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Numerical method of Rayleigh-Stokes problem for heated generalized second grade fluid with fractional derivative *

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Abstract In this paper, we consider the Rayleigh-Stokes problem for a heated generalized second grade fluid (RSP-HGSGF) with fractional derivative. An effective numerical method for approximating RSP-HGSGF in a bounded domain is presented. The stability and convergence of the method are analyzed. Numerical examples are presented to show the application of the present technique.

Key words Rayleigh-Stokes problem, numerical method, stability, convergence

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Introduction

The mechanics of non-Newtonian fluids present a special challenge to engineers, physicists and mathematicians, and the motion of these fluids plays an important role not only in theory but also in many industrial applications. In the last two decades, the interest for non-Newtonian fluids have been drawing attention from researchers because of their practical importance. Specifically, these occur in the extrusion of polymer fluids, cooling of a metallic plate in a bath, exotic lubricants, artificial and natural gels, and colloidal and suspension solutions[1-4].

The governing differential equation corresponding to the non-Newtonian fluids motions, in the absence of a pressure gradient in the flow direction, as it results from [4-6] is

$$(\nu + \alpha\partial_t)\partial_x^2 u(x,t) = \partial_t u(x,t), \quad x, t > 0, \tag{1}$$

where $u(x,t)$ is the velocity, $\nu = \mu/\rho$ (μ is the coeffient of the viscosity, and ρ is the constant density of the fluid) is the kinematic viscosity of the fluid, and $\alpha = \alpha_1/\rho$ (α_1 is the normal stress modulus).

Zierep and Fetecau[7] discussed the energetic balance in the Rayleigh-Stokes problem for a Maxwell fluid for several initial and/or boundary conditions. Fetecau and Zierep[5] found the exact solutions both for the Stokes' problem and for the Rayleigh-Stokes problem within the context of the fluids of second grade, and the adequate Navier-Stokes solutions appeared as a limiting case of their solutions.

Recently, fractional calculus has encountered much success in the description of constitutive relations of viscoelastic fluids. The starting point of the fractional derivative model of a viscoelastic fluid is usually a classical differential equation which is modified by replacing the

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time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operator. This generalization allows one to define precisely non-integer order integrals or derivatives. Shen et al.[8] obtained the Rayleigh-Stokes problem for a generalized second grade fluid subject to a flow on a heated flat plate and within a heated edge. They obtained exact solutions of the velocity and temperature fields by using the Fourier sine transform and the fractional Laplace transform. Xue and Nie[9] extended the Rayleigh-Stokes problem to that of a heated generalized second grade fluid with fractional derivative in a porous half-space.

In this paper, we consider the accuracy and stability of an implicit numerical approximation scheme (INAS) for the Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative[8] with a forcing term (RSP-HGSGF):

$$\frac{\partial u(x,z,t)}{\partial t} = (\nu + \alpha D_t^{1-\beta})\Delta u(x,z,t) + f(x,z,t), \quad (x,z) \in \Omega, \quad 0 < t < T \tag{2}$$

with the boundary condition

$$u(x,z,t) = \Psi(x,z,t), \quad (x,z) \in \partial\Omega, \tag{3}$$

and the initial condition

$$u(x,z,0) = \omega(x,z), \quad (x,z) \in \Omega, \tag{4}$$

where Δ is the Laplace operator, $\Omega = \{(x,z) \,|\, 0 \le x \le a_x,\ 0 \le z \le a_z\}$, $\partial\Omega$ is the boundary of the domain Ω, the constants $v, \alpha > 0$, and the symbol $D_t^{1-\beta}$ $(0 < \beta < 1)$ means the Riemman-Liouville fractional derivative of order $1-\beta$ defined by

$$D_t^{1-\beta} u(x,z,t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,z,\eta)}{(t-\eta)^{1-\beta}} d\eta.$$

Here, $\Gamma(\cdot)$ is the gamma function.

Fractional differential equations have been discussed in viarous fields. However, analytic solutions to most fractional differential equations are not usually expressed explicitly, and many authors resort to numerical methods. Liu et al.[10-11] considered the space fractional partial differential equation and transformed the equation into a system of ordinary differential equations that were then solved using backward differentiation formulas. Shen and Liu[12] proposed an explicit finite difference approximation for the space fractional diffusion equation and gave an error analysis. Roop[13] investigated the computational aspects for the Galerkin approximation using continuous piecewise polynomial basis functions on a regular triangulation of a bounded domain in R^2. Chen et al.[14] proposed a Fourier method for the fractional diffusion equation discribing the sub-diffusion and the Galilei invariant fractional advection diffusion equation. Chen et al.[15] and Wu[16] discussed Stokes' first problem for a heated generalized second grade fluid with fractional derivative using two different methods, respectively, which was the limiting case of Rayleigh-Stokes problem proposed in [8]. They proposed implicit and explicit numerical approximation schemes to solve it. Then the stability and the convergence were discussed. The main purpose of this paper is to solve the Rayleigh-Stokes problem (2) by introducing an effective numerical method.

The rest of this paper is organized as follows. In Section 2, an effective numerical method is proposed. In Sections 3 and 4, the stability and convergence analyses are discussed. Finally, in Section 5, some numerical results are given to testify our method and theoretical analyses.

1 Mathematical preliminaries

In this section, we introduce some definitions and mathematical notations that will be used in later sections and state their corresponding properties.

Firstly, we give the definition of the temporal fractional integral.

Definition 1.1 *Let $y(t) \in L^1(a,b)$. The integral*

$$ {}_aI_t^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{y(\eta)}{(t-\eta)^{1-\beta}} d\eta, \quad t > a, \tag{5}$$

where $\beta > 0$ is called the Riemann-Liouville fractional integral of order β[17].

In this paper, we refer to $t \in [0,T]$ and $0 < \beta < 1$. In order to compute ${}_0I_t^\beta y(t)$, we begin to discretize the temporal domain $[0,T]$ by placing a grid over the domain. For convenience, we use a uniform grid, with grid spacing $\tau = T/n$. If we wish to refer to one of the points in the grid, we call the points $t_k = k\tau,\ k = 0, 1, \cdots, n$. So, for $k = 1, 2, \cdots, n$,

$$ {}_0I_t^\beta y(t_k) = \frac{1}{\Gamma(\beta)} \int_0^{t_k} \frac{y(\eta)}{(t_k-\eta)^{1-\beta}} d\eta = \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k-1} \int_{t_{k-1-j}}^{t_{k-j}} \frac{y(\eta)}{(t_k-\eta)^{1-\beta}} d\eta. \tag{6}$$

Hence, we have

$$\begin{aligned} &\left| {}_0I_t^\beta y(t_k) - \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k-1} \int_{t_{k-1-j}}^{t_{k-j}} \frac{y(t_{k-j})}{(t_k-\eta)^{1-\beta}} d\eta \right| \\ &\le \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k-1} \int_{t_{k-1-j}}^{t_{k-j}} \frac{|y(\eta) - y(t_{k-j})|}{(t_k-\eta)^{1-\beta}} d\eta \\ &\le Ck^\beta \tau^{\beta+1}. \end{aligned} \tag{7}$$

Thus, we have

Lemma 1.1 *If $y(t) \in C^1[0,T]$, then*

$$ {}_0I_t^\beta y(t_k) = \frac{\tau^\beta}{\Gamma(\beta+1)} \sum_{j=0}^{k-1} b_j y(t_{k-j}) + R_{k,\beta}, \tag{8}$$

where

$$ b_j = (j+1)^\beta - j^\beta, \quad j = 0, 1, \cdots, n, \tag{9}$$

and $|R_{k,\beta}| \le Ct_k^\beta \tau$.

