^2,\quad \xi=x+\delta(\lambda^2-4\mu), \tag{21}$$

$$u(\xi)=-24\delta\mu-12\delta\mu^2\Big(\frac{G}{G'}\Big)^2-12\delta\Big(\frac{G}{G'}\Big)^{-2},\quad \xi=x+16\delta\mu t, \tag{22}$$

$$u(\xi)=8\delta\mu-12\delta\mu^2\Big(\frac{G}{G'}\Big)^2-12\delta\Big(\frac{G}{G'}\Big)^{-2},\quad \xi=x-16\delta\mu t, \tag{23}$$

$$u(\xi)=-12\delta\mu-12\delta\Big(\frac{G}{G'}\Big)^{-2},\quad \xi=x+4\delta\mu t, \tag{24}$$

$$u(\xi)=-4\delta\mu-12\delta\Big(\frac{G}{G'}\Big)^{-2},\quad \xi=x-4\delta\mu t, \tag{25}$$

$$u(\xi)=-2\delta(\lambda^2+2\mu)-12\lambda\delta\frac{G'}{G}-12\delta\Big(\frac{G'}{G}\Big)^2,\quad \xi=x+2\delta(\lambda^2+2\mu)t, \tag{26}$$

$$u(\xi)=-12\delta\mu-12\lambda\delta\frac{G'}{G}-12\delta\Big(\frac{G'}{G}\Big)^2,\quad \xi=x+12\delta\mu t. \tag{27}$$

Substituting the general solutions of (5) into (20) and (21), we can obtain the travelling wave solutions of (8) as follows:

When $\lambda^2-4\mu>0$,

$$u_1(\xi)=\frac{48\delta\mu M^2c_1c_2\operatorname{sech}^2(\frac{1}{2}M\xi)}{(-\lambda c_1-Mc_2+(\lambda c_2+Mc_1)\tanh(\frac{1}{2}M\xi))^2}, \tag{28}$$

where $M=\sqrt{\lambda^2-4\mu}$ and $\xi=x-\delta(\lambda^2-4\mu)t$;

$$\begin{aligned}u_2(\xi)=-2\delta(\lambda^2+2\mu)&-\frac{48\delta\mu^2(c_1-c_2\tanh(\frac{1}{2}M\xi)^2)}{(-\lambda c_1-Mc_2+(\lambda c_2+Mc_1)\tanh(\frac{1}{2}M\xi))^2}\\&-\frac{24\lambda\mu\delta(c_1-c_2\tanh(\frac{1}{2}M\xi))}{-Mc_2-\lambda c_1+(\lambda c_2+Mc_1)\tanh(\frac{1}{2}M\xi)},\end{aligned} \tag{29}$$

where $M=\sqrt{\lambda^2-4\mu}$ and $\xi=x+\delta(\lambda^2-4\mu)t$.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\xi) = \frac{12\delta\mu(\lambda^2(c_1 + c_2\xi)^2 - 4(c_2^2 + \mu(c_1 + c_2\xi)^2))}{(-2c_2 + \lambda(c_1 + c_2\xi))^2}, \tag{30}$$

where $\xi = x - \delta(\lambda^2 - 4\mu)t = x$;

$$u_4(\xi) = -2\delta(\lambda^2 + 2\mu) - \frac{48\delta\mu^2(c_1 + c_2\xi)^2}{(-2c_2 + \lambda(c_1 + c_2\xi))^2} + \frac{24\delta\lambda\mu(c_1 + c_2\xi)}{-2c_2 + \lambda(c_1 + c_2\xi)}, \tag{31}$$

where $\xi = x + \delta(\lambda^2 - 4\mu)t = x$.

