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# Bifurcation of piecewise-linear nonlinear vibration system of vehicle suspension *

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**Abstract** A kinetic model of the piecewise-linear nonlinear suspension system that consists of a dominant spring and an assistant spring is established. Bifurcation of the resonance solution to a suspension system with two degrees of freedom is investigated with the singularity theory. Transition sets of the system and 40 groups of bifurcation diagrams are obtained. The local bifurcation is found, and shows the overall characteristics of bifurcation. Based on the relationship between parameters and the topological bifurcation solutions, motion characteristics with different parameters are obtained. The results provides a theoretical basis for the optimal control of vehicle suspension system parameters.

**Key words** vehicle suspension system, singularity theory, piecewise-linear nonlinear system, bifurcation

**Chinese Library Classification** TB535, O322
**2000 Mathematics Subject Classification** 34K18, 34A34, 55P40

## Introduction

There are a great deal of nonlinear factors in the vehicle system. With the development of the electricity-liquid control, computer technology, sensors, microprocessor, and new materials, the capability, the control technology and the configuration of vehicle suspension have been improved unceasingly. This requires a nonlinear investigation on the vehicle suspension system to meet the industry development[1-3]. Therefore, it is significant to establish and study a nonlinear model of the vehicle suspension system by the modern nonlinear dynamic theory and the Hopf-bifurcation theory for further study.

## 1 Vehicle suspension system model

A double mass simplified model of the vehicle suspension system is shown in Fig. 1[4]. Here $M$, $m$ are the mass above the spring and below the spring, respectively; $k_1, k_2$ and $c_1, c_2$ are the equivalent stiffness coefficients and damping coefficients, respectively; $k_3, k_4$ and $c_3, c_4$ are

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the tire's equivalent stiffness coefficients and damping coefficients, respectively. The governing differential equations are[5]

$$
\begin{aligned}
&\frac{Mb^2+I}{(a+b)^2}\ddot{x}_1+\frac{Mab-I}{(a+b)^2}\ddot{x}_2+c_1\dot{x}_1-c_1\dot{x}_3+k_1x_1-k_3x_3+g_1(\dot{x}_1,\dot{x}_3)+g_2(x_1,x_3)=0,\\
&\frac{Mab-I}{(a+b)^2}\ddot{x}_1+\frac{Mb^2+I}{(a+b)^2}\ddot{x}_2+c_2\dot{x}_2-c_2\dot{x}_4+k_2x_2-k_4x_4+g_3(\dot{x}_2,\dot{x}_4)+g_4(x_2,x_4)=0,\\
&m\ddot{x}_3-c_1\dot{x}_1+(c_1+c_3)\dot{x}_3-k_1x_1+(k_1+k_3)x_3+g_1(\dot{x}_1,\dot{x}_3)+g_2(x_1,x_3)=F_3,\\
&m\ddot{x}_4-c_2\dot{x}_2+(c_2+c_4)\dot{x}_4-k_2x_2+(k_2+k_4)x_4+g_3(\dot{x}_2,\dot{x}_4)+g_4(x_2,x_4)=F_4,
\end{aligned}
$$

where $g_1, g_2, g_3$, and $g_4$ are nonlinear functions.

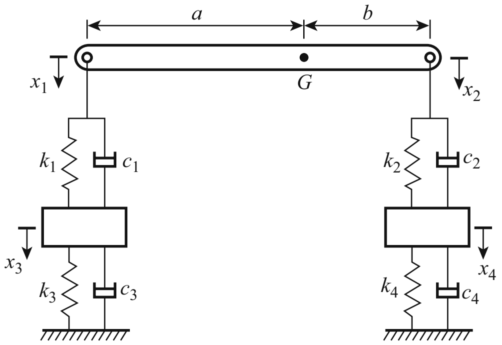



**Fig. 1** A simplified model of the vehicle suspension

We assume that $k_0, c_0$ are the equivalent stiffness coefficient and damping coefficient of the tire, respectively; $z_2$ is the absolute displacement of $M$; $z_{12}$ is the relative displacement of $m$ versus $M$; $z_0$ is the unevenness function caused by road; the suspension system, consisting of a dominant spring and an assistant spring, is a function of the piecewise-linear force and the damping force. Then, the nonlinear functions $g_1, g_2, g_3$, and $g_4$ can be simplified to