Lemma 1.2 *In* (9), *the coefficients b_k $(k = 0, 1, 2, \cdots)$ satisfy the following properties*:

(i) $b_0 = 1$, *and* $b_0 > b_1 > \cdots > b_k > \cdots > b_n > 0$.

(ii) *There exists a positive constant $C > 0$ such that $\tau \le Cb_k\tau^\beta$ with $k = 1, 2, 3, \cdots$.*

Proof Let $\psi_1(x) = x^\beta$ and $\psi_2(x) = (x+1)^\beta - x^\beta$. It is easily seen that $\psi_1(x)$ is monotone increasing and $\psi_2(x)$ is monotone decreasing when $x > 0$. Thus, (i) holds.

As for (ii), using

$$ \lim_{n\to\infty} \frac{n^{\beta-1}}{b_n} = \lim_{n\to\infty} \frac{n^{-1}}{(1+n^{-1})^\beta - 1} = \frac{1}{\beta}, $$

then there exists a positive constant C_1 such that $n^{\beta-1}/b_n \le C_1$, i.e.,

$$ n^{-1} \le C_1 b_n n^{-\beta} \le C_1 b_k n^{-\beta}. $$

Thus, from $\tau = T/n$, the inequality (ii) is obtained.

For $k=0,1,\cdots,n-1$, we have

$$
\begin{aligned}
{}_0I_t^\beta y(t_{k+1}) - {}_0I_t^\beta y(t_k) &= \frac{1}{\Gamma(\beta)}\Big[\int_0^{t_{k+1}} \frac{y(\eta)}{(t_{k+1}-\eta)^{1-\beta}}d\eta - \int_0^{t_k} \frac{y(\eta)}{(t_k-\eta)^{1-\beta}}d\eta\Big] \\
&= \frac{1}{\Gamma(\beta)}\Big[\int_0^{\tau} \frac{y(\eta)}{(t_{k+1}-\eta)^{1-\beta}}d\eta + \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \frac{y(\eta+\tau)-y(\eta)}{(t_k-\eta)^{1-\beta}}d\eta\Big].
\end{aligned}
$$

Suppose that $y(t)\in C^2[0,T]$. When $0\le\eta\le\tau$,

$$
|y(\eta)-y(\tau)| = |y'(\xi)(\eta-\tau)| \le C\tau, \quad \eta\le\xi\le\tau.
$$

When $t_j\le\eta\le t_{j+1}$, we can obtain

$$
\begin{aligned}
y(\eta+\tau)-y(\eta) &= y(t_{j+2})-y(t_{j+1}) + (y'(\varsigma_j+\tau)-y'(\varsigma_j))(\eta-t_{j+1}) \\
&= y(t_{j+2})-y(t_{j+1}) + y''(\rho_j)\tau(\eta-t_{j+1}), \qquad (10)
\end{aligned}
$$

where $\eta\le\varsigma_j\le t_{j+1}$ and $\varsigma_j\le\rho_j\le\varsigma_j+\tau$. Hence, $|y(\eta+\tau)-y(\eta)-[y(t_{j+2})-y(t_{j+1})]\,|\le C\tau^2$.

Thanks to Lemma 1.2, we have proved the following result.

Lemma 1.3 *If* $y(t)\in C^2[0,T]$, *then*

$$
{}_0I_t^\beta y(t_{k+1}) - {}_0I_t^\beta y(t_k) = \frac{\tau^\beta}{\Gamma(\beta+1)}\Big[y(t_{k+1}) + \sum_{j=0}^{k-1}(b_{j+1}-b_j)y(t_{k-j})\Big] + R_{k,\beta}^{(2)}, \qquad (11)
$$

where $|R_{k,\beta}^{(2)}|\le Cb_k\tau^{1+\beta}$.

2 An implicit numerical approximation scheme for the RSP-HGSGF

In this section, we construct an effective numerical method for the RSP-HGSGF (2) with the boundary and initial conditions (3) and (4).

Let $\Lambda=[0,a_x]\times[0,a_z]\times[0,T]$. Then define the function space

$$
\Phi(\Lambda) = \left\{u(x,z,t)|\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial z^2}\in C^2(\Lambda),\frac{\partial^5 u}{\partial x^4\partial t},\frac{\partial^5 u}{\partial z^4\partial t}\in C(\Lambda)\right\}.
$$

We suppose the RSP-HGSGF (2)–(4) has a solution $u(x,z,t)\in\Phi(\Lambda)$.

We now discretize space and time by grid points and time instants as follows:

$$
\begin{aligned}
&x_i = ih_x, \quad i=0,1,2,\cdots,m, \quad h_x=a_x/m,\\
&z_j = jh_z, \quad j=0,1,2,\cdots,n, \quad h_z=a_z/n,\\
&t_k = k\tau, \quad k=0,1,2,\cdots,K, \quad \tau=T/K,
\end{aligned}
$$

where h_x, h_z, and τ are the space and time steps, respectively. For convenience, we denote

$$
\begin{aligned}
\delta_x^2 u(x,z,t) &= u(x+h_x,z,t)-2u(x,z,t)+u(x-h_x,z,t),\\
\delta_z^2 u(x,z,t) &= u(x,z+h_z,t)-2u(x,z,t)+u(x,z-h_z,t).
\end{aligned}
$$

Integrating both sides of (2) from t_k to t_{k+1},

$$
\begin{aligned}
u(x_i,z_j,t_{k+1}) = u(x_i,z_j,t_k) &+ \int_{t_k}^{t_{k+1}}(\nu\Delta u(x_i,z_j,\eta)+f(x_i,z_j,\eta))d\eta \\
&+ {}_0I_t^\beta\Delta u(x_i,z_j,t_{k+1}) - {}_0I_t^\beta\Delta u(x_i,z_j,t_k). \qquad (12)
\end{aligned}
$$

Use the following approximation:

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}(\nu\Delta u(x_i,z_j,\eta)+f(x_i,z_j,\eta))d\eta\\&=\nu\tau\Delta u(x_i,z_j,t_{k+1})+\tau f(x_i,z_j,t_{k+1})+R_{11}\\&=\nu\tau(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)u(x_i,z_j,t_{k+1})+R_{12}+\tau f(x_i,z_j,t_{k+1})+R_{11}\\&=\nu\tau(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)u(x_i,z_j,t_{k+1})+\tau f(x_i,z_j,t_{k+1})+R_1,\end{aligned}$$

where

$$\begin{aligned}R_{11}&=\int_{t_k}^{t_{k+1}}\left[\nu(\Delta u(x_i,z_j,\eta)-\Delta u(x_i,z_j,t_{k+1}))+f(x_i,z_j,\eta)-f(x_i,z_j,t_{k+1})\right]d\eta,\\R_{12}&=\nu\tau\left[\Delta u(x_i,z_j,t_{k+1})-(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)u(x_i,z_j,t_{k+1})\right].\end{aligned}$$

Note that

$$\begin{aligned}&\nu\Delta u(x_i,z_j,\eta)+f(x_i,z_j,\eta)\\&=\nu(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2})u(x_i,z_j,t_{k+1})+f(x_i,z_j,t_{k+1})\\&\quad+\left[\nu(\frac{\partial^3}{\partial x^2\partial t}+\frac{\partial^3}{\partial z^2\partial t})u(x_i,z_j,\xi)+\frac{\partial}{\partial t}f(x_i,z_j,\xi)\right](\eta-t_{k+1}),\end{aligned}\tag{13}$$

where $t_k\le\eta\le\xi\le t_{k+1}$.

It is apparent that $|R_{11}|\le C\tau^2$ and $|R_{12}|\le C\tau(h_x^2+h_z^2)$. Hence, we have

$$|R_1|\le C\tau(\tau+h_x^2+h_y^2).$$

From the above results and Lemma 1.3, we have

$$\begin{aligned}u(x_i,z_j,t_{k+1})&=u(x_i,z_j,t_k)+\nu\tau(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)u(x_i,z_j,t_{k+1})\\&\quad+\tau f(x_i,z_j,t_k)+rb_k(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)u(x_i,z_j,\tau)\\&\quad+r\sum_{s=0}^{k-1}b_{k-s-1}[\frac{1}{h_x^2}\delta_x^2(u(x_i,z_j,t_{s+2})-u(x_i,z_j,t_{s+1}))\\&\quad+\frac{1}{h_z^2}\delta_z^2(u(x_i,z_j,t_{s+2})-u(x_i,z_j,t_{s+1}))]+R_{i,j}^{k+1},\end{aligned}$$

where $r=\frac{\alpha\tau^\beta}{\Gamma(\beta+1)}$, and

$$|R_{i,j}^{k+1}|\le C(b_k\tau^\beta+\tau)(\tau+h_x^2+h_z^2).\tag{14}$$

Let $\boldsymbol{R}^k=[R_{1,1}^k,\cdots,R_{1,n-1}^k,R_{2,1}^k,\cdots,R_{2,n-1}^k,\cdots,R_{m-1,1}^k,\cdots,R_{m-1,n-1}^k]^{\mathrm{T}}$. By using Lemma 1.2 and (14), we obtain the following lemma.

Lemma 2.1 *Suppose that* $\|\boldsymbol{R}^k\|=\sqrt{h_xh_z\sum\limits_{i=1}^{m-1}\sum\limits_{j=1}^{n-1}|R_{ij}^k|^2}$. *If* $u(x,z,t)\in\Phi(\Lambda)$ *is the solution of* (2)–(4), *then we have*

$$\|\boldsymbol{R}^k\|\le Cb_k\tau^\beta(\tau+h_x^2+h_z^2).$$

Let $u_{i,j}^k$ be the numerical approximation to $u(x_i, z_j, t_k)$ and $f_{i,j}^k = f(x_i, z_j, t_k)$. Introduce the following notations:

$$\delta_x^2 u_{i,j}^k = u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k, \quad \Delta_x u_{i,j}^k = u_{i+1,j}^k - u_{i,j}^k,$$
$$\delta_z^2 u_{i,j}^k = u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k, \quad \Delta_z u_{i,j}^k = u_{i,j+1}^k - u_{i,j}^k.$$

We obtain the following implicit numerical approximation scheme (INAS):

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^k + \nu\tau(\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2)u_{i,j}^{k+1} + \tau f_{i,j}^{k+1} + rb_k(\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2)u_{i,j}^1 \\ &\quad + r\sum_{s=0}^{k-1} b_{k-s-1}[\frac{1}{h_x^2}\delta_x^2(u_{i,j}^{s+2} - u_{i,j}^{s+1}) + \frac{1}{h_z^2}\delta_z^2(u_{i,j}^{s+2} - u_{i,j}^{s+1})], \end{aligned} \tag{15}$$

which can be rewritten as

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^k + \nu\tau(\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2)u_{i,j}^{k+1} + \tau f_{i,j}^{k+1} + r[\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2]u_{i,j}^{k+1} \\ &\quad + r\sum_{s=0}^{k-1}(b_{s+1} - b_s)(\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2)u_{i,j}^{k-s}, \end{aligned} \tag{16}$$

for $i = 1, 2, \cdots, m-1,\ j = 1, 2, \cdots, n-1$, and $k = 0, 1, 2, \cdots, K-1$.

The boundary and initial conditions are

$$\begin{cases} u_{i,j}^k = \Psi(x_i, z_j, t_k), \quad (x_i, z_j) \in \partial\Omega, \quad 0 \le i \le m, \quad 0 \le j \le n, \quad 1 \le k \le K, \\ u_{i,j}^0 = \omega(ih_x, jh_z), \quad i = 0, 1, \cdots, m, \quad j = 0, 1, \cdots, n. \end{cases} \tag{17}$$

Then, we rewrite (16) and (17) as the following matrix forms:

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}^1 = \boldsymbol{u}^0 + \tau\boldsymbol{f}^1 + r_x\boldsymbol{w}_x^1 + r_z\boldsymbol{w}_z^1, \\ \boldsymbol{A}\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \tau\boldsymbol{f}^{k+1} + r_x\boldsymbol{w}_x^{k+1} + r_z\boldsymbol{w}_z^{k+1} + \boldsymbol{g}^{k+1}, \quad k > 0, \\ \boldsymbol{u}^0 = \boldsymbol{\varUpsilon}, \end{cases} \tag{18}$$

where

$$\boldsymbol{u}^k = [u_{1,1}^k, u_{1,2}^k, \cdots, u_{1,n-1}^k, u_{2,1}^k, u_{2,2}^k, \cdots, u_{2,n-1}^k, \cdots, u_{m-1,1}^k, u_{m-1,2}^k, \cdots, u_{m-1,n-1}^k]^{\mathrm{T}},$$
$$\boldsymbol{w}_x^k = [u_{0,1}^k, u_{0,2}^k, \cdots, u_{0,n-1}^k, 0, 0, \cdots, 0, \cdots, u_{m,1}^k, u_{m,2}^k, \cdots, u_{m,n-1}^k]^{\mathrm{T}},$$
$$\boldsymbol{w}_z^k = [u_{1,0}^k, 0, \cdots, u_{1,n}^k, u_{2,0}^k, 0, \cdots, u_{2,n}^k, \cdots, u_{m-1,0}^k, 0, \cdots, u_{m-1,n}^k]^{\mathrm{T}},$$
$$\boldsymbol{\varUpsilon} = [\omega_{1,1}, \omega_{1,2}, \cdots, \omega_{1,n-1}, \omega_{2,1}, \omega_{2,2}, \cdots, \omega_{2,n-1}, \cdots, \omega_{m-1,1}, \omega_{m-1,2}, \cdots, \omega_{m-1,n-1}]^{\mathrm{T}},$$

and

$$\boldsymbol{g}^{k+1} = r\sum_{s=0}^{k-1}(\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_z^2}\delta_z^2)\boldsymbol{u}^{k-s}, \quad \omega_{i,j} = \omega(ih_x, jh_z), \quad r_x = \frac{r + \nu\tau}{h_x^2},\ r_z = \frac{r + \nu\tau}{h_z^2}.$$

The form of $\boldsymbol{f}^k$ is the same as that of $\boldsymbol{u}^k$.

The matrix $\boldsymbol{A}$ in (18) is block tridiagonal. Its diagonal blocks are triangular matrix, which can be written as the following form:

$$\begin{bmatrix} 1 + 2(r_x + r_z) & -r_z & \cdots & 0 & 0 \\ -r_z & 1 + 2(r_x + r_z) & \cdots & 0 & 0 \\ 0 & -r_z & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 + 2(r_x + r_z) & -r_z \\ 0 & 0 & \cdots & -r_z & 1 + 2(r_x + r_z) \end{bmatrix}. \tag{19}$$

The lower-angular and super-angular blocks are equal to $-r_x\boldsymbol{E}$ and $-r_z\boldsymbol{E}$, respectively, where E denotes the unit matrix.

We can see that the matrix $\boldsymbol{A}$ is a strictly diagonally dominant symmetric matrix with positive diagonal terms and nonpositive off-diagonal terms. Then, we obtain the following theorem.

Theorem 2.1 *The discretization matrix* $\boldsymbol{A}$ *is invertible, and the system* (16) *and* (17) *has a unique solution.*

3 Stability of the INAS

For

$$\boldsymbol{v}=[v_{1,1},v_{1,2},\cdots,v_{1,n-1},v_{2,1},v_{2,2},\cdots,v_{2,n-1},\cdots,v_{m-1,1},v_{m-1,2},\cdots,v_{m-1,n-1}]^{\mathrm{T}},$$
$$\boldsymbol{w}=[w_{1,1},w_{1,2},\cdots,w_{1,n-1},w_{2,1},w_{2,2},\cdots,w_{2,n-1},\cdots,w_{m-1,1},w_{m-1,2},\cdots,w_{m-1,n-1}]^{\mathrm{T}},$$

we define the inner product $(\boldsymbol{v},\boldsymbol{w})=\sum\limits_{i=1}^{m-1}\sum\limits_{j=1}^{n-1}v_{i,j}w_{i,j}h_xh_z$ and the norm $\|\boldsymbol{v}\|_2=\sqrt{(v,v)}$.

We suppose that $\widetilde{u}_{i,j}^k$ $(i=0,1,2,\cdots,m;\ j=0,1,2,\cdots,n;\ k=0,1,2,\cdots,K)$ is the approximation solution of (16) and (17). Then, the error $\varepsilon_{i,j}^k=\widetilde{u}_{i,j}^k-u_{i,j}^k$ satisfies

$$\begin{aligned}\varepsilon_{i,j}^{k+1}&=\varepsilon_{i,j}^{k}+\nu\tau\Big(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2\Big)\varepsilon_{i,j}^{k+1}+r\Big(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2\Big)\varepsilon_{i,j}^{k+1}\\&\quad+r\sum_{s=0}^{k-1}(b_{s+1}-b_s)\Big(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2\Big)\varepsilon_{i,j}^{k-s},\end{aligned}\tag{20}$$

$$\varepsilon_{0,j}^k=\varepsilon_{m,j}^k=\varepsilon_{i,0}^k=\varepsilon_{i,n}^k=0\quad(i=0,1,\cdots,m;\quad j=0,1,\cdots,n;\quad k=1,2,\cdots,K).\tag{21}$$

Let $\boldsymbol{E}^k=[\varepsilon_{1,1}^k,\varepsilon_{1,2}^k,\cdots,\varepsilon_{1,n-1}^k,\varepsilon_{2,1}^k,\varepsilon_{2,2}^k,\cdots,\varepsilon_{2,n-1}^k,\cdots,\varepsilon_{m-1,1}^k,\varepsilon_{m-1,2}^k,\cdots,\varepsilon_{m-1,n-1}^k]^{\mathrm{T}}$. Introduce the notations

$$r_{1x}=\frac{\nu\tau}{h_x^2},\quad r_{1z}=\frac{\nu\tau}{h_z^2},\quad r_{2x}=\frac{r}{h_x^2}=\frac{\alpha\tau^\beta}{\Gamma(\beta+1)h_x^2},\quad r_{2z}=\frac{r}{h_z^2}=\frac{\alpha\tau^\beta}{\Gamma(\beta+1)h_z^2}.$$

Multiplying (20) by $\varepsilon_{i,j}^{k+1}h_xh_z$, and summing i from 1 to $m-1$ and j from 1 to $n-1$, respectively, we have

$$\begin{aligned}\|\boldsymbol{E}^{k+1}\|_2^2&=(\boldsymbol{E}^{k+1},\boldsymbol{E}^{k})+r_{1x}(\delta_x^2\boldsymbol{E}^{k+1},\boldsymbol{E}^{k+1})+r_{1z}(\delta_z^2\boldsymbol{E}^{k+1},\boldsymbol{E}^{k+1})\\&\quad+r_{2x}(\delta_x^2\boldsymbol{E}^{k+1},\boldsymbol{E}^{k+1})+r_{2z}(\delta_z^2\boldsymbol{E}^{k+1},\boldsymbol{E}^{k+1})\\&\quad+r_{2x}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(\delta_x^2\boldsymbol{E}^{k-s},\boldsymbol{E}^{k+1})+r_{2z}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(\delta_z^2\boldsymbol{E}^{k-s},\boldsymbol{E}^{k+1}),\end{aligned}$$

