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Stabilized Crouzeix-Raviart element for the coupled Stokes and Darcy problem *

Min-fu FENG (冯民富)[1], Rui-sheng QI (祁瑞生)[1],
Rui ZHU (朱瑞)[1], Bing-tao JU (鞠炳焘)[2]

(1. Department of Mathematics, Sichuan University, Chengdu 610064, P. R. China;
2. Department of Information Technology, CNOOC Energy Technology and Services Limited Beijing Branch Company, Beijing 100027, P. R. China)

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Abstract This paper introduces a new stabilized finite element method for the coupled Stokes and Darcy problem based on the nonconforming Crouzeix-Raviart element. Optimal error estimates for the fluid velocity and pressure are derived. A numerical example is presented to verify the theoretical predictions.

Key words Beavers-Joseph-Saffman condition, mass conservation, balance of force, coupled Stokes and Darcy problem

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1 Introduction

Our research in this paper begins with the model of the coupled Stokes and Darcy problem. The model is based on the Stokes equation in the fluid coupled across an interface with the Darcy equation for the filtration velocity in the porous media. It is widely applied in industries. For example, we can use it to simulate the procedure of the pollutants discharged into streams, lakes, and rivers and making their way into the water supply, and to determine whether a dam built over a river is solid from the amount of water filtration through the dam.

Recently, the coupled problem has become a research topic from mathematics and numerical analysis viewpoint[1-6]. However, there still exist several main difficulties in the solution to the coupled problem by using the finite element approximation. First, the coupled problem owns a common restraint with the Stokes equation and the Darcy equation that velocity and pressure spaces must satisfy the inf-sup condition of Babuška[7] and Brezzi[8]. Second, finite element discretizations with choices of space in two regions are different for many methods[5,9-10]. Thus, different spaces in two regions lead to many difficulties in mathematical theory and numerical analysis. On the other hand, in [1, 4], it was argued that finite element discretizations based on the same finite element space for both regions will have some advantages with respect to implementation. However, the construction of the elements[1,4] is rather complicated and

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Corresponding author Min-fu FENG, Professor, Ph. D., E-mail: fmf@wtjs.cn

therefore less attractive for engineering purposes. Third, the technical difficulty is how to treat the interface condition, especially, the mass conservation condition. Hence, it is interesting to use one simple element that could be implemented easily to approximate the coupled problem.

A seemingly promising candidate for such an element is the Crouzeix-Raviart (CR) element, which has the following nice properties: in combination with piecewise constant pressure, it satisfies the inf-sup condition and element-wise conservation of mass. Hence, the CR element is widely used in many problems, such as the Darcy-Stokes problem[11], the Stokes problem[12], and the elasticity problem[13-14]. In [11], in order to ensure convergence in the Darcy limit and fulfill Korn's inequalities for the Darcy-Stokes problem that is not an interface problem, a similar stabilization like [13] is introduced. For the complexity of the coupled Stokes and Darcy problem, when the CR element is used, besides those difficulties in the Darcy-Stokes problem, there are other difficulties in treating the interface condition, especially, in the mass conservation condition. In this paper, we apply the CR element to the coupled Stokes and Darcy problem with an added stabilization, which is also similar to the one in [13]. In order to treat the interface condition well, a new interpolation that is not a CR interpolation on the entire domain is introduced. We give the proof of stability and convergence of our method. Finally, we present a numerical example to show the performance of the method on the coupled problem.

2 Model of problem and notation

Let Ω be a polygonal domain in R^2, which is subdivided into subdomains $\Omega^{1,i}$ and $\Omega^{2,j}$ $(i = 1, 2, \cdots, m_1;\ j = 1, 2, \cdots, m_2)$ (see Fig. 1). The subdomains $\Omega^{1,\bullet}$ and $\Omega^{2,\bullet}$ are assumed to be bounded connected polygonal domains such that

$$\Omega^{1,i} \bigcap \Omega^{1,j} = \emptyset, \quad \Omega^{2,i} \bigcap \Omega^{2,j} = \emptyset \quad \text{for} \quad i \neq j,$$

and

$$\Omega^{1,\bullet} \bigcap \Omega^{2,\bullet} = \emptyset, \quad \Omega_1 = \bigcup_{i=1}^{m_1} \Omega^{1,i}, \quad \Omega_2 = \bigcup_{i=1}^{m_2} \Omega^{2,i}.$$

Denote by Γ_{ij} the interface between two subdomains $\Omega^{1,i}$ and $\Omega^{2,j}$, and let $\Gamma = \cup \Gamma_{ij}$ and $\Gamma_i = \partial\Omega_i / \Gamma$. Denote by $u = (u_1, u_2)$ the fluid velocity and by $p = (p_1, p_2)$ the fluid pressure, where $u_i = u|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$.