$$
f_1(z_{12})=\begin{cases}0, & z_{12}<e,\\ k_2z_{12}, & z_{12}\geq e,\end{cases}\qquad f_2(\dot{z}_{12})=\begin{cases}0, & z_{12}<e,\\ c_2\dot{z}_{12}, & z_{12}\geq e.\end{cases}
$$

Considering the symmetry of the system, the governing differential equations can be simplified to

$$
\begin{cases}m_1(\ddot{z}_2+\ddot{z}_{12})+f_1(z_{12})+f_2(\dot{z}_{12})+k_1z_{12}+c_1\dot{z}_{12}=-k_0(z_2+z_{12}-z_0)-c_0(\dot{z}_2+\dot{z}_{12}-\dot{z}_0),\\ m_2\ddot{z}_2-f_1(z_{12})-f_2(\dot{z}_{12})-k_1z_{12}-c_1\dot{z}_{12}=0,\end{cases} \tag{1}
$$

where $m_1$ and $m_2$ are the mass above the spring and below the spring, respectively.

## 2 Approximate analytical solution

The first-order approximate equations of the governing differential equations of the vehicle suspension system in the case of primary resonance, i.e., $\lambda_2=\omega$, are obtained by the Krylov-Bogoliulov (KB) method[6]:

$$
\frac{dY}{dt}=\delta_{\mathrm{e}}(Y)Y-\frac{rk_0P}{2m_1\Delta_2\lambda_2}\cos(\theta-\alpha'), \tag{2}
$$

$$
\frac{d\theta}{dt}=p_{\mathrm{e}}(Y)-\omega+\frac{rk_0P}{2m_1\Delta_2\lambda_2 Y}\sin(\theta-\alpha'), \tag{3}
$$

where $Y$ is the vibration amplitude of the suspension system, $\theta$ is the phase, $\lambda_2$ is the second-order natural frequency of the suspension system, $\delta_{\rm e}(Y)$ is the equivalent linear attenuate exponent, and $p_{\rm e}(Y)$ is the equivalent linear natural frequency. Furthermore, $\delta_{\rm e}(Y)$ and $p_{\rm e}(Y)$ are given by

$$\begin{aligned}\delta_{\rm e}(Y) =&\frac{1}{\Delta_2}\{-\frac{1}{2}[\frac{1}{m_2b_1}(a_1-\lambda_2^2)-\frac{m_1+m_2}{m_1m_2}](c_0+c_1+c_2)H(\alpha,Z)+\frac{1}{a_2}(a_1-\lambda_2^2)\\ &-\frac{1}{2}[\frac{c_0}{m_1}+\frac{c_1}{m_1}+\frac{c_1}{m_2}+\frac{c_1}{m_2b_1}(a_1-\lambda_2^2)]\},\end{aligned}\tag{4}$$

$$p_{\rm e}(Y) =\lambda_2+\frac{1}{2\Delta_2\lambda_2}[\frac{1}{m_2b_1}(a_1-\lambda_2^2)-\frac{m_1+m_2}{m_1m_2}](k_0+k_1+k_2)H(\beta,Z),\tag{5}$$