i.e.,

$$\begin{aligned}\|\boldsymbol{E}^{k+1}\|_2^2&=(\boldsymbol{E}^{k+1},\boldsymbol{E}^{k})-r_{1x}\Big(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k+1}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k+1}\|_2^2\Big)\\&\quad-r_{1z}\Big(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k+1}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k+1}\|_2^2\Big)-r_{2x}\Big(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k+1}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k+1}\|_2^2\Big)\\&\quad-r_{2z}\Big(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k+1}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k+1}\|_2^2\Big)\end{aligned}$$

$$
\begin{aligned}
&+ r_{2x}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(-\sum_{j=1}^{n-1}\varepsilon_{1,j}^{k-s}\varepsilon_{1,j}^{k+1}h_xh_z-(\Delta_x\boldsymbol{E}^{k-s},\Delta_x\boldsymbol{E}^{k+1}))\\
&+ r_{2z}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(-\sum_{i=1}^{m-1}\varepsilon_{i,1}^{k-s}\varepsilon_{i,1}^{k+1}h_xh_z-(\Delta_z\boldsymbol{E}^{k-s},\Delta_z\boldsymbol{E}^{k+1}))\\
\le{}& \frac{1}{2}(\|\boldsymbol{E}^{k+1}\|_2^2+\|\boldsymbol{E}^{k}\|_2^2)-r_{1x}(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k+1}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k+1}\|_2^2)\\
&- r_{1z}(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k+1}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k+1}\|_2^2)-r_{2x}(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k+1}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k+1}\|_2^2)\\
&- r_{2z}(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k+1}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k+1}\|_2^2)\\
&+ \frac{r_{2x}}{2}\sum_{s=0}^{k-1}(b_s-b_{s+1})\Big[\sum_{j=1}^{n-1}(|\varepsilon_{1,j}^{k-s}|^2h_xh_z+|\varepsilon_{1,j}^{k+1}|^2h_xh_z)+\|\Delta_x\boldsymbol{E}^{k-s}\|_2^2+\|\Delta_x\boldsymbol{E}^{k+1}\|_2^2\Big]\\
&+ \frac{r_{2z}}{2}\sum_{s=0}^{k-1}(b_s-b_{s+1})\Big[\sum_{i=1}^{m-1}(|\varepsilon_{i,1}^{k-s}|^2h_xh_z+|\varepsilon_{i,1}^{k+1}|^2h_xh_z)+\|\Delta_z\boldsymbol{E}^{k-s}\|^2+\|\Delta_z\boldsymbol{E}^{k+1}\|^2\Big].
\end{aligned}
$$

Noting that $\sum\limits_{s=0}^{k-1}(b_s-b_{s+1})=1-b_k$ and $b_k>0$, we have

$$
\begin{aligned}
&\|\boldsymbol{E}^{k+1}\|_2^2+r_{2x}\sum_{s=0}^{k}b_s(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k+1-s}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k+1-s}\|_2^2)\\
&+ r_{2z}\sum_{s=0}^{k}b_s(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k+1-s}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k+1-s}\|_2^2)\\
\le{}& \|\boldsymbol{E}^{k}\|_2^2+r_{2x}\sum_{s=0}^{k-1}b_s(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k-s}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k-s}\|_2^2)\\
&+ r_{2z}\sum_{s=0}^{k-1}b_s(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k-s}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k-s}\|_2^2).
\end{aligned}\tag{22}
$$

Defining the energy norm

$$
\begin{aligned}
\|\boldsymbol{E}^{k}\|_{\mathrm{E}}^2={}&\|\boldsymbol{E}^{k}\|_2^2+r_{2x}\sum_{s=0}^{k-1}b_s(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^{k-s}|^2h_xh_z+\|\Delta_x\boldsymbol{E}^{k-s}\|_2^2)\\
&+ r_{2z}\sum_{s=0}^{k-1}b_s(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^{k-s}|^2h_xh_z+\|\Delta_z\boldsymbol{E}^{k-s}\|_2^2),
\end{aligned}\tag{23}
$$

we have

$$
\|\boldsymbol{E}^{k+1}\|_2^2\le\|\boldsymbol{E}^{k+1}\|_E^2\le\|\boldsymbol{E}^{k}\|_E^2\le\cdots\le\|\boldsymbol{E}^{1}\|_E^2.
$$

As

$$
\varepsilon_{i,j}^1=\varepsilon_{i,j}^0+r_{1x}\delta_x^2\varepsilon_{i,j}^1+r_{1z}\delta_z^2\varepsilon_{i,j}^1+r_{2x}\delta_x^2\varepsilon_{i,j}^1+r_{2z}\delta_z^2\varepsilon_{i,j}^1,
$$

we have

$$\begin{aligned}\|\boldsymbol{E}^1\|_2^2 =& (\boldsymbol{E}^1,\boldsymbol{E}^0)+(r_{1x}+r_{2x})(\delta_x^2\boldsymbol{E}^1,\boldsymbol{E}^1)+(r_{1z}+r_{2z})(\delta_z^2\boldsymbol{E}^1,\boldsymbol{E}^1)\\ \le& \frac{1}{2}(\|\boldsymbol{E}^1\|_2^2+\|\boldsymbol{E}^0\|_2^2)-(r_{1x}+r_{2x})(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^1|^2h_xh_z)+\|\Delta_x\boldsymbol{E}^1\|_2^2)\\ &-(r_{1z}+r_{2z})(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^1|^2h_xh_z)+\|\Delta_z\boldsymbol{E}^1\|_2^2).\end{aligned}\tag{24}$$

Then,

$$\begin{aligned}\|\boldsymbol{E}^1\|_E^2 =& \|\boldsymbol{E}^1\|_2^2+r_{2x}(\sum_{j=1}^{n-1}|\varepsilon_{1,j}^1|^2h_xh_z+\|\Delta_x\boldsymbol{E}^1\|_2^2)\\ &+r_{2z}(\sum_{i=1}^{m-1}|\varepsilon_{i,1}^1|^2h_xh_z+\|\Delta_z\boldsymbol{E}^1\|_2^2)\\ \le& \|\boldsymbol{E}^0\|_2^2.\end{aligned}\tag{25}$$

Therefore, $\|\boldsymbol{E}^{k+1}\|_2^2\le\|\boldsymbol{E}^0\|_2^2$.

We can obtain the following theorem of stability.

Theorem 3.1 *The implicit numerical approximation scheme* (16) *is unconditionally stable.*

4 Convergence of the INAS

Suppose that $u(x,z,t)$ is the solution of the RSP-HGSGF (2)–(4) and $u(x,z,t)\in\Phi(\Lambda)$. Let $u(x_i,z_j,t_k)$ $(i=0,1,2,\cdots,m;\ j=0,1,2,\cdots,n;\ k=0,1,2,\cdots,K)$ be the exact solution of (2)–(4) at the mesh point (x_i,z_j,t_k).