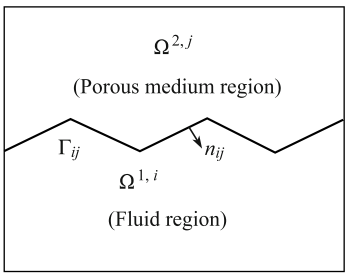


Fig. 1 The diagram of the domain

We assume that the flow in domain Ω_1 is the Stokes equation. Therefore, the following equations are satisfied:

$$\begin{cases} -2\mu \nabla \cdot \epsilon(u_1) + \nabla p_1 = f_1 \quad \text{in} \ \ \Omega_1, \\ \nabla \cdot u_1 = g_1 \quad \text{in} \ \ \Omega_1, \\ u_1 = 0 \quad \text{on} \ \ \Gamma_1, \end{cases} \tag{1}$$

where $\epsilon(u_1)$ is the strain tensor defined by $\epsilon(u_1) = \frac{1}{2}(\nabla u_1 + \nabla u_1')$.

In the region Ω_2, the flow pressure and velocity satisfy the the single phase Darcy flow equations:

$$\begin{cases} \mu u_2 + k\nabla p_2 = f_2 \quad \text{in } \ \Omega_2, \\ \nabla \cdot u_2 = g_2 \quad \text{in } \ \Omega_2, \\ u_2 \cdot n = 0 \quad \text{on } \ \Gamma_2, \end{cases} \tag{2}$$

where k is the symmetric, positive definite tensor bounded below and above uniformly, f_i represent the body forces, $\mu > 0$ denotes the viscosity of the fluid, and n is the outward unit normal to Γ_2.

As the pressure is unique up to an additive constant, we assume

$$\int_\Omega p dx = 0. \tag{3}$$

Let n and τ be the unit normal and tangential vectors to Γ outward of Ω_1, respectively. The conditions at the interface Γ are

$$u_1 \cdot n = u_2 \cdot n \quad \text{on } \ \Gamma, \tag{4}$$

$$2\mu n \cdot \epsilon(u_1) \cdot n = p_1 - p_2 \quad \text{on } \ \Gamma, \tag{5}$$

$$2\mu n \cdot \epsilon(u_1) \cdot \tau = k^{-\frac{1}{2}} \alpha u_1 \cdot \tau \quad \text{on } \ \Gamma. \tag{6}$$

Here, (4) represents the mass conservation, (5) represents the balance force, and the condition Beaver-Joseph-Saffman law (6) is the most accepted condition[15-17] and includes a friction constant $k > 0$ that can be determined experimentally.

We introduce the Hilbert spaces

$$W := \{v \in H_0(\text{div}, \Omega) : \ v|_{\Omega_1} \in [H^1(\Omega_1)]^2, v|_{\Gamma_1} = 0\},$$

$$V := \{v \in H_0(\text{div}, \Omega) : \ v|_{\Omega_i} \in [H^1(\Omega_i)]^2, v|_{\Gamma_1} = 0\},$$

and

$$Q := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \ \int_\Omega q dx = 0\right\}.$$

For $i = 1, 2$, let ξ_h^i be a non-degenerate quasi-uniform triangulation of Ω_i, Γ_h^i be the union of the boundaries (except some boundaries belonging to Γ) of all elements in ξ_h^i, and h_i denote the maximum diameter of elements in ξ_h^i. We assume the meshes at interfaces in the sense: any edge $e \in \partial K^1 \cap \Gamma$, where K^1 $(\in \xi_h^1)$ belongs to one and only one element K^2 $(\in \xi_h^2)$. We introduce the nonconforming Crouzeix-Raviart finite element space:

$$V_h := \left\{v : \ v|_K \in [P_1(K)]^2, \ \forall K \in \xi_h^1 \cap \xi_h^2; \ \int_e [v] ds = 0, \ e \in (\Gamma_h^1 \cup \Gamma_h^2)/\partial\Omega_2; \right.$$
$$\left. \int_e [v \cdot n] ds = 0, \ e \in \partial\Omega_2 \right\},$$

where the jumps $[v]$ and $[v \cdot n]$ on a face e are defined by

$$[v](x) := \begin{cases} \lim\limits_{s \to 0+} (v(x + sn) - v(x - sn)), & \text{if } \ e \in (\Gamma_h^1 \cup \Gamma_h^2)/\partial\Omega, \\ \lim\limits_{s \to 0+} -v(x - sn), & \text{if } \ e \in \partial\Omega, \end{cases}$$