When $\lambda^2 - 4\mu < 0$,

$$u_5(\xi) = -12\delta\mu\Big(1 + \frac{2\lambda(c_1 + c_2\tan(\frac{1}{2}M\xi))}{M(c_2 - c_1\tan(\frac{1}{2}M\xi))} + \frac{4\mu(c_1 + c_2\tan(\frac{1}{2}M\xi))^2}{M^2(c_2 - c_1\tan(\frac{1}{2}M\xi))^2}\Big), \tag{32}$$

where $M = \sqrt{\lambda^2 - 4\mu}$ and $\xi = x - \delta(\lambda^2 - 4\mu)t$;

$$u_6(\xi) = -2\delta(\lambda^2 + 2\mu) - \frac{24\lambda\delta\mu(c_1 + c_2\tan(\frac{1}{2}M\xi))}{M(c_2 - c_1\tan(\frac{1}{2}M\xi))} - \frac{48\delta\mu^2(c_1 + c_2\tan(\frac{1}{2}M\xi))^2}{M^2(c_2 - c_1\tan(\frac{1}{2}M\xi))^2}, \tag{33}$$

where $M = \sqrt{\lambda^2 - 4\mu}$ and $\xi = x + \delta(\lambda^2 - 4\mu)t$.

Substituting the general solutions of (5) into (22), we can obtain the travelling wave solutions of (8) as follows:

When $\mu > 0$,

$$u_7(\xi) = -\frac{12\delta\mu(c_1^2 + c_2^2)^2\sec^4(\sqrt{\mu}\xi)}{(c_2 - c_1\tan(\sqrt{\mu}\xi))^2(c_1 + c_2\tan(\sqrt{\mu}\xi))^2}, \tag{34}$$

where $\xi = x + 16\delta\mu t$.

When $\mu < 0$,

$$u_8(\xi) = \frac{12\delta\mu(c_1^2 - c_2^2)^2\text{sech}^4(\sqrt{-\mu}\xi)}{(c_2 + c_1\tanh(\sqrt{-\mu}\xi))^2(c_1 + c_2\tanh(\sqrt{-\mu}\xi))^2}, \tag{35}$$

where $\xi = x + 16\delta\mu t$.

Substituting the general solutions of (5) into (23) , we can obtain the travelling wave solutions of (8) as follows:

When $\mu > 0$,

$$u_9(\xi) = 4\delta\mu\Big(2 - \frac{3(c_2 - c_1\tan(\sqrt{\mu}\xi))^2}{(c_1 + c_2\tan(\sqrt{\mu}\xi))^2} - \frac{3(c_1 + c_2\tan(\sqrt{\mu}\xi))^2}{(c_2 - c_1\tan(\sqrt{\mu}\xi))^2}\Big), \tag{36}$$

where $\xi = x - 16\delta\mu t$.

When $\mu < 0$,

$$u_{10}(\xi) = 4\delta\mu\Big(2 + \frac{3(c_2 + c_1\tanh(\sqrt{-\mu}\xi))^2}{(c_1 + c_2\tanh(\sqrt{-\mu}\xi))^2} + \frac{3(c_1 + c_2\tanh(\sqrt{-\mu}\xi))^2}{(c_2 + c_1\tanh(\sqrt{-\mu}\xi))^2}\Big), \tag{37}$$

where $\xi = x - 16\delta\mu t$.

Substituting the general solutions of (5) into (24), we can obtain the travelling wave solutions of (8) as follows:

When $\mu > 0$,

$$u_{11}(\xi) = -\frac{12\delta\mu(c_1^2 + c_2^2)}{(c_1 \cos(\sqrt{\mu}\xi) + c_2 \sin(\sqrt{\mu}\xi))^2}, \tag{38}$$

where $\xi = x + 4\delta\mu t$.

When $\mu < 0$,

$$u_{12}(\xi) = -\frac{12\delta\mu(c_1^2 - c_2^2)}{(c_1 \cosh(\sqrt{-\mu}\xi) + c_2 \sinh(\sqrt{-\mu}\xi))^2}, \tag{39}$$

where $\xi = x + 4\delta\mu t$.

Substituting the general solutions of (5) into (25), we can obtain the travelling wave solutions of (8) as follows:

When $\mu > 0$,

$$u_{13}(\xi) = -4\delta\mu\Big(1 + \frac{3(c_2 - c_1 \tan(\sqrt{\mu}\xi))^2}{(c_1 + c_2 \tan(\sqrt{\mu}\xi))^2}\Big), \tag{40}$$

where $\xi = x - 4\delta\mu t$.

When $\mu < 0$,

$$u_{14}(\xi) = -4\delta\mu\Big(1 - \frac{3(c_2 + c_1 \tanh(\sqrt{-\mu}\xi))^2}{(c_1 + c_2 \tanh(\sqrt{-\mu}\xi))^2}\Big), \tag{41}$$

where $\xi = x - 4\delta\mu t$.