Define

$$y_{i,j}^k=u(x_i,z_j,t_k)-u_{i,j}^k$$

and $\boldsymbol{Y}^k=[y_{1,1}^k,y_{1,2}^k,\cdots,y_{1,n-1}^k,y_{2,1}^k,y_{2,2}^k,\cdots,y_{2,n-1}^k,\cdots,y_{m-1,1}^k,y_{m-1,2}^k,\cdots,y_{m-1,n-1}^k]^{\mathrm{T}}$. Substituting $u_{i,j}^k=u(x_i,z_j,t_k)-y_{i,j}^k$ into (16) leads to

$$\begin{aligned}y_{i,j}^{k+1} =& y_{i,j}^k+\nu\tau(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)y_{ij}^{k+1}+r(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)y_{i,j}^{k+1}\\ &+r\sum_{s=0}^{k-1}(b_{s+1}-b_s)(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_z^2}\delta_z^2)y_{i,j}^{k-s}+R_{ij}^{k+1},\end{aligned}\tag{26}$$

where $i=1,2,\cdots,m-1;\ j=1,2,\cdots,n-1$; $k=0,1,\cdots,K-1$; and

$$\begin{aligned}&y_{i,j}^0=0\quad(i=0,1,\cdots,m;\quad j=0,1,\cdots,n),\\ &y_{0,j}^k=y_{m,j}^k=y_{i,0}^k=y_{i,n}^k=0\quad(0\le i\le m;\quad 0\le j\le n;\quad 0\le k\le K).\end{aligned}\tag{27}$$

By multiplying (26) by $y_{i,j}^{k+1}h_xh_z$, and summing i from 1 to $m-1$ and j from 1 to $n-1$, respectively, we obtain

$$\begin{aligned}\|\boldsymbol{Y}^{k+1}\|_2^2 =& (\boldsymbol{Y}^{k+1},\boldsymbol{Y}^k) + r_{1x}(\delta_x^2\boldsymbol{Y}^{k+1},\boldsymbol{Y}^{k+1}) + r_{1z}(\delta_z^2\boldsymbol{Y}^{k+1},\boldsymbol{Y}^{k+1})\\ &+ r_{2x}(\delta_x^2\boldsymbol{Y}^{k+1},\boldsymbol{Y}^{k+1}) + r_{2z}(\delta_z^2\boldsymbol{Y}^{k+1},\boldsymbol{Y}^{k+1}) + r_{2x}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(\delta_x^2\boldsymbol{Y}^{k-s},\boldsymbol{Y}^{k+1})\\ &+ r_{2z}\sum_{s=0}^{k-1}(b_{s+1}-b_s)(\delta_z^2\boldsymbol{Y}^{k-s},\boldsymbol{Y}^{k+1}) + (\mathbf{R}^{k+1},\boldsymbol{Y}^{k+1}).\end{aligned} \tag{28}$$

For $s=0,1,\cdots,k+1$, we have

$$(\delta_x^2\boldsymbol{Y}^s,\boldsymbol{Y}^{k+1}) = -\sum_{j=1}^{k-1} y_{1,j}^{k+1}y_{1,j}^s h_x h_z - (\Delta_x\boldsymbol{Y}^s,\Delta_x\boldsymbol{Y}^{k+1}).$$

Using $|vw|\le \sigma v^2 + \frac{1}{4\sigma}w^2$ with $\sigma>0$, we obtain

$$|(\boldsymbol{R}^{k+1},\boldsymbol{Y}^{k+1})| \le \Big(\frac{r_{1x}h_x^2}{a_x^2} + \frac{r_{1z}h_z^2}{a_z^2}\Big)\|\boldsymbol{Y}^{k+1}\|_2^2 + \frac{1}{\frac{4r_{1x}h_x^2}{a_x^2}+\frac{4r_{1z}h_z^2}{a_z^2}}\|\boldsymbol{R}^{k+1}\|_2^2. \tag{29}$$

Similar to the proof of the stability,

$$\begin{aligned}&\|\boldsymbol{Y}^{k+1}\|_2^2\\ \le& \frac{1}{2}(\|\boldsymbol{Y}^{k+1}\|_2^2 + \|\boldsymbol{Y}^k\|_2^2) + \frac{r_{2x}}{2}\sum_{s=1}^{k-1}(b_s-b_{s+1})\Big(\sum_{j=1}^{n-1}|y_{1,j}^{k-s}|^2h_xh_z + \|\Delta_x\boldsymbol{Y}^{k-s}\|_2^2\Big)\\ &+ \frac{r_{2z}}{2}\sum_{s=1}^{k-1}(b_s-b_{s+1})\Big(\sum_{i=1}^{m-1}|y_{i,1}^{k-s}|^2h_xh_z + \|\Delta_z\boldsymbol{Y}^{k-s}\|_2^2\Big)\\ &+ \Big(\frac{r_{1x}h_x^2}{a_x^2}b_k + \frac{r_{1z}h_z^2}{a_z^2}b_k\Big)\|\boldsymbol{Y}^{k+1}\|_2^2 + \frac{1}{\frac{4r_{2x}h_x^2b_k}{a_x^2}+\frac{4r_{2z}h_z^2b_k}{a_z^2}}\|\boldsymbol{R}^{k+1}\|_2^2.\end{aligned} \tag{30}$$

Lemma 4.1 *Given* $\|\boldsymbol{Y}^k\|_r^2 = \max\limits_{1\le i\le m-1}\sum\limits_{j=1}^{n-1}|y_{i,j}^k|^2h_z$, *then*

$$\|\boldsymbol{Y}^k\|_2^2 \le a_x\|\boldsymbol{Y}^k\|_r^2 \le \frac{a_x^2}{2h_x^2}\Big(\sum_{j=1}^{n-1}|y_{1,j}^k|^2h_xh_z + \|\Delta_x\boldsymbol{Y}^k\|_2^2\Big).$$

Given $\|\boldsymbol{Y}^k\|_c^2 = \max\limits_{1\le j\le n-1}\sum\limits_{i=1}^{m-1}|y_{i,j}^k|^2h_x$, *then*

$$\|\boldsymbol{Y}^k\|_2^2 \le a_z\|\boldsymbol{Y}^k\|_c^2 \le \frac{a_z^2}{2h_z^2}\Big(\sum_{i=1}^{m-1}|y_{i,1}^k|^2h_xh_z + \|\Delta_z\boldsymbol{Y}^k\|_2^2\Big).$$

Proof Suppose that $\sum\limits_{j=1}^{n-1}|y_{i_0,j}^k| = \max\limits_{1\le i\le m-1}\sum\limits_{j=1}^{n-1}|y_{i,j}^k|$. From $y_{i_0,j}^k = y_{1,j}^k + \sum\limits_{i=0}^{i_0-1}\Delta_x y_{i,j}^k$ and $y_{i_0,j}^k = -\sum\limits_{i=i_0}^{m-1}\Delta_x y_{i,j}^k$, we obtain $2|y_{i_0,j}^k| \le |y_{1,j}^k| + \sum\limits_{i=1}^{m-1}|\Delta_x y_{i,j}^k|$.