and

$$[v\cdot n](x) := \begin{cases} \lim\limits_{s\to 0+}(v(x+sn)-v(x-sn))\cdot n, & \text{if } e\in\Gamma_h^2\cup\Gamma, \\ \lim\limits_{s\to 0+} -v(x-sn)\cdot n, & \text{if } e\in\partial\Omega_2/\Gamma, \end{cases}$$

where n is a normal unit vector on e and $x\in e$. If $e\in\partial\Omega$, we choose the orientation of n to be outward with respect to Ω; otherwise n has an arbitrary but fixed orientation. Further, we introduce the following spaces:

$$Q_h := \left\{q : q|_K \in P_0(K),\ \int_\Omega q dx = 0\right\},$$
$$W_h^i := \left\{v : v|_K \in P_1(K)^2,\ \forall K\in\xi_h^i,\ \int_e [v]ds = 0,\ e\in\Gamma_h^i/\partial\Omega_i\right\},\ i=1,2,$$

and

$$X^i = X|_{\Omega_i},\ i=1,2,$$

where X may be V_h or V.

Lemma 2.1 *Assume Γ_{ij} $(i=1,2,\cdots,m_1;\ j=1,2,\cdots,m_2)$ are polygonal interfaces of Ω_1 and Ω_2. Then, there exists an interpolation operator R_h from V to V_h.*

Proof Let r_h^i denote the Crouzeix-Raviart interpolation

$$r_h^i : [H^1(\Omega_i)]^2 \to W_h^i, \quad i=1,2,$$

and let

$$R_h|_{V^i} = r_h^i|_{V^i}, \quad i=1,2.$$

Obviously, R_h is an interpolation from V to V_h only if

$$\int_e [R_h v\cdot n]ds = 0, \quad \forall v\in W, \quad \forall e\in\Gamma.$$

In fact,

$$\int_e [R_h v\cdot n]ds = \int_e [R_h v]ds\cdot n = \int_e [v]ds\cdot n = \int_e [v\cdot n]ds = 0, \quad \forall e\in\Gamma,$$

where n is a normal vector on e.

Remark 2.1 Though $R_h^i|_{\Omega_i}$ is the Crouzeix-Raviart interpolation in the domain Ω_i, R_h is not the CR one in the entire domain from the definition of V_h.

Letting the discretely divergence-free space Z_h as

$$Z_h := \{v_h\in W_h :\ (\nabla\cdot v_h, q_h) = 0,\ \forall q_h\in Q_h\},$$

we shall show that the spaces V_h and Q_h satisfy the inf-sup condition for the coupled problem, and hence Z_h is nonempty.

3 Finite element formulation

In order to formulate our finite element method, we first introduce the weak formulations of (1)–(6).

Let

$$a(u,v) = 2\mu\int_{\Omega_1}\epsilon(u_1):\epsilon(v_1)dx + \mu\int_{\Omega_2} k^{-1}u_2\cdot v_2 dx + \mu\alpha\int_\Gamma k^{-\frac{1}{2}}(u_1\cdot\tau)(v_1\cdot\tau)ds,$$

where $\epsilon : \epsilon = \sum_{ij} \epsilon_{ij}\epsilon_{ij}$.

Consider the bilinear form

$$B[(u,p),(v,q)] = a(u,v) - (p, \nabla \cdot v)_\Omega + (q, \nabla \cdot u). \tag{7}$$

Then, the weak formulations (1)–(6) take the form: find $(u,p) \in W \otimes Q$ such that

$$B[(u,p),(v,q)] = (f,v) + (g,q), \quad \forall\, (v,q) \in W \otimes Q, \tag{8}$$

where $f|_{\Omega_i} = f_i$ and $g|_{\Omega_i} = g_i$. Note that all the free interface conditions (4)–(6) are expressed weakly.

We introduce the following bilinear form on which we will base our finite element method:

$$B_h[(u,p),(v,q)] = a_h(u,v) + b(v,p) - b(u,q) + j(u,v), \tag{9}$$

where

$$a_h(u,v) = \sum_{K\in\xi_h^1} \int_K \mu\epsilon(u_1) : \epsilon(v_1)dx + \int_{\Omega_2} \mu k^{-1} u_2 \cdot v_2 dx + \int_\Gamma \mu k^{-\frac{1}{2}} \alpha(u_1\cdot\tau)\cdot(v_1\cdot\tau)ds,$$

$$b(v,p) = -(\nabla\cdot v, p)_h = -\sum_{K\in\xi_h^1\cup\xi_h^2} \int_K p\nabla\cdot v dx,$$

and

$$j(u,v) = \sum_{e\in\Gamma_h^1} \gamma_{\rm s} \int_e \frac{\mu}{h_e}[u][v]ds + \sum_{e\in\Gamma_h^2\cup\Gamma} \gamma_{\rm d} \int_e \frac{\mu}{h_e}[u\cdot n][v\cdot n]ds,$$

where $\gamma_{\rm s} > 0$ and $\gamma_{\rm d} > 0$ are stabilization parameters and $h_e = |e|$.

