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# Series solutions for the stagnation flow of a second-grade fluid over a shrinking sheet *

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**Abstract** This study derives the analytic solutions of boundary layer flows bounded by a shrinking sheet. With the similarity transformations, the partial differential equations are reduced into the ordinary differential equations which are then solved by the homotopy analysis method (HAM). Two-dimensional and axisymmetric shrinking flow cases are discussed.

**Key words** stagnation flow, second-grade fluid, shrinking sheet

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**2000 Mathematics Subject Classification** 76A05

## Introduction

Investigation of flows for non-Newtonian fluids is a topic of great interest nowadays. This is due to their various industrial and engineering applications. Several models of non-Newtonian fluids have been developed. Amongst these, the second-grade fluid model has been widely studied by several researchers[1-7]. Moreover, the boundary layer flows of non-Newtonian fluids are important in the aerodynamic extrusion of plastic sheets, glass fiber, paper production, and manufacturing of polymer sheets, etc. Since the pioneering work done by Sakiadis[8], various attempts discussing the boundary layer flows over the stretching[9-12] sheets have been presented. An existing literature also indicates that few studies[13-15] have been presented regarding the boundary layer flows bouded by the shrinking sheets.

In the view of aforesaid discussions, the goal here is to discuss the stagnation flows of a second-grade fluid by shrinking surfaces. Besides, heat transfer effects are also considered. The series solutions for two-dimensional and axisymmetric cases have been developed by using a powerful technique known as the homotopy analysis method (HAM)[16-20]. The convergence of the series solutions is obtained. Graphical results are presented and discussed.

## 1 Formulation of the problem

Consider a two-dimensional incompressible second-grade stagnation flow towards a shrinking sheet. The velocity components along the $x$-, $y$-, and $z$-axes are denoted by $u$, $v$, and $w$, respectively. The potential stagnation flow at infinity is taken as $u = ax$, $w = -az$, whereas on the stretching (shrinking) surface, the velocities are considered as $u = b(x + c)$, $w = 0$. Here,

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$b > 0$ is the stretching rate, $b < 0$ is the shrinking rate, and $-c$ is the location of the stretching origin. The flow and heat characteristics are governed by the following equations[7,15] :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho}(u\frac{\partial^3 u}{\partial x\partial z^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z} + w\frac{\partial^3 u}{\partial z^3}), \tag{2}$$

$$\rho c_p(u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z}) = k\nabla^2 T, \tag{3}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, $\rho$ is the density, $\alpha_1$ is the second-grade parameter, $c_p$ is the specific heat, $T$ is the temperature, and $k$ $(> 0)$ is the thermal conductivity where dissipation is neglected.

## 2 Two-dimensional stagnation flow

Define[15]

$$\eta = \sqrt{\frac{a}{\nu}}\, z, \quad \theta = \frac{T - T_\infty}{T_0 - T_\infty} \tag{4}$$

$$u = axf'(\eta) + bch(\eta), \quad v = 0, \quad w = -\sqrt{a\nu}f(\eta), \tag{5}$$

Equation (1) is satisfied identically, and Eqs. (2) and (3) yield

$$f''' + ff'' - f'^2 + 1 + \beta\left(2f'f''' + f''^2 - ff^{(4)}\right) = 0, \tag{6}$$

$$h'' + fh' - f'h + \beta(hf''' + f'h'' + f''h' - fh''') = 0, \tag{7}$$

$$\theta'' + Prf\theta' = 0, \tag{8}$$

where $\beta = \frac{\alpha_1 a}{\rho\nu}$, and $Pr = \frac{\nu}{k}$.

The associated boundary conditions are

$$\begin{cases} f(0) = 0, \quad f'(0) = \dfrac{b}{a} = \alpha, \quad f'(\infty) = 1, \\ h(0) = 1, \quad h(\infty) = 0, \\ \theta(0) = 1, \quad \theta(\infty) = 0, \end{cases} \tag{9}$$

in which the prime denotes the derivative with respect to $\eta$.