where

$$H(\alpha,Z)=\frac{1}{Z\pi}(\alpha-1)(Z\sec\frac{1}{Z}-\sqrt{1-\frac{1}{Z^2}}),\quad H(\beta,Z)=\frac{1}{Z\pi}(\beta-1)(Z\sec\frac{1}{Z}+\sqrt{1-\frac{1}{Z^2}}),$$
$$\alpha=\frac{c_0+c_1}{c_0+c_1+c_2},\quad \beta=\frac{k_0+k_1}{k_0+k_1+k_2},\quad Z=\frac{Y}{e},\quad a_1=\frac{k_0+k_1}{m_1}+\frac{k_1}{m_1},\quad a_2=-\frac{k_0}{m_1},$$
$$\Delta_2=1+\frac{1}{a_2b_1}(a_1-\lambda_2^2)^2,\quad b_1=\frac{k_1}{m_2},\quad \alpha'=\arctan\frac{c_0\omega}{k_0},\quad P=\sqrt{1+(\frac{c_0\omega}{k_0})^2}.$$

Assuming that $2\lambda_2\approx 2\omega$ in (2) and $2\lambda_2\approx p_{\rm e}(Y)+\omega$ in (3), we can obtain stationary solutions of the system as follows:

$$Y=\frac{\frac{rk_0P}{m\Delta_2}}{\sqrt{[p_{\rm e}^2(Y)-\omega^2]^2+4\omega^2\delta_{\rm e}^2(Y)}},\tag{6}$$

$$\theta=\tan\frac{2\omega\delta_{\rm e}(Y)}{\omega^2-p_{\rm e}^2(Y)}+\alpha'.\tag{7}$$

We choose the system parameters on the basis of practical cases[7] as $m_1=156.8$ kg, $m_2=1\,274$ kg, $k_0=5\,670$ N/cm, $k_1=1\,620$ N/cm, $k_2=4\,860$ N/cm, $c_0=567$ N·s/cm, $c_1=62.8$ N·s/cm, $c_2=25.6$ N·s/cm. When $e=4$ cm, $r=2$ cm, the amplitude-frequency curve and the phase-frequency curve of (6) and (7) are shown in Fig. 2.

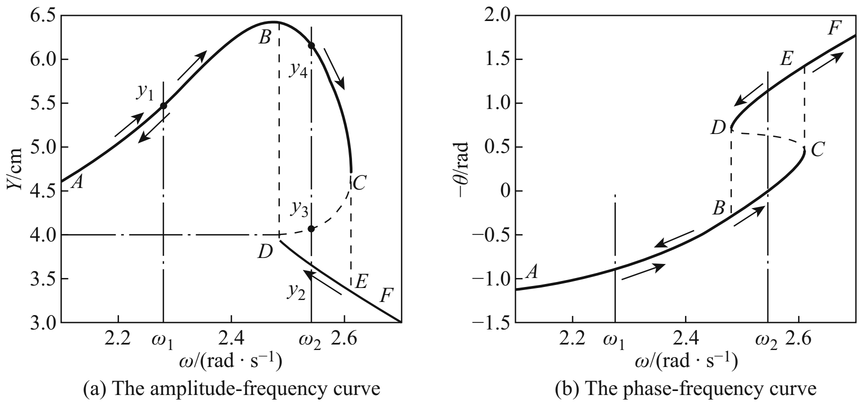



**Fig. 2** The amplitude-frequency curve and the phase-frequency curve of the suspension system (here, $e=4$ cm and $r=2$ cm)

## 3 Bifurcation of stationary solution

From Fig. 2, when $\omega < \omega_D$, there is only one steady steady-state solution from (2) and (3). When $\omega_D < \omega < \omega_E$, there are two steady steady-state solutions and an unsteady steady-state solution, and the unsteady steady-state solution is a saddle-node[8]. When $\omega > \omega_C$, there is only one stationary solution. The physical realization of the stationary solution is determined by the initial condition.

When the frequency of the outside disturbance force decreases slowly, the leap happens from point $D$ to point $B$. Similarly, when the frequency increases slowly, the leap happens from point $C$ to point $E$. Therefore, there are bifurcations at points $D$ and $C$.