We propose the following finite element formulation: find $(u_h, p_h) \in V_h \otimes Q_h$ such that

$$B_h[(u_h,p_h),(v_h,q_h)] = (f,v_h) + (g,q_h), \quad \forall (v_h,q_h) \in V_h \otimes Q_h. \tag{10}$$

Remark 3.1 Unlike some stabilized terms[18-20] for the Stokes problems, the term $j(u,v)$ is used to control the rigid rotation which causes a lack of coercivity for the Crouzeix-Rarvriant approximation rather than overcome the inf-sup condition restraint.

Lemma 3.1 *For* $\forall v \in Z_h$, *we have*

$$\nabla \cdot v|_K = 0, \quad \forall K \in \xi_h^1 \cup \xi_h^2.$$

Proof For $v \in Z_h$, we get

$$\sum_{K\in\xi_h^1\cup\xi_h^2} \int_K \nabla\cdot v dx = \sum_{e\in\Gamma_h^1\cup\Gamma_h^2\cup\Gamma} \int [v\cdot n]ds = 0.$$

Thus, $\nabla \cdot v \in Q_h$. Taking $q = \nabla \cdot v$ yields

$$0 = \sum_{K\in\xi_h^1\cup\xi_h^2} \int_K |\nabla\cdot v|^2 dx = \sum_{K\in\xi_h^1\cup\xi_h^2} \mathrm{meas}(K)|\nabla\cdot v|_K^2.$$

The lemma follows.

4 Stability

In this section, we show the inf-sup stability with the suggested jump term.

We define the norm

$$\|(u,p)\|_B^2 := a_h(u,u) + j(u,u) + \mu\|\nabla\cdot u_2\|_{\Omega_2}^2 + \frac{1}{\mu}\|p\|_0^2.$$

Theorem 4.1 *For $\|(v_h,q_h)\|_B \neq 0$, there holds*

$$\sup_{(v_h,q_h)\in V_h\times Q_h} \frac{B_h[(u_h,p_h),(v_h,q_h)]}{\|(v_h,q_h)\|_B} \geq \beta\|(u_h,p_h)\|_B, \tag{11}$$

where β is a constant independent of the meshsize.

Proof We will prove this inf-sup condition in two steps. First, we prove that there exists $(w_h,q_h)\in V_h\otimes Q_h$ such that

$$c\|(u_h,p_h)\|_B^2 \leq B_h[(u_h,p_h),(w_h,q_h)]. \tag{12}$$

Then, we prove

$$\|(w_h,q_h)\|_B \geq c\|(u_h,p_h)\|_B. \tag{13}$$

Step 1 We have

$$B_h[(u_h,p_h),(u_h,p_h)] = a_h(u_h,u_h).$$

For any $p_h\in Q_h\subset L_0^2(\Omega)$, there exists $v_p\in[H_0^1(\Omega)]^2$ such that

$$-\nabla\cdot v_p = p_h, \tag{14}$$

$$\|v_p\|_1 \leq c\|p_h\|_0. \tag{15}$$

Since

$$(\nabla\cdot v_p,1)_h = (\nabla\cdot R_hv_p,1)_h,$$

we find

$$B_h\left[(u_h,p_h),\left(-\frac{R_hv_p}{\mu},0\right)\right] = -\frac{1}{\mu}a_h(u_h,R_hv_p) - \frac{1}{\mu}j(u_h,R_hv_p) + \frac{1}{\mu}\|p_h\|_0^2.$$

Further, we have

$$-\frac{1}{\mu}j(u_h,R_hv_p) \leq \frac{\varepsilon_1}{2}j(u_h,u_h) + \frac{1}{2\varepsilon_1\mu^2}j(R_hv_p,R_hv_p), \tag{16}$$

and by use of the trace inequality

$$\|v\|_{L_2(\partial K)}^2 \leq c\left(h_K^{-1}\|v\|_{L^2(K)}^2 + h_K\|v\|_{H^1(K)}^2\right), \quad \forall\, v\in H^1(K) \tag{17}$$