Using the Cauchy-Schwarz inequality, we have

$$4|y_{i_0,j}^k|^2 \le m(|y_{1,j}^k|^2 + \sum_{i=1}^{m-1} |\Delta_x y_{i,j}^k|^2) \le \frac{2a_x}{h_x}(|y_{1,j}^k|^2 + \sum_{i=1}^{m-1} |\Delta_x y_{i,j}^k|^2).$$

Therefore, $\|\boldsymbol{Y}^k\|_r^2 = \sum_{j=1}^{n-1} |y_{i_0,j}^k|^2 h_z \leqslant \frac{a_x}{2h_x^2}(\sum_{j=1}^{n-1} |y_{1,j}^k|^2 h_x h_z + \|\Delta_x \boldsymbol{Y}^k\|_2^2)$.

The second part is similar to the proof of the first part.

Applying Lemma 4.1, we have

$$\begin{aligned}
&(\frac{r_{2x}h_x^2}{a_x^2}b_k + \frac{r_{2z}h_z^2}{a_z^2}b_k)\|\boldsymbol{Y}^{k+1}\|_2^2 \\
&\le \frac{r_{2x}b_k}{2}(\sum_{j=1}^{n-1} |y_{1,j}^{k+1}|^2 h_x h_z + \|\Delta_x \boldsymbol{Y}^{k+1}\|_2^2) + \frac{r_{2z}b_k}{2}(\sum_{i=1}^{m-1} |y_{i,1}^{k+1}|^2 h_x h_z + \|\Delta_z \boldsymbol{Y}^{k+1}\|_2^2).
\end{aligned} \tag{31}$$

Hence, from (30) and (31), we have

$$\begin{aligned}
\|\boldsymbol{Y}^{k+1}\|_2^2 \le &\frac{1}{2}(\|\boldsymbol{Y}^{k+1}\|_2^2 + \|\boldsymbol{Y}^k\|_2^2) \\
&+ \frac{r_{2x}}{2}\sum_{s=1}^{k-1}(b_s - b_{s+1})(\sum_{j=1}^{n-1} |y_{1,j}^{k-s}|^2 h_x h_z + \|\Delta_x \boldsymbol{Y}^{k-s}\|_2^2) \\
&+ \frac{r_{2z}}{2}\sum_{s=1}^{k-1}(b_s - b_{s+1})(\sum_{i=1}^{m-1} |y_{i,1}^{k-s}|^2 h_x h_z + \|\Delta_z \boldsymbol{Y}^{k-s}\|_2^2) \\
&+ \frac{1}{\frac{4r_{2x}h_x^2 b_k}{a_x^2} + \frac{4r_{2z}h_z^2 b_k}{a_z^2}}\|\boldsymbol{R}^{k+1}\|_2^2.
\end{aligned}$$

According to $r_{2x} = \frac{\alpha\tau^\beta}{\Gamma(\beta+1)h_x^2}$ and $r_{2z} = \frac{\alpha\tau^\beta}{\Gamma(\beta+1)h_z^2}$, we have

$$\frac{1}{\frac{4r_{2x}h_x^2 b_k}{a_x^2} + \frac{4r_{2z}h_z^2 b_k}{a_z^2}}\|\boldsymbol{R}^{k+1}\|_2^2 = \frac{1}{2b_k\frac{\alpha\tau^\beta}{\Gamma(\beta+1)}(\frac{1}{a_x^2} + \frac{1}{a_z^2})}\|\boldsymbol{R}^{k+1}\|_2^2 \le c\tau^\beta b_k(\tau + h_x^2 + h_z^2)^2.$$

Let

$$\begin{aligned}
\gamma_k = &\|\boldsymbol{Y}^k\|_2^2 + r_{2x}\sum_{s=0}^{k-1} b_s(\sum_{j=1}^{n-1} |y_{1,j}^{k-s}|^2 h_x h_z + \|\Delta_x \boldsymbol{Y}^{k-s}\|_2^2) \\
&+ r_{2z}\sum_{s=0}^{k-1} b_s(\sum_{i=1}^{m-1} |y_{i,1}^{k-s}|^2 h_x h_z + \|\Delta_z \boldsymbol{Y}^{k-s}\|_2^2),
\end{aligned}$$

and then

$$\gamma_{k+1} \le \gamma_k + C\tau^\beta b_k(\tau + h_x^2 + h_z^2)^2.$$

Hence, we obtain

$$\gamma_{k+1} \le C\sum_{s=0}^{k} b_s\tau^\beta(\tau + h_x^2 + h_z^2)^2.$$

Noting that $\sum_{s=0}^{k} b_s\tau^\beta = (k+1)^\beta\tau^\beta \le T^\beta$ and $\|\boldsymbol{Y}^{k+1}\|_2^2 \le \gamma_{k+1}$, we have

$$\|\boldsymbol{Y}^{k+1}\|_2^2 \le CT^\beta(\tau + h_x^2 + h_z^2)^2.$$

Consequently, the following theorem of convergence is obtained.

Theorem 4.1 *Let $u(x,z,t) \in \Phi(\Lambda)$ be the solution of the RSP-HGSGF* (2)–(4). *Then, the INAS* (16) *is convergent, and there exists a positive constant $C > 0$ such that*

$$\|Y^{k+1}\|_2 \leq C(\tau + h_x^2 + h_z^2), \quad k = 0, 1, \cdots, K-1.$$

5 Numerical examples

In this section, two numerical examples are presented to support our theoretical analysis.

Example 1 Consider the following RSP-HGSGF:

$$\begin{cases} \dfrac{\partial u(x,z,t)}{\partial t} = (\nu + \alpha D_t^{1-\beta})\Delta u(x,z,t), \quad (x,z) \in \Omega = [0,2] \times [0,2], \quad t > 0, \\ u(x,z,t)|_{\partial\Omega} = 0, \\ u(x,z,0) = \delta(0.8,0.8) = \begin{cases} 200, \quad (x,z) = (0.8,0.8), \\ 0, \quad (x,z) \neq (0.8,0.8), \quad (x,z) \in \Omega, \end{cases} \end{cases} \tag{32}$$

where $\nu = \alpha = 0.1$.

The INAS is used to solve (32). The numerical simulation of the process is shown in Figs. 1–3. As the time t increases, one observes that the source diffuses.