The singularity theory is used to analyze the different vibration characteristics of the system with different parameters. By solving (2) and (3), when the suspension system is in resonance, the period bifurcation equation is obtained:

$$Y^2\{[p_{\mathrm{e}}^2(Y)-\omega^2]^2+4\omega^2\delta_{\mathrm{e}}^2(Y)\}=\frac{rk_0B}{m\Delta_2}. \tag{8}$$

When $Z=Z_0$, expanding Eq. (8) by the Taylor progression method, yields

$$\begin{aligned}
&l_2^4f_2^4Z^6+(4l_1l_2f_2^3+4l_2^4f_1f_2^3)Z^5+(6l_1^2l_2^2f_2^2+12l_1l_2^3f_1f_2^2+6l_2^4f_1^2f_2^2-2l_2^2f_2^2\lambda_2^2\\
&-2l_2^2f_2^2\sigma^2+4j_2^2g_2^2\lambda_2^2+4j_2^2g_2^2\sigma^2+4l_2^2f_2^2\lambda_2\sigma-8j_2^2g_2^2\lambda_2\sigma)Z^4+(8j_2^2g_1g_2\lambda_2^2\\
&+12l_1l_2^3f_1^2f_2+12l_1^2l_2^2f_1f_2+8j_1j_2g_2\lambda_2^2-4l_2^2f_1f_2\lambda_2^2-4l_1l_2f_2\lambda_2^2+8l_1l_2f_2\lambda\sigma\\
&+4l_1^3l_2f_2-4l_1l_2f_2\sigma^2+4l_2^4f_1^3f_2+8l_2^2f_1f_2\lambda_2\sigma-16j_2^2g_1g_2\lambda_2\sigma-16j_1j_2g_2\lambda_2\sigma\\
&-4l_2^2f_1f_2\sigma^2+8j_1j_2g_2\sigma^2+8j_2^2g_1g_2\sigma^2)Z^3+(l_2^4f_1^4+4j_1^2\lambda_2^2+4j_1^2\sigma^2-4\lambda_2^3\sigma\\
&-2l_1^2\lambda_2^2-2l_1^2\sigma^2+4\lambda_2\sigma^3+6\lambda_2^2\sigma^2+\lambda_2^4+8l_1l_2f_1\lambda_2\sigma+4j_2^2g_1^2\lambda_2^2-2l_2^2f_1^2\sigma^2\\
&+l_1^4+4l_1^3l_2f_1+4l_1^2\lambda_2\sigma+4l_1l_2^3f_1^3+4j_2^2g_1^2\sigma^2-8j_1^2\lambda_2\sigma+6l_1^2l_2^2f_1^2-2l_2^2f_1^2\lambda_2^2\\
&-4l_1l_2f_1\lambda_2^2-4l_1l_2f_1\sigma^2+4l_2^2f_1^2\lambda_2\sigma+8j_1j_2g_1\lambda_2^2-8j_2^2g_1^2\lambda_2\sigma+8j_1j_2g_1\sigma^2\\
&+\sigma^4-16j_1j_2g_1\lambda_2\sigma)Z^2-\frac{rk_0B}{m_1\Delta_2}=0,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
g_1&=\frac{(\alpha-1)(Z_0^2\sqrt{Z_0^2-1}\pi-2Z_0^2\sqrt{Z_0^2-1}\arcsin\frac{1}{Z_0}+2Z_0^2-2)}{2\pi Z_0^2\sqrt{Z_0^2-1}},\\
g_2&=\frac{2(\alpha-1)(Z_0^2-1)}{\pi Z_0^3\sqrt{Z_0^2-1}},\\
f_1&=\frac{(\beta-1)(Z_0^2\sqrt{Z_0^2-1}\pi-2Z_0^2\sqrt{Z_0^2-1}\arcsin\frac{1}{Z_0}+2Z_0^2+2)}{2\pi Z_0^2\sqrt{Z_0-1}},\\
f_2&=\frac{2(\beta-1)}{\pi Z_0^3\sqrt{Z_0^2-1}},\\
j_1&=(2a_1b_1m_1m_2-2b_1m_1m_2\lambda_2^2-a_1a_2b_1c_0m_2-a_1a_2b_1c_1m_2-a_1a_2b_1c_1m_1-a_1a_2c_1m_1\\
&\quad+a_2b_1c_0m_2\lambda_2^2+a_2b_1c_1m_2\lambda_2^2+a_2b_1c_1m_1\lambda_2^2+a_2c_1m_1\lambda_2^2)/(2a_2b_1m_1m_2\Delta_2),\\
j_2&=(-a_1c_0m_1-a_1c_1m_1-a_1c_2m_1+c_0m_1\lambda_2^2+c_1m_1\lambda_2^2+c_2m_1\lambda_2^2+b_1c_0m_1+b_1c_0m_2\\
&\quad+b_1c_1m_1+b_1c_1m_2+b_1c_2m_1+b_1c_2m_2)/(2b_1m_1m_2\Delta_2),\\
l_1&=\lambda_2,
\end{aligned}$$