(see [21]), we find

$$\begin{aligned}\frac{1}{\mu^2}j(R_hv_p,R_hv_p) &= \frac{1}{\mu^2}j(R_hv_p-v_p,R_hv_p-v_p)\\ &\leq \frac{c}{\mu}\sum_{K\in\xi_h^1\cup\xi_h^2}\left(\frac{1}{h_K^2}\|R_hv_p-v_p\|_{L_2(K)}^2 + \|R_hv_p-v_p\|_{H^1(K)}^2\right)\\ &\leq \frac{c}{\mu}\|v_p\|_1^2 \leq \frac{C_p}{\mu}\|p_h\|_0^2.\end{aligned}$$

We also have

$$-\frac{1}{\mu}a_h(u_h, R_h v_p) \le \frac{\varepsilon_2}{2}a_h(u_h, u_h)+\frac{1}{2\varepsilon_2\mu^2}a_h(R_h v_p, R_h v_p)$$
$$\le \frac{\varepsilon_2}{2}a_h(u_h, u_h)+\frac{C_\mu}{2\varepsilon_2\mu}\|p_h\|_0^2. \tag{18}$$

Consequently, we have

$$\gamma B_h[(u_h, p_h), (u_h, p_h)] + B_h\left[(u_h, p_h), \left(-\frac{1}{\mu}R_h v_p, 0\right)\right]$$
$$\ge \left(\gamma - \frac{\varepsilon_2}{2}\right) a_h(u_h, u_h) + \left(\gamma - \frac{\varepsilon_1}{2}\right) j(u_h, u_h) + \frac{1}{\mu}\left(1 - \frac{C_p}{2\varepsilon_1} - \frac{C_\mu}{2\varepsilon_2}\right)\|p_h\|_0^2.$$

If we choose $\varepsilon_1 = 2C_p$, $\varepsilon_2 = 2C_\mu$, and $\gamma = \sum \varepsilon_j$, (12) follows with $w_h = \gamma u_h - \frac{1}{\mu}R_h v_p$ and $q_h = \gamma p_h + \mu\nabla\cdot u_2^h$.

Step 2 Using the same arguments once more, we immediately find

$$\|(w_h, q_h)\|_B = \left\|\left(\gamma u_h - \frac{1}{\mu}R_h v_p, \gamma p_h + \mu\nabla\cdot u_h\right)\right\|_B$$
$$\le \|\gamma(u_h, p_h)\|_B + \left\|\left(\frac{1}{\mu}R_h v_p, \mu\nabla\cdot u_h\right)\right\|_B$$
$$\le c\|(u_h, p_h)\|_B.$$

The proof is complete.

5 Convergence

In fact, the problem (10) is equivalent to the problem: find $(u_h, p_h) \in V_h \otimes Q_h$ such that

$$a_h(u_h, v_h) + j(u_h, v_h) - b(v_h, p_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{19}$$
$$b(u_h, q_h) = (g, q_h), \quad \forall q_h \in Q_h. \tag{20}$$

On Z_h, using the notation $\langle u, v\rangle_e$ for the $L_2(e)$ scalar product, we have

$$a_h(u, v_h) + j(u, v_h)$$
$$= \sum_{K\in\xi_h^1}\int_K 2\mu\epsilon(u_1) : \epsilon(v_1^h)dx + \mu\int_{\Omega_2} k^{-1}\mu u_2\cdot v_2^h ds + \int_\Gamma \mu k^{-\frac{1}{2}}\alpha(u_1\cdot\tau)\cdot(v_1^h\cdot\tau)ds$$
$$= (f, v_1^h)_{\Omega_1} + \sum_{e\in\Gamma_h^1}\langle 2\mu\epsilon(u_1)n - p_1 n,\ [v_h]\rangle_e - \sum_{e\in\Gamma_h^2\cup\Gamma}\langle p_2,\ [v_h\cdot n]\rangle_e.$$

Thus, the following Galerkin orthogonality relation holds:

$$a_h(u - u_h, v_h) + j(u - u_h, v_h) = \sum_{e\in\Gamma_h^1}\langle 2\mu\epsilon(u_1)n - p_1 n,\ [v_h]\rangle_e - \sum_{e\in\Gamma_h^2\cup\Gamma}\langle p_2,\ [v_h\cdot n]\rangle_e. \tag{21}$$

We next define the norm

$$|||u||| := (a_h(u_h, u_h) + j(u_h, u_h))^{\frac{1}{2}}. \tag{22}$$

Now we will estimate the error $|||(u - u_h)|||$. Set $e_h := R_h u - u_h$. From (20), we get

$$0 = b(u - R_h u, q_h) = b(u_h - R_h u, q_h), \quad \forall q_h \in Q_h.$$

Hence,

$$e_h \in Z_h, \quad \nabla \cdot e_h|_K = 0, \quad \forall K \in \xi_h^1 \cup \xi_h^2.$$

Assuming

$$u|_{\Omega_i} \in [H^2(\Omega_i)]^2, \quad p|_{\Omega_i} \in H^1(\Omega_i) \quad \text{for} \quad i = 1, 2,$$

we have the following two results.