Substituting the general solutions of (5) into (26), we can obtain the travelling wave solutions of (8) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{15}(\xi) = -\frac{2\delta M(-4c_1 + c_2(\cosh(M\xi) + \sinh(M\xi)))\mathrm{sech}^2(\frac{1}{2}M\xi)}{(c_1 + c_2 \tanh(\frac{1}{2}M\xi))^2}. \tag{42}$$

When $\lambda^2 - 4\mu = 0$,

$$u_{16}(\xi) = \frac{\delta(\lambda^2(c_1 + c_2\xi)^2 - 4(3c_2^2 + \mu(c_1 + c_2\xi)^2))}{(c_1 + c_2\xi)^2}. \tag{43}$$

When $\lambda^2 - 4\mu < 0$,

$$u_{17}(\xi) = \delta\Big(-2(\lambda^2+2\mu) - \frac{3M(c_2 - c_1 \tan(\frac{1}{2}M\xi))^2}{(c_1 + c_2 \tan(\frac{1}{2}M\xi))^2} - \frac{6\lambda M(c_2 - c_1 \tan(\frac{1}{2}M\xi))}{c_1 + c_2 \tan(\frac{1}{2}M\xi)}\Big). \tag{44}$$

In (42)–(44), $M = \sqrt{\lambda^2 - 4\mu}$, $\xi = x + 2\delta(\lambda^2 + 2\mu)t$, and $\lambda = \pm 2\mathrm{i}\sqrt{2\mu}$.

Substituting the general solutions of (5) into (27), we can obtain the travelling wave solutions of (8) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{18} = \frac{12\delta c_1 M\,\mathrm{sech}^2(\frac{1}{2}M\xi)}{(c_1 + c_2 \tanh(\frac{1}{2}M\xi))^2}. \tag{45}$$

When $\lambda^2 - 4\mu = 0$,

$$u_{19} = \frac{3\delta(\lambda^2 c_1 - 4(c_2 + \mu c_1)^2)}{(c_1 + c_2\xi)^2}. \tag{46}$$

When $\lambda^2 - 4\mu < 0$,

$$u_{20} = 3\delta\Big(-4\mu - \frac{M(c_2 - c_1 \tan(\frac{1}{2}M\xi))^2}{(c_1 + c_2 \tan(\frac{1}{2}M\xi))^2} - \frac{2\lambda M(c_2 - c_1 \tan(\frac{1}{2}M\xi))}{c_1 + c_2 \tan(\frac{1}{2}M\xi)}\Big). \tag{47}$$

In (45)–(47), $M = \sqrt{\lambda^2 - 4\mu}$, $\xi = x + 12\delta\mu t$, and $\lambda = \pm 2\mathrm{i}\sqrt{2\mu}$.

Remark 2 In particular, if c_1, c_2, λ, and μ are taken as special values, the various known results in literatures can be rediscovered. For instance, if we set $c_2 = 0$, $c_1 \neq 0$, and $\mu < 0$, the solutions $u_{12}(\xi)$ and $u_{14}(\xi)$ can be written as

$$u_{21}(\xi) = -12\delta\mu \,\mathrm{sech}^2(\sqrt{-\mu}\xi), \tag{48}$$

where $\xi = x + 4\delta\mu t$, and

$$u_{22}(\xi) = -4\delta\mu + 12\delta\mu \tanh^2(\sqrt{-\mu}\xi), \tag{49}$$

where $\xi = x - 4\delta\mu t$.

Remark 3 $u_{21}(\xi)$ and $u_{22}(\xi)$ are the well-known solutions of the KdV equation (8) (see [1]). Besides, the solutions $u_j(\xi)$ $(j = 1, 2, \cdots, 20)$ have not been given in [1–3, 6–7, 9–11, 18–21] and other references.

4 Conclusions and discussions

The new auxiliary equation method has been used to find new travelling wave solutions to the (1+1)-dimensional KdV equations in this paper. Twenty-two exact travelling wave solutions with parameters c_1 and c_2 are successfully obtained, including some new and well-known solutions. Compared with other methods, our method is more direct, more concise, more effective, and easier for calculations. Furthermore, this method includes the methods of [13–17], and it is better than those methods. It is also a promising method of solving other nonlinear partial differential equations in mathematical physics. More importantly, this method can be used to obtain the exact travelling wave solutions and the non-travelling wave solutions of some high-dimensional or high-order nonlinear evolution equations. It will be given in other articles.