$$l_2 = (a_1k_0m_1 + a_1k_1m_1 + a_1k_2m_1 - b_1k_0\lambda_2^2 - b_1k_1\lambda_2^2 - b_1k_2\lambda_2^2 - b_1k_0m_1 - b_1k_1m_1 - b_1k_2m_1 - b_1k_0m_2 - b_1k_1m_2 - b_1k_2m_2)/(2b_1m_1m_2\Delta_2\lambda_2).$$

Omitting the higher order items of (9), we can obtain the bifurcation equation

$$Z^4 + \alpha_1\sigma Z^3 + \alpha_2 Z^3 + \alpha_3\sigma Z^2 + \alpha_4 Z^2 - \alpha_5 = 0, \tag{10}$$

in which

$$\alpha_1 = \frac{n_1}{n_6}, \quad \alpha_2 = \frac{n_2}{n_6}, \quad \alpha_3 = \frac{n_3}{n_6}, \quad \alpha_4 = \frac{n_4}{n_6}, \quad \alpha_5 = \frac{n_5}{n_6},$$

and

$$\begin{aligned}
n_1 &= 8l_1l_2f_2\lambda + 8l_2^2f_1f_2\lambda_2 - 16j_1j_2g_2\lambda_2 - 16j_2^2g_1g_2\lambda_2,\\
n_2 &= 12l_1l_2^3f_1^2f_2 + 12l_1^2l_2^2f_1f_2 + 8j_2^2g_1g_2\lambda_2^2 + 4l_2^4f_1^3f_2 + 4l_1^3l_2f_2 + 8j_1j_2g_2\lambda_2^2\\
&\quad - 4l_1l_2f_2\lambda_2^2 - 4l_2^2f_1f_2\lambda_2^2,\\
n_3 &= -4\lambda_2^3 + 8l_1l_2f_1\lambda_2 + 4l_1^2\lambda_2 - 8j_1^2\lambda_2 + 4l_2^2f_1^2\lambda_2 - 8j_2^2g_1^2\lambda_2 - 16j_1j_2g_1\lambda_2,\\
n_4 &= l_2^4f_1^4 + 4j_1^2\lambda_2^2 - 2l_1^2\lambda_2^2 + \lambda_2^4 + l_1^4 + 4j_2^2g_1^2\lambda_2^2 + 4l_1^3l_2f_1 + 4l_1l_2^3f_1^3 + 6l_1^2l_2^2f_1^2\\
&\quad - 2l_2^2f_1^2\lambda_2^2 - 4l_1l_2f_1\lambda_2^2 + 8j_1j_2g_1\lambda_2^2,\\
n_5 &= \frac{rk_0B}{m\Delta_2},\\
n_6 &= l_1^2l_2^2f_2^2 + 12l_1l_2^3f_1f_2^2 + 6l_2^4f_1^2f_2^2 - 2l_2^2f_2^2\lambda_2^2 + 4j_2^2g_2^2\lambda_2^2.
\end{aligned}$$