Theorem 5.1 *There holds*

$$\begin{aligned} |||u - u_h||| \le c \Big(&\mu^{\frac{1}{2}} h_1 \|u_1\|_{[H^2(\Omega_1)]^2} + h_2 \|u_2\|_{[H^2(\Omega_2)]^2} \\ &+ \mu^{-\frac{1}{2}} h_1 \|p_1\|_{H^1(\Omega_1)} + \mu^{-\frac{1}{2}} h_2 \|p_2\|_{H^1(\Omega_2)} \Big). \end{aligned} \tag{23}$$

Proof Define π_1 as the projection onto piecewise constants. By Korn's inequality for piecewise H^1 vector fields,

$$\sum_{K \in \xi_1^h} \|v_h^1\|_{1,K}^2 \le c \Big(\sum_{K \in \xi_h^1} \|\epsilon(u_1^h)\|_{0,K}^2 + j(v_h, v_h) \Big),$$

(see [22]), and the Galerkin orthogonality (21), we have

$$\begin{aligned} c|||e_h|||^2 &\le a_h(e_h, e_h) + j(e_h, e_h) \le |a_h(R_h u - u, e_h)| + |j(R_h u - u, e_h)| \\ &\quad + \Big| \sum_{e \in \Gamma_h^1} \langle 2\mu\epsilon(u_1) \cdot n - p_1 \cdot n, \ [e_h] \rangle_e \Big| + \Big| \sum_{e \in \Gamma_h^2 \cup \Gamma} \langle p_2, \ [e_h \cdot n] \rangle_e \Big| \\ &\le |||u - R_h u||| \ |||e_h||| + \Big| \sum_{e \in \Gamma_h^1} \langle 2\mu(\epsilon(u_1) - \pi_1 \epsilon(u)) \cdot n, \ [e_h] \rangle_e \Big| \\ &\quad + \Big| \sum_{e \in \Gamma_h^1} \langle (p_1 - \pi_1 p_1) \cdot n, \ [e_h] \rangle_e \Big| + \Big| \sum_{e \in \Gamma_h^2 \cup \Gamma} \langle p_2 - \pi_1 p_2, \ [e_h \cdot n] \rangle_e \Big| \\ &\le |||u - R_h u||| \ |||e_h||| + \sum_{e \in \Gamma_h^1} \|2\mu^{\frac{1}{2}} h_e^{\frac{1}{2}} (\epsilon(u_1) - \pi_1 \epsilon(u_1)) \cdot n\|_{L_2(e)} |||e_h||| \\ &\quad + \sum_{e \in \Gamma_h^1} \|\mu^{-\frac{1}{2}} h_e^{\frac{1}{2}} (p_1 - \pi_1 p_1)\|_{L_2(e)} |||e_h||| + \sum_{e \in \Gamma_h^2 \cup \Gamma} \|\mu^{-\frac{1}{2}} h_e^{\frac{1}{2}} (p_2 - \pi_1 p_2)\|_{L_2(e)} |||e_h|||. \end{aligned} \tag{24}$$

By use of the trace of inequalities (17) and (24), we find

$$\sum_{e \in \Gamma_h^1} \Big\| h_e^{\frac{1}{2}} (\epsilon(u_1) - \pi_1 \epsilon(u_1)) \cdot n \Big\|_{L_2(e)}^2 \le c \sum_{K \in \xi_h^1} h_K^2 \|u_1\|_{[H^2(K)]^2}^2$$

and

$$\sum_e \Big\| h_e^{\frac{1}{2}} (p - \pi_1 p) \Big\|_{L_2(e)}^2 \le c \sum_K h_K^2 \|p\|_{H^1(K)}^2,$$

which yields

$$|||e_h||| \le c \Big(|||u - R_h u||| + \mu^{\frac{1}{2}} h_1 \|u_1\|_{[H^2(\Omega_1)]^2} + \mu^{-\frac{1}{2}} h_1 \|p_1\|_{H^1(\Omega_1)} + \mu^{-\frac{1}{2}} h_2 \|p_2\|_{H^1(\Omega_2)} \Big).$$

Finally, (23) follows from the trace inequality and the interpolation theory[12] for R_h and triangle inequality.