### 2.1 Solutions for two-dimensional stagnation flows towards a shrinking sheet

For the HAM solution, we select the following initial approximations and auxiliary linear operators:

$$f_{01}(\eta) = (1-\alpha)(\exp(-\eta) - 1) + \eta, \tag{10}$$

$$h_{01}(\eta) = \exp(-\eta), \tag{11}$$

$$\theta_{01}(\eta) = \exp(-\eta), \tag{12}$$

$$\mathcal{L}_{01}\left[\widehat{f}(\eta;p)\right] = \frac{\partial^3 \widehat{f}(\eta;p)}{\partial\eta^3} + \frac{\partial^2 \widehat{f}(\eta;p)}{\partial\eta^2}, \tag{13}$$

$$\mathcal{L}_{02}\left[\widehat{h}(\eta;p)\right] = \frac{\partial^2 \widehat{h}(\eta;p)}{\partial\eta^2} + \frac{\partial \widehat{h}(\eta;p)}{\partial\eta}, \tag{14}$$

$$\mathcal{L}_{03}\left[\widehat{\theta}(\eta;p)\right] = \frac{\partial^2 \widehat{\theta}(\eta;p)}{\partial\eta^2} + \frac{\partial \widehat{\theta}(\eta;p)}{\partial\eta}, \tag{15}$$

where the subscript 01 means the first case, and other subscripts denote the next cases. The properties satisfied by the auxiliary linear operator are

$$\mathcal{L}_{01}\left[C_1 + C_2\eta + C_3\mathrm{e}^{-\eta}\right] = 0, \tag{16}$$
$$\mathcal{L}_{02}\left[C_4 + C_5\mathrm{e}^{-\eta}\right] = 0, \tag{17}$$
$$\mathcal{L}_{03}\left[C_6 + C_7\mathrm{e}^{-\eta}\right] = 0, \tag{18}$$

where $C_i$ $(i = 1, 2, \cdots, 7)$ are the arbitrary constants. If $p \in [0, 1]$ is an embedding parameter and $\hbar_i$ $(i = 1, 2, 3)$ are non-zero auxiliary parameters, then the zeroth order and $m$th order deformation problems are as follows:

The zeroth order deformation problems:

$$(1-p)\,\mathcal{L}_{01}\left[\widehat{f}(\eta;p) - f_{01}(\eta)\right] = p\hbar_1\mathcal{N}_1\left[\widehat{f}(\eta;p)\right], \tag{19}$$
$$(1-p)\,\mathcal{L}_{02}\left[\widehat{h}(\eta;p) - h_{01}(\eta)\right] = p\hbar_2\mathcal{N}_2\left[\widehat{h}(\eta;p), \widehat{f}(\eta;p)\right], \tag{20}$$
$$(1-p)\,\mathcal{L}_{03}\left[\widehat{\theta}(\eta;p) - \theta_{01}(\eta)\right] = p\hbar_3\mathcal{N}_3\left[\widehat{\theta}(\eta;p), \widehat{f}(\eta;p)\right], \tag{21}$$
$$\widehat{f}(0;p) = 0, \quad \widehat{f}'(0;p) = \alpha, \quad \widehat{f}'(\infty;p) = 1, \tag{22}$$
$$\widehat{h}(0;p) = 1, \quad \widehat{h}(\infty;p) = 0, \tag{23}$$
$$\widehat{\theta}(0;p) = 1, \quad \widehat{\theta}(\infty;p) = 0, \tag{24}$$