## 4 Singularity analysis

According to the Chen-Langford (C-L) method[9], we obtain the transition sets of (8) as follows:

The bifurcation set $B$ is

$$\alpha_3^4 - \alpha_1\alpha_2\alpha_3^3 + \alpha_1^2\alpha_3^2\alpha_4 - \alpha_1^4\alpha_5 = 0.$$

The lagging point set $H$ is

$$Z^4 + \alpha_1\sigma Z^3 + \alpha_2 Z^3 + \alpha_3\sigma Z^2 + \alpha_4 Z^2 - \alpha_5 = 0,$$

in which $\sigma = -(8Z + 3\alpha_2)/(3\alpha_1)$, $-6\alpha_1 Z^2 - 8\alpha_3 Z + 3\alpha_1\alpha_4 - 3\alpha_2\alpha_3 = 0$.

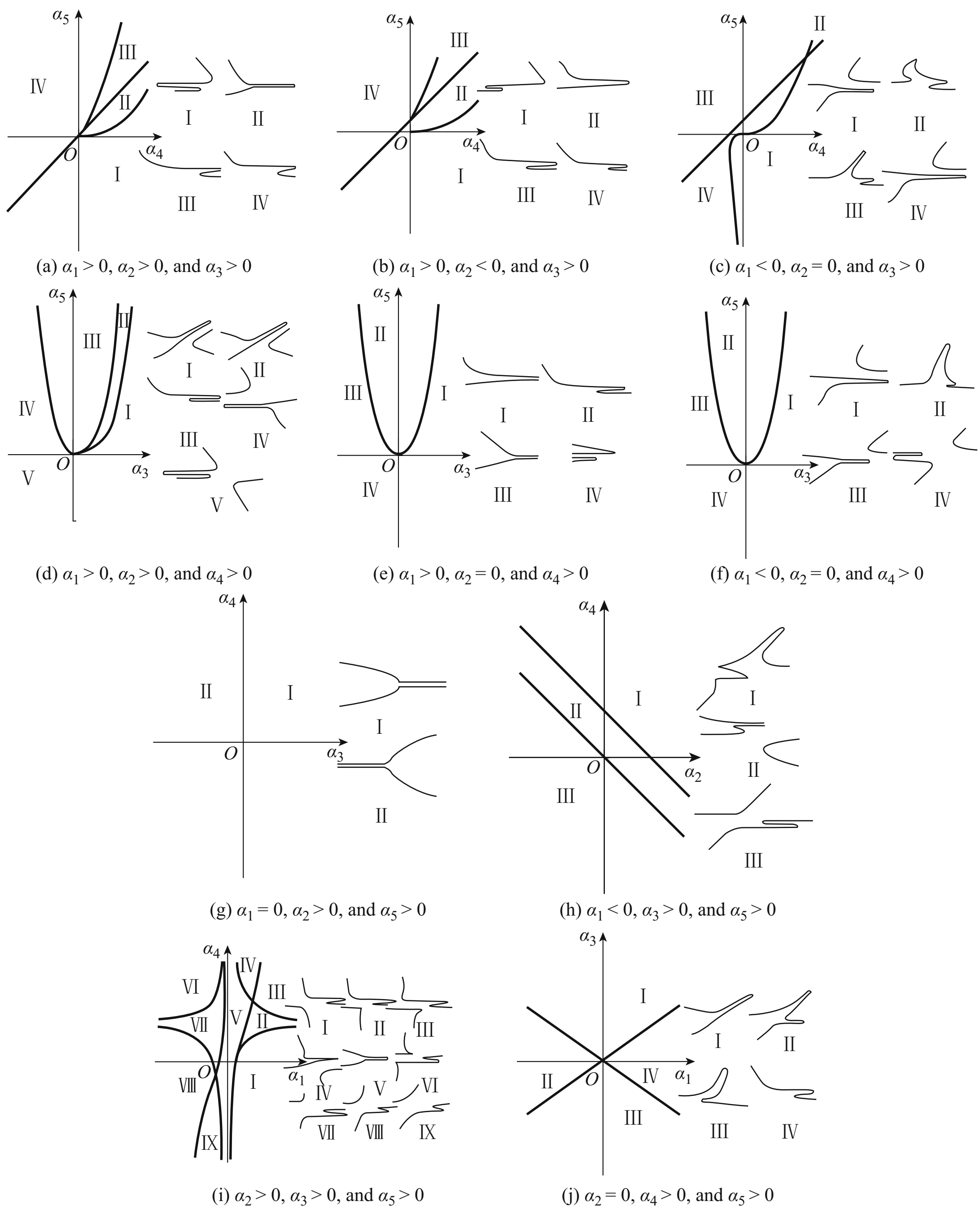
The double limit point set $D$ is an empty set.