Theorem 5.2 *There holds*

$$\|p - p_h\| \leq c(\mu h_1 \|u_1\|_{[H^2(\Omega_1)]^2} + \mu h_2 \|u_2\|_{[H^2(\Omega_2)]^2} + h_1 \|p_1\|_{H^1(\Omega_1)} + h_2 \|p_2\|_{H^1(\Omega_2)}).$$

Proof We split the error into

$$\|p - p_h\|_0 \leq \|p - \pi_2 p\|_0 + \|\pi_2 p - p_h\|_0,$$

where $\pi_2 p$ is the L^2-projection of p onto Q_h. For the first part, we have the standard estimate

$$\|p - \pi_2 p\|_{0,\Omega_i} \leq c h_i \|p\|_{H^1(\Omega_i)}.$$

For the second part, we proceed as follows. By the subjectivity of the divergence operator (see [23]), there exists $v_p \in [H_0^1(\Omega)]^2$ such that

$$\nabla \cdot v_p = \pi_2 p - p_h, \quad \|v_p\|_1 \leq c\|\pi_2 p - p_h\|_0.$$

Using the orthogonality of the L^2-projection, we obtain

$$\begin{aligned}\|\pi_2 p - p_h\|_0^2 &= (\pi_2 p - p_h, \nabla \cdot v_p)_h = (\pi_2 p - p_h, \nabla \cdot R_h v_p)_h \\ &= (p - p_h, \nabla \cdot R_h v_p)_h = a_h(u - u_h, R_h v_p) + j(u - u_h, R_h v_p) \\ &\quad - \sum_{e \in \Gamma_h^1} \langle 2\mu\epsilon(u_1) \cdot n - p_1 \cdot n,\ [R_h v_p] \rangle_e + \sum_{e \in \Gamma_h^2 \cup \Gamma} \langle p_2,\ [R_h v_p \cdot n] \rangle_e.\end{aligned}$$

Using the trace inequality, we find

$$\begin{aligned}& a_h(u - u_h, R_h v_p) + j(u - u_h, R_h v_p) - \sum_{e \in \Gamma_h^1} \langle 2\mu\epsilon(u_1) \cdot n - p_1 \cdot n,\ [R_h v_p] \rangle_e \\ & + \sum_{e \in \Gamma_h^2 \cup \Gamma} \langle p_2,\ [R_h v_p \cdot n] \rangle_e \\ \leq\ & c\Bigg(|||u - R_h u|||^2 + \sum_{e \in \Gamma_h^1} \mu \|\epsilon(u_1) - \pi_1 \epsilon(u_1)\|_{L_2(e)}^2 + \sum_{e \in \Gamma_h^1} \mu^{-1} \|(p_1 - \pi_1 p_1) n\|_{L_2(e)}^2 \\ & + \sum_{e \in \Gamma_h^2 \cup \Gamma} \mu^{-1} \|(p_2 - \pi_1 p_2)\|_{L_2(e)}^2 \Bigg)^{\frac{1}{2}} |||R_h v_p||| \\ \leq\ & c\Bigg(\mu^{\frac{1}{2}} |||u - R_h u||| + \sum_{e \in \Gamma_h^1} \mu \|\epsilon(u_1) - \pi_1 \epsilon(u_1)\|_{L_2(e)} \\ & + \sum_{e \in \Gamma_h^1 \cup \Gamma_h^2 \cup \Gamma} \|p - \pi_1 p\|_{L_2(e)} \Bigg) \|\pi_2 p - p_h\|_0.\end{aligned}$$

Divide both sides by $|||\pi_0 p - p_h|||$ and conclude the proof.

6 Numerical experiment

For simplicity, we choose $\mu = \alpha = 1$ and take the permeability tensor K as the identity. In the following numerical example, we choose the stabilization parameters $r_{\mathrm{s}} = 3$ and $r_{\mathrm{d}} = 1$.

We consider an artificial example. In domains $\Omega = [0\ 1] \times [0\ 1]$, the flow is governed by the Stokes equation on $\Omega_1 = [0\ \frac{1}{2}] \times [0\ 1]$ with the given pressure and velocity fields

$$p = -\sin(\pi x) + \frac{1}{\pi}, \quad u = \left(\pi \sin^2(\pi x)\sin(2\pi y), -\pi\sin(2\pi x)\sin^2(\pi y)\right),$$

and by the Darcy equation on $\Omega_2 = [\frac{1}{2}\ 1] \times [0\ 1]$ with the ith exact solution given by

$$p = -\sin(\pi x) + \frac{1}{\pi}, \quad u = \left(\pi \sin^2(\pi x)\sin(2\pi y), \pi\sin(\pi y)\right).$$