$$\begin{aligned}\mathcal{N}_1\left[\widehat{f}(\eta;p)\right] = {} & \frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3} + \widehat{f}(\eta;p)\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2} - \left(\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\right)^2 + 1 \\ & + \beta\left[2\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3} + \left(\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}\right)^2 - \widehat{f}(\eta;p)\frac{\partial^4\widehat{f}(\eta;p)}{\partial\eta^4}\right],\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{N}_2\left[\widehat{f}(\eta;p), \widehat{h}(\eta;p)\right] = {} & \frac{\partial^2\widehat{h}(\eta;p)}{\partial\eta^2} + \widehat{f}(\eta;p)\frac{\partial\widehat{h}(\eta;p)}{\partial\eta} - \widehat{h}(\eta;p)\frac{\partial\widehat{f}(\eta;p)}{\partial\eta} \\ & + \beta\left[\widehat{h}(\eta;p)\frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3} + \frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\frac{\partial^2\widehat{h}(\eta;p)}{\partial\eta^2}\right] \\ & + \beta\left[\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}\frac{\partial\widehat{h}(\eta;p)}{\partial\eta} - \widehat{f}(\eta;p)\frac{\partial^3\widehat{h}(\eta;p)}{\partial\eta^3}\right],\end{aligned} \tag{26}$$

$$\mathcal{N}_3\left[\widehat{\theta}(\eta;p), \widehat{f}(\eta;p)\right] = \frac{\partial^2\widehat{\theta}(\eta;p)}{\partial\eta^2} + Pr\widehat{f}(\eta;p)\frac{\partial\widehat{\theta}(\eta;p)}{\partial\eta}. \tag{27}$$

The $m$th-order deformation problems:

$$\mathcal{L}_{01}\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \hbar_1\mathcal{R}_{1m}(\eta), \tag{28}$$
$$\mathcal{L}_{02}\left[h_m(\eta) - \chi_m h_{m-1}(\eta)\right] = \hbar_2\mathcal{R}_{2m}(\eta), \tag{29}$$
$$\mathcal{L}_{03}\left[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)\right] = \hbar_3\mathcal{R}_{3m}(\eta), \tag{30}$$
$$\begin{cases} f_m(0) = 0, \ f'_m(0) = 0, \ f'_m(\infty) = 0, \\ h_m(0) = 0, \ h_m(\infty) = 0, \\ \theta_m(0) = 0, \ \theta_m(\infty) = 0, \end{cases} \tag{31}$$

$$\mathcal{R}_{1m}(\eta) = f'''_{m-1}(\eta) + (1-\chi_m) + \sum_{k=0}^{m-1} \left(f_{m-1-k} f''_k - f'_{m-1-k} f'_k\right) + \beta \sum_{k=0}^{m-1} \left(2f'_{m-1-k} f'''_k + f''_{m-1-k} f''_k - f_{m-1-k} f^{(4)}_k\right), \tag{32}$$

$$\mathcal{R}_{2m}(\eta) = h''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left(h'_{m-1-k} f_k - h_{m-1-k} f'_k\right) + \beta \sum_{k=0}^{m-1} \left(h_{m-1-k} f'''_k + f'_{m-1-k} h''_k + h'_{m-1-k} f''_k - f_{m-1-k} h'''_k\right), \tag{33}$$

$$\mathcal{R}_{3m}(\eta) = \theta''_{m-1}(\eta) + Pr \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k, \tag{34}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{35}$$

The symbolic software Mathematica is used to get the solutions to Eqs. (28)–(30) up to the first few orders of approximations. It is found that the solutions for $f$, $h$, and $\theta$ are

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M\to\infty} \left[\sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{2M+1} \mathrm{e}^{-(n+1)\eta} \left(\sum_{m=n-1}^{2M} \sum_{k=1}^{2m+1-n} a^k_{m,n} \eta^k\right)\right], \tag{36}$$

$$h(\eta) = \sum_{m=0}^{\infty} h_m(\eta) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+1} \mathrm{e}^{-(n+1)\eta} \left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} c^k_{m,n} \eta^k\right)\right], \tag{37}$$

$$\theta(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+1} \mathrm{e}^{-(n+1)\eta} \left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} b^k_{m,n} \eta^k\right)\right]. \tag{38}$$