The transition set in the breaking parameter interspaces is

$$\Sigma = B \cup H \cup D.$$

Because the parameter interspace is a five-dimensional space, it is difficult to show the transition set in space directly. Therefore, the bifurcation behaviors are mainly discussed in the projective plane of the parameter interval. In addition, because all the breaking parameters are small in quantity, the paper gives various bifurcations with different parameter values. In every projective plane, the curves of the bifurcation set and the lagging point set divide the plane into different subintervals. The corresponding bifurcation curves of two points chosen in the same subinterval are topologically equivalent, i.e., they have the permanence. Any two points in the different subintervals are not topologically equivalent.

By the numerical analysis, the transition sets and 40 groups of bifurcation curves are obtained. Some transition sets in the different parameter projective planes and the corresponding bifurcation curves are shown in Fig. 3. Therefore, various permanence bifurcation curves in the different fields are obtained. From Fig. 3, there are abundant bifurcation behaviors. For example, the bifurcation of field I in Fig. 3(a) is of the double limit point set, fields II and III are of the lagging point set, and the bifurcation in Fig. 3(g) is of the branch. Besides these typical bifurcation types, there are some more complicated bifurcation types, whose characteristics are discussed below based on the physical meaning of the system.



(a) $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_3 > 0$ (b) $\alpha_1 > 0$, $\alpha_2 < 0$, and $\alpha_3 > 0$ (c) $\alpha_1 < 0$, $\alpha_2 = 0$, and $\alpha_3 > 0$

(d) $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_4 > 0$ (e) $\alpha_1 > 0$, $\alpha_2 = 0$, and $\alpha_4 > 0$ (f) $\alpha_1 < 0$, $\alpha_2 = 0$, and $\alpha_4 > 0$

(g) $\alpha_1 = 0$, $\alpha_2 > 0$, and $\alpha_5 > 0$ (h) $\alpha_1 < 0$, $\alpha_3 > 0$, and $\alpha_5 > 0$

(i) $\alpha_2 > 0$, $\alpha_3 > 0$, and $\alpha_5 > 0$ (j) $\alpha_2 = 0$, $\alpha_4 > 0$, and $\alpha_5 > 0$

**Fig. 3** Bifurcation diagrams

**Remark 1** The breaking parameters of field II in Fig. 3(f) are $\alpha_1 < 0, \alpha_2 = 0, \alpha_4 > 0$, and $\alpha_5 > 0$, that is, the system's equivalent damping coefficient $c_2$ is very small, whereas the spring's equivalent stiffness $k_2$ is relatively big, and the exciting force is quite great. The bifurcation curve is shown in Fig. 4. In this case, two leaps happen in a small area when the bifurcation parameter decreases. At point $B$, the humorous response solution leaps to point $C$, then goes to point $D$, leaps to point $E$, and finally goes to point $F$.