References

[1] Li, Z. B. *The Travelling Wave Solutions of Nonlinear Mathematical and Physical Equations* (in Chinese), Science Press, Beijing (2006)

[2] Liu, S. S. and Liu, S. D. *Nonlinear Equations of the Physics* (in Chinese), Peking University Press, Beijing (2000)

[3] Li, Y. S. *Soliton and Integerable System* (in Chinese), Shanghai Scientific and Technological Education Publishing House, Shanghai (1999)

[4] Wazwaz, A. M. The tanh method for travelling wave solutions of nonlinear equations. *Appl. Math. Comput.* **154**(3), 713–723 (2004)

[5] Mansour, M. B. A. Existence of traveling wave solutions for a nonlinear dissipative equation. *Applied Mathematics and Mechanics* (*English Edition*) **30**(4), 513–516 (2009) DOI 10.1007/s10483-009-0411-6

[6] Lou, S. Y. and Tang, X. Y. *The Method of Nonlinear Mathematical Physics* (in Chinese), Science Press, Beijing (2006)

[7] Deng, Z. G. and Ma, H. P. Optimal error estimates for Fourier spectral approximation of the generalized KdV equation. *Applied Mathematics and Mechanics* (*English Edition*) **30**(1), 29–38 (2009) DOI 10.1007/s10483-009-0104-1

[8] Taogetusang and Sirendaoerji. Construction of the exact solitary wave solutions of the non-linear evolution equations using a new auxiliary equation (in Chinese). *Acta Scientiarum Naturalium Universitatis Neimongol* **35**(3), 0246–0251 (2004)

[9] Zhang, S. and Xia, T. C. A generalized new auxiliary equation method and its applications to nonlinear partial differential equations. *Phys. Lett. A* **363**, 356–360 (2007)

[10] Li, B. A., You, G. W., and Qin, Q. Periodic wave solutions and solitary wave solutions to BBM equation (in Chinese). *Joural of Henan University of Science and Technology* (*Natural Science*), **25**(5), 70–73 (2004)

[11] Wazwaz, A. M. A sine-cosine method for handling nonlinear wave equations. *Math. Comput. Modelling* **40**, 499–508 (2004)

[12] Zhang，S. Application of exp-function method for (3+1)-dimensional nonlinear evolution equations. *Computers Mathematics with Applications* **56**(5), 1451–1456 (2008)

[13] Wang, M. L., Li, X. Z., and Zhang, J. L. The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **372**, 417–423 (2008)

[14] Bekir, A. Application of the $(\frac{G'}{G})$-expansion method for nonlinear evolution equations. *Phys. Lett. A* **372**, 3400–3406 (2008)

[15] Zhang, J., Wei, X. L., and Lu, Y. J. A generalized $(\frac{G'}{G})$-expansion method and its applications. *Phys. Lett. A* **372**, 3653–3658 (2008)

[16] Zhang, S., Tong, J. L., and Wang, W. A generalized $(\frac{G'}{G})$-expansion method for the mKdV equation with variable coefficients. *Phys. Lett. A* **372**, 2254–2257 (2008)

[17] Bian, C. Q., Pang, J., Jin, L. H., and Ying, X. M. Solving two fifth order strong nonlinear evolution equations by using the $(\frac{G}{G'})$-expansion method. *Communications in Nonlinear Science and Numerical Simulation* **15**, 2337–2343 (2010)

[18] Wazwaz, A. M. Compactons, solitons and periodic solutions for variants of the KDV and the KP equations. *Appl. Math. Comput.* **161**(2), 561–575 (2005)

[19] Schiesser, W. E. Method of lines solution of the Korteweg-de Vries equation. *Comput. Math. Appl.* **28**, 147–154 (1994)

[20] Yomba, E. The extended Fan's sub-equation method and its application to KdV-MKdV, BKK and variant Boussinesq equations. *Phys. Lett. A* **336**, 463–476 (2005)

[21] Wazwaz, A. M. Distinct variants of the KdV equation with compact and noncompact structures. *Appl. Math. Comput.* **150**, 365–377 (2004)