## 3 Axisymmetric stagnation flows towards an axisymmetric shrinking surface

With the following transformations[15]:

$$\begin{cases} \eta(x,y) = \sqrt{\dfrac{a}{\nu}}\ z, \quad \theta = \dfrac{T - T_\infty}{T_0 - T_\infty}, \\ u = axg'(\eta) + bcl(\eta), \quad v = ayg'(\eta), \quad w = -2\sqrt{a\nu}\ g(\eta), \end{cases} \tag{39}$$

Eqs. (2) and (3) give

$$g''' + 2gg'' - g'^2 + 1 + \beta\left(2g'g''' + g''^2 - 2gg^{(4)}\right) = 0, \tag{40}$$

$$l'' + 2gl' - g'l + \beta\left(lg''' + g'l'' + g''l' - 2gl'''\right) = 0, \tag{41}$$

$$\theta'' + 2Prg\theta' = 0. \tag{42}$$

The relevant boundary conditions are[15]

$$\begin{cases} g(0) = 0, \quad g'(0) = \dfrac{b}{a} = \alpha, \quad g'(\infty) = 1, \\ l(0) = 1, \quad l(\infty) = 0, \\ \theta(0) = 1, \quad \theta(\infty) = 0. \end{cases} \tag{43}$$

Adopting the procedure of the previous subsection, we have the following solutions:

$$g(\eta)=\sum_{m=0}^{\infty}g_m(\eta)=\lim_{M\to\infty}\left[\sum_{m=0}^{M}a_{m,0}^0+\sum_{n=1}^{2M+1}\mathrm{e}^{-(n+1)\eta}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n}a_{m,n}^k\eta^k\right)\right], \tag{44}$$

$$l(\eta)=\sum_{m=0}^{\infty}l_m(\eta)=\lim_{M\to\infty}\left[\sum_{n=1}^{2M+1}\mathrm{e}^{-(n+1)\eta}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n}c_{m,n}^k\eta^k\right)\right], \tag{45}$$

$$\theta(\eta)=\sum_{m=0}^{\infty}\theta_m(\eta)=\lim_{M\to\infty}\left[\sum_{n=1}^{2M+1}\mathrm{e}^{-(n+1)\eta}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n}b_{m,n}^k\eta^k\right)\right]. \tag{46}$$

## 4 Results and discussions

The analytic solutions of the two problems have been computed in Eqs. (36)–(38) and (44)–(46) containing non-zero auxiliary parameters $\hbar_i$ $(i=1,2)$ which can adjust and control the convergence of the solutions. The $\hbar$-curve[15] is defined as a horizontal line segment above the valid region of all possible values of $\hbar$. In the present cases, the 19th and 20th orders of $\hbar_i$-curves are plotted in Figs. 1 and 2. From these figures, we can see that the admissible ranges for $\hbar_i$ are $-1.8\le\hbar_1\le-0.3$ and $-1.9\le\hbar_2\le-0.3$. To assure the convergence of the HAM solution, the values of $\hbar_i$ should be chosen from these regions. The region for the values of $\hbar_i$ is dependent on the values of the involved parameters.

The velocities $f'$, $h$, and temperature $\theta$ for the two-dimensional stagnation flow are plotted in Figs. 3–5. Figure 3 shows the influence of second-grade parameter $\beta$ on $f$ for a two-dimensional