**Remark 2** The breaking parameters of field I in Fig. 3(h) are $\alpha_1 < 0, \alpha_2 > 0, \alpha_4 > 0$, and $\alpha_5 > 0$, i.e., the system's equivalent damping coefficient $c_2$ and the spring's equivalent stiffness $k_2$ are both quite small. The bifurcation is shown in Fig. 5. In this case, the humorous response has the lagging phenomenon when the frequency is small. When the bifurcation parameter increases slowly, it leaps from the humorous solution to point $A$, goes to point $B$, leaps to point $C$, goes to point $D$, then leaps to point $F$.

Based on the above discussions, it is not difficult to find that the action of the assistant spring's equivalent damping coefficient is significant. If the damping coefficient is very small, the system may have leaping phenomena easily. Therefore, adjusting the damping coefficient properly to minish the system's vibration can avoid unnecessary leaping phenomena and keep the running-vehicle's stability and security.

By using the data of field III in Fig. 3(j), the resonance curve obtained from Eq. (8) is compared with that of the transition set obtained from the corresponding field with the same parameters to validate the theoretic analysis method imported above. The results are shown in Figs. 6 and 7.

Figure 6 shows the resonance curve obtained from (8), and Fig. 7 shows the bifurcation curve of field III in Fig. 3(j). It is easy to find that the two curves are qualitatively the same. Besides, fields II and III in Fig. 3(c), field I in Fig. 3(b) and 3(g), and field III in Fig. 3(j) can also reflect the vibration characteristics of the common used parameter interval of the system.

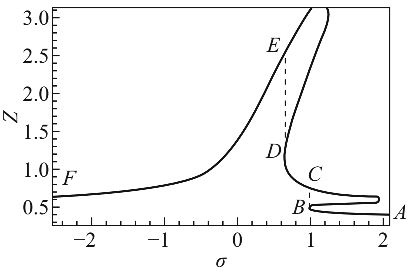



**Fig. 4** Amplificatory bifurcation diagram of field II in Fig. 3(f)

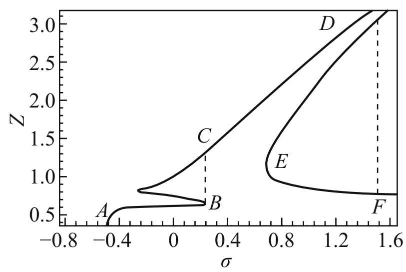



**Fig. 5** Amplificatory bifurcation diagram of field I in Fig. 3(h)

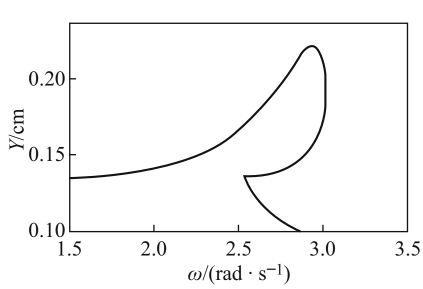



**Fig. 6** Bifurcation diagram obtained by the original bifurcation equation under the same parameter

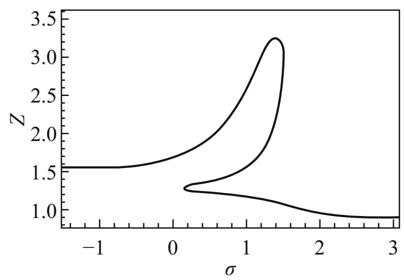



**Fig. 7** Amplificatory bifurcation diagram of field III in Fig. 3(j)

## 5 Conclusions

In this paper, we investigate the resonance solution's bifurcation of the vehicle suspension system with two degrees of freedom by using the averaging method and singularity theory, and find the complicated local bifurcation behaviors. It is helpful to get the system's bifurcation characteristics uniformly and comprehensively. The relationship between the system's parameters and its topological bifurcation solution is established. The system's vibration characteristics with different parameters are analyzed. It provides a theoretical basis for the suspension parameter optimal control.

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