The example does not satisfy the Beavers-Joseph-Samman condition, but it is reasonable with the form

$$2\mu n\epsilon(u_1)\tau = u_1\tau + g \quad \text{on} \quad \Gamma.$$

The used computational mesh is shown in Fig. 2, and the computational results are shown in Table 1 and Figs. 2 and 3. Table 1 shows that the convergence of our method in L^2-norm is of the second-order accuracy for the velocity and of the first-order accuracy for the pressure, and that the convergence accuracy of the Galerkin method approximates to zero with the decrease of the scale of the mesh. Figures 3 and 4 show the velocity and pressure distributions with and without the normal jump stabilization.

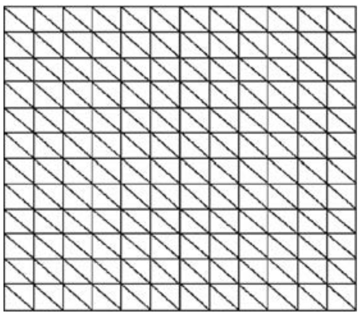
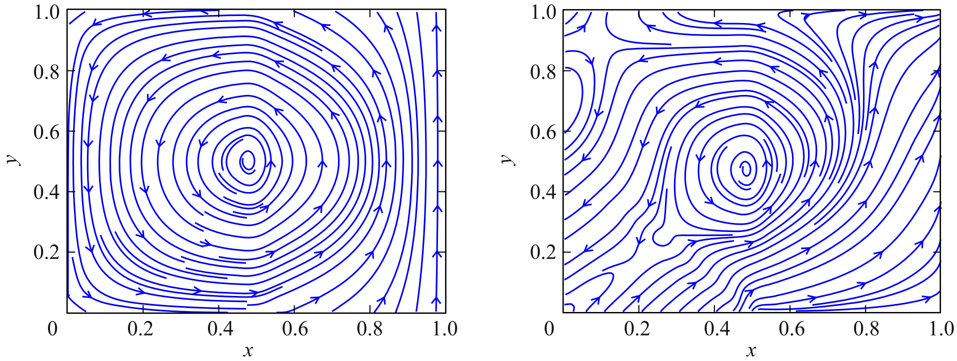


Fig. 2 The computational mesh

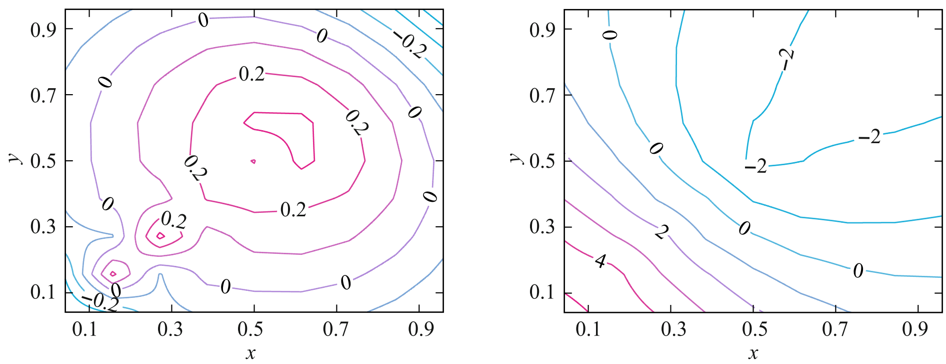
Table 1 L^2-norm convergence of the velocity and pressure for the coupled problem

	h	$\|u-u_h\|_0$	Rate	$\|p-p_2\|_0$	Rate
Stabilized method	$\frac{1}{4}$	0.448 4	–	1.108 5	–
	$\frac{1}{8}$	0.130 5	1.780 7	0.456 5	1.279 9
	$\frac{1}{16}$	0.037 1	1.814 6	0.186 4	1.292 2
	$\frac{1}{32}$	0.009 8	1.920 6	0.085 6	1.122 7
Galerkin method	$\frac{1}{4}$	1.817 8	–	6.006 2	–
	$\frac{1}{8}$	1.773 4	0.035 7	4.174 6	0.660 9
	$\frac{1}{16}$	1.780 5	−0.005 8	3.718 8	0.166 8
	$\frac{1}{32}$	1.783 0	−0.002 0	3.601 7	0.046 2



(a) With normal jump stabilization (b) Without normal jump stabilization

Fig. 3 Approximate solutions of velocity with and without the normal jump stabilization



(a) With normal jump stabilization (b) Without normal jump stabilization

Fig. 4 Approximate solutions of pressure with and without the normal jump stabilization

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