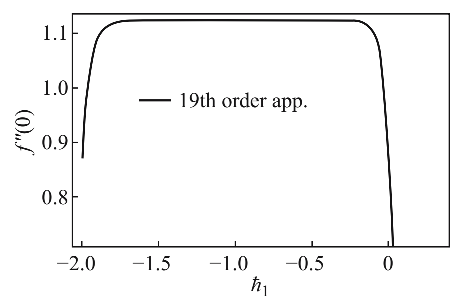


**Fig. 1** The $\hbar$ curve of $f$ for the two-dimensional stagnation flow

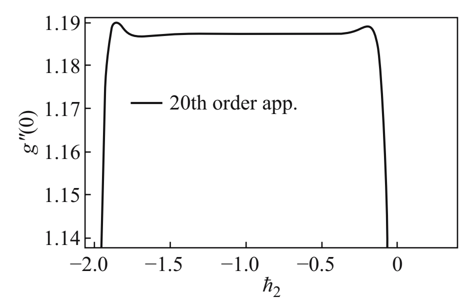


**Fig. 2** The $\hbar$ curve of $g$ for the axisymmetric stagnation flow towards an axisymmetric shrinking surface

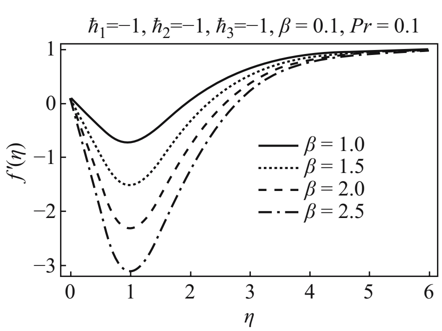


**Fig. 3** Influence of second-grade parameter $\beta$ on $f$ for a two-dimensional stagnation flow

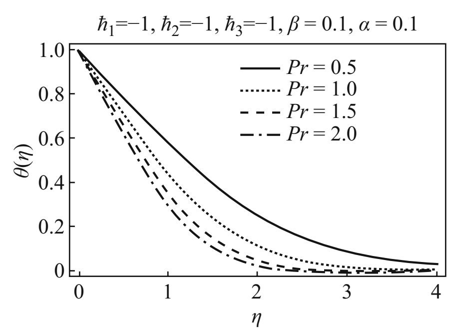


**Fig. 4** Influence of Prandtl number $(Pr)$ on $\theta$ for a two-dimensional stagnation flow

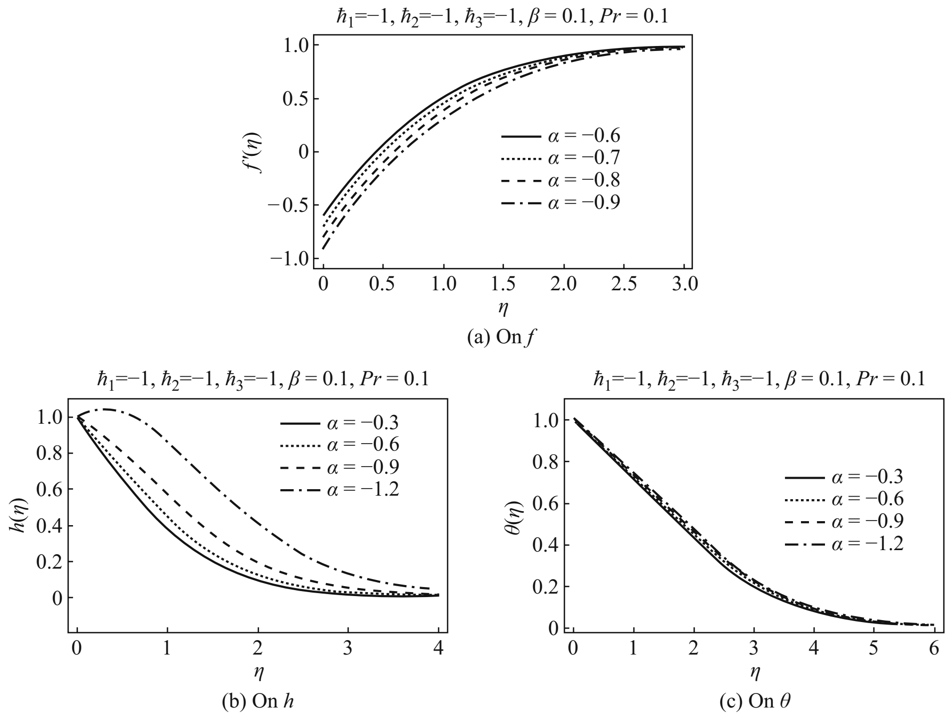


**Fig. 5** Influence of shrinking parameter $\alpha$ on $f$, $h$, and $\theta$ for the two-dimensional stagnation flow

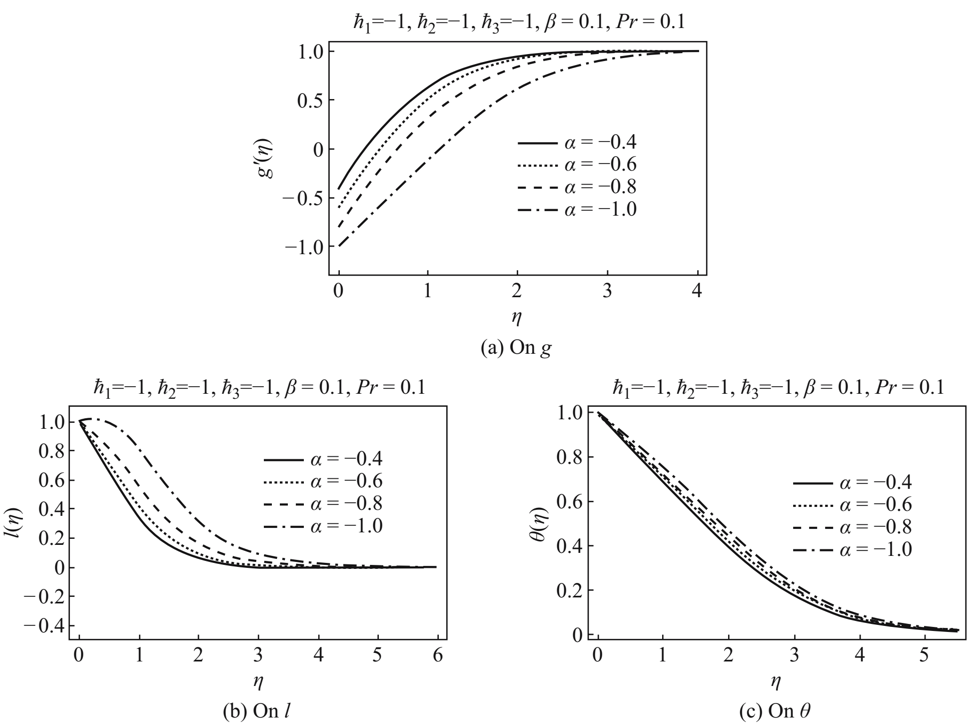


**Fig. 6** Influence of shrinking parameter $\alpha$ on $g$, $l$, and $\theta$ for axisymmetric stagnation flows towards an axisymmetric shrinking surface

stagnation flow, and Fig. 4 shows the influnce of the Prandtl number ($Pr$) on $\theta$ for a two-dimensional stagnation flow. From these figures, we can see that $f'$ decreases for large second-grade parameter $\beta$, $\theta$ decreases with an increase in $Pr$, whereas the boundary layer thickness decreases with the increase in $\eta$. In Fig. 5, the nondimensional velocities $f'$, $h$, and $\theta$ against $\eta$ for different values of shrinking parameter $\alpha < 0$ are displayed. From the figure, we can see that $f'$ increases with an increase in $\alpha$ whereas $h$ and $\theta$ decreases with the increase in $\alpha$.

Figure 6 showns $g'$, $l$, and $\theta$ against $\eta$ for various values of $\alpha$, $\beta$, and $Pr$ for axisymmetric stagnation flows on an axisymmetric shrinking surface. From this figure, we can see that $g'$ increases with the increase in shrinking parameter $\alpha$, whereas $l$ and $\theta$ decrease with the increase in $\alpha$ and the boundary layer thickness increases with the increase in $\eta$.

## 5 Concluding remarks

In this paper, stagnation flows of a second-grade fluid due to a shrinking sheet are considered. The series solutions valid for both the two-dimensional and axisymmetric shrinking sheets are obtained by using the HAM. The convergence of the results is shown. The results are presented graphically and the effect of the parameters is discussed. It is concluded that the HAM provides a simple and easy way to control and adjust the convergence region for strong nonlinearity and is applicable to highly nonlinear problems.

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