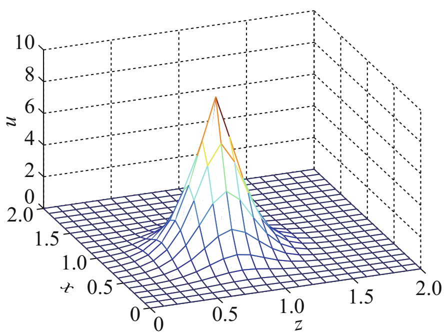

Fig. 1 Numerical solution at $t = 0.1$

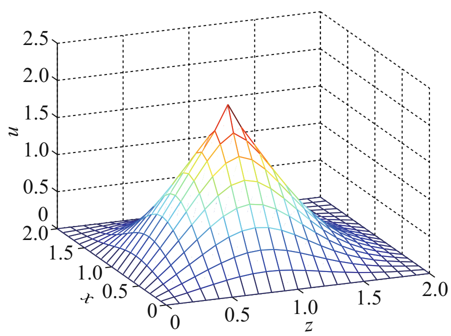

Fig. 2 Numerical solution at $t = 0.5$

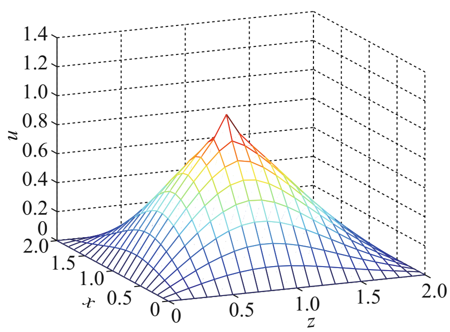

Fig. 3 Numerical solution at $t = 1.0$

In order to show the approximation order of the INAS, we construct an example with an analytic solution.

Example 2 The following RSP-HGSGF with an analytic solution is considered:

$$\begin{cases} \dfrac{\partial u(x,z,t)}{\partial t} = (1+D_t^{1-\beta})\Delta u(x,z,t) + f(x,z,t), \quad (x,z)\in\Omega=[0,1]\times[0,1], \quad t>0, \\ u(x,z,t)|_{\partial\Omega} = \mathrm{e}^{x+z}t^{1+\beta}, \\ u(x,z,0)=0, \quad (x,z)\in\Omega, \end{cases} \tag{33}$$

where $f(x,z,t) = \mathrm{e}^{x+z}((1+\beta)t^{\beta} - \frac{\Gamma(2+\beta)}{2\Gamma(1+2\beta)}t^{2\beta} - 2t^{1+\beta})$. The exact solution of (33) is $u(x,z,t) = \mathrm{e}^{x+z}t^{1+\beta}$.

Let $\chi = \{(i,j,k)\,|0\le i\le m,\ 0\le j\le n,\ 0\le k\le K\}$. The maximum absolute error between the exact solution u and the numerical solutions $U = u_{ij}^n$ is defined as follows:

$$||u-U||_\infty = \max_{(i,j,k)\in\chi}\{|u(x_i,z_j,t_n) - u_{ij}^n|\}.$$

The maximum absolute error between the exact solution and the numerical solutions by the INAS, with spatial and temporal steps $\tau = h_x = h_z = 1/10, 1/15, 1/20, 1/25$ and $\tau = h_x^2 = h_z^2$ and $h_x = h_z = 1/10, 1/15, 1/20, 1/25$ at time $t = 1.0$, are listed in Tables 1 and 2, respectively, which are in good agreement with the theory analysis. Figures 4 and 5 show the exact solution and the numerical solutions by the INAS at $x = 0.5, t = 1.0$ and $z = 0.75, t = 1.0$, respectively.

Table 1 The maximum absolute error of the INAS

$\tau = h_x = h_z$	h_i/h_{i+1}	β=0.4	Rate	β=0.7	Rate	β=0.9	Rate
$\frac{1}{10}$	–	3.131 2E−3	–	7.019 2E−3	–	9.579 1E−3	–
$\frac{1}{15}$	$\frac{1}{10}/\frac{1}{15} = 1.5$	2.268 9E−3	1.38	4.754 2E−3	1.48	6.381 5E−3	1.50
$\frac{1}{20}$	$\frac{1}{15}/\frac{1}{20} \approx 1.33$	1.794 4E−3	1.26	3.599 4E−3	1.32	4.784 4E−3	1.33
$\frac{1}{25}$	$\frac{1}{20}/\frac{1}{25} = 1.25$	1.493 1E−3	1.20	2.905 5E−3	1.24	3.834 5E−3	1.25

Table 2 The maximum absolute error of the INAS

$h_x = h_z$	$\tau = h_x^2 = h_z^2$	$(h_i/h_{i+1})^2$	β=0.4	Rate	β=0.7	Rate	β=0.9	Rate
$\frac{1}{10}$	$\frac{1}{100}$	–	7.767 3E−4	–	1.072 8E−3	–	1.279 3E−3	–
$\frac{1}{15}$	$\frac{1}{225}$	$(\frac{1}{10}/\frac{1}{15})^2 = 2.25$	3.583 1E−4	2.16	4.822 3E−4	2.22	5.721 6E−4	2.23
$\frac{1}{20}$	$\frac{1}{400}$	$(\frac{1}{15}/\frac{1}{20})^2 \approx 1.78$	2.117 3E−4	1.69	2.744 5E−4	1.76	3.221 2E−4	1.77
$\frac{1}{25}$	$\frac{1}{625}$	$(\frac{1}{20}/\frac{1}{25})^2 = 1.56$	1.385 7E−4	1.52	1.749 7E−4	1.56	2.043 3E−4	1.57

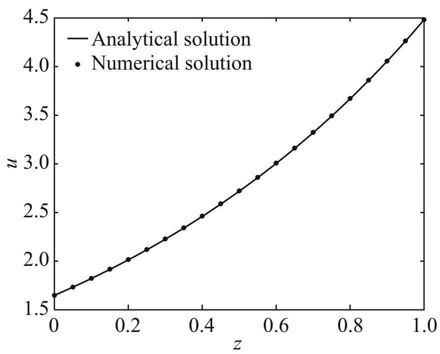

Fig. 4 Comparison of the exact and numerical solutions at $x = 0.5, t = 1.0$

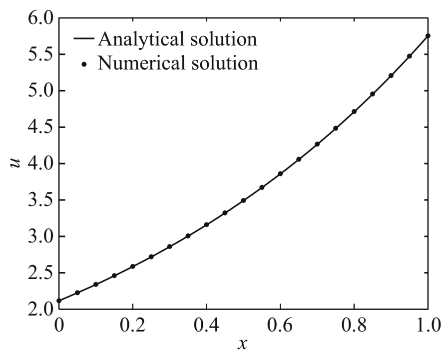

Fig. 5 Comparison of the exact and numerical solutions at $z = 0.75, t = 1.0$

6 Conclusions

In this paper, an INAS for the RSP-HGSGF has been described and demonstrated. The stability, consistency, and convergence of the INAS for the RSP-HGSGF have been discussed. These methods and analytical techniques can also be extended to some high-dimensional fractional partial differential equations.

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