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Analytical solution to stagnation-point flow and heat transfer over a stretching sheet based on homotopy analysis *

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Abstract This paper is concerned with two-dimensional stagnation-point steady flow of an incompressible viscous fluid towards a stretching sheet whose velocity is proportional to the distance from the slit. The governing system of partial differential equations is first transformed into a system of dimensionless ordinary differential equations. Analytical solutions of the velocity distribution and dimensionless temperature profiles are obtained for different ratios of free stream velocity and stretching velocity, Prandtl number, Eckert number and dimensionality index in series forms using homotopy analysis method(HAM). It is shown that a boundary layer is formed when the free stream velocity exceeds the stretching velocity, and an inverted boundary layer is formed when the free stream velocity is less than the stretching velocity. Graphs are presented to show the effects of different parameters.

Key words boundary-layer, heat transfer, stagnation point, stretching sheet, homotopy analysis method

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Introduction

Flow and heat transfer phenomena over a stretching sheet have received great attention during the last decades owing to the abundance of their practical applications in chemical and manufacturing processes, such as polymer extrusion, drawing of copper wires, continuous casting of metals, wire drawing, and glass blowing. The two-dimensional steady flow due to stretching of a sheet is particularly interesting because there is a closed form solution, which has been obtained by Crane[1]. Brady and Acrivos [2] investigated the similarity exact solutions of the steady flow inside a stretching channel and inside a stretching cylinder. Jacobi[3] reported numerical results for a stretched surface with uniform motion. It is worth mentioning that the related problems of a stretched sheet with a linear velocity and different thermal boundary conditions have been studied, theoretically, numerically and experimentally, by many researchers

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such as Gupta and Gupta[4], Hussaini et al.[5], McLeod and Rajagopal[6], Chen and Char[7], Riley and Weidman[8], and Mahapatra and Gupta[9]. Recently, Khan[10] studied heat transfer in a viscoelastic fluid flow over a stretching surface with heat source/sink, suction/blowing and radiation.

In this paper we examine analytically the stagnation-point flow and heat transfer over a stretching surface, which had been considered numerically by Mahapatra and Gupta[9]. The method we employed here is based on the homotopy analytical method (HAM)[11-14] of solving non-linear equations which has already been applied to some other problems[15-18].

1 Mathematical formulation

Consider the steady flow of a viscous and incompressible fluid near the stagnation point of a flat sheet coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the x-axis so that the local tangential velocity is $u_{\mathrm{w}} = cx$, where c is a positive constant. The external stream is set into an impulsive motion from rest with the velocity $u_{\mathrm{e}} = ax$, where $a > 0$ is a constant. The velocity distribution in the frictionless potential flow in the neighborhood of the stagnation at $x = y = 0$ is given by $u = ax, v = -ay$. Under these assumptions, the governing equations can be written as follows:

$$\frac{\partial(x^k u)}{\partial x} + \frac{\partial(x^k v)}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(\frac{u^k}{x^k}) + \frac{\partial^2 u}{\partial y^2}], \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu[\frac{\partial^2 v}{\partial x^2} + \frac{k}{x}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y^2}], \tag{3}$$

$$\rho c_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = \kappa\frac{\partial^2 T}{\partial y^2} + \mu(\frac{\partial u}{\partial y})^2, \tag{4}$$

subject to boundary conditions

$$u(x,0) = cx, \quad v(x,0) = 0, \quad u(x,\infty) = u_e = ax, \quad T(x,0) = T_{\mathrm{w}}, \quad T(x,\infty) = T_\infty. \tag{5}$$

Here, k is the dimensionality index. When $k = 1$, Eqs. (1–5) are the axially symmetric stagnation-point flow; while $k = 0$, they are the plane flow. x and y are the Cartesian coordinates with the origin at the stagnation point along and normal to the plate, respectively. u and v are the velocity components along the x-axes and y-axes, respectively. T_{w} is the wall temperature, T_∞ is the temperature of the fluid far from the sheet. ρ is the density, ν is the kinematic viscosity, c_p is the specific heat capacity and κ the thermal conductivity.

Near the sheet, we assume that the flow field is given by the stream function: $\Psi = \frac{x^{k+1}}{k+1}F(y)$. Then the velocity components are

$$u = \frac{1}{x^k}\frac{\partial \Psi}{\partial y} = \frac{x}{k+1}F'(y), \quad v = -\frac{1}{x^k}\frac{\partial \Psi}{\partial x} = -F(y). \tag{6}$$

Substituting (6) into Eq. (2), the x-momentum equation gives

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = x(\nu\frac{F'''}{k+1} + \frac{FF''}{k+1} - \frac{(F')^2}{(k+1)^2}). \tag{7}$$

We assume

$$P_0 - P = \frac{1}{2}\rho[x^2(\nu\frac{F'''}{k+1} + \frac{FF''}{k+1} - \frac{(F')^2}{(k+1)^2}) + K(y)], \tag{8}$$

where P_0 is the stagnation pressure. When (6) and (8) are inserted in the y-momentum equation, it is easy to obtain

$$\nu F''' + FF'' - \frac{(F')^2}{k+1} = -a^2(k+1), \tag{9}$$

which is subjected to the boundary conditions:

$$F(0) = 0, \quad F'(0) = (k+1)c, \quad F'(+\infty) = a(k+1). \tag{10}$$

Further, introducing the following dimensionless quantities and transformations:

$$f(\eta) = \frac{F(y)}{(c\nu)^{\frac{1}{2}}}, \quad \eta = y(\frac{c}{\nu})^{\frac{1}{2}}, \quad \overline{T} = \frac{T - T_\infty}{T_{\mathrm{w}} - T_\infty}, \quad \overline{T}(x,\eta) = \Theta(\eta) + \frac{cx^2}{\nu}\theta(\eta), \tag{11}$$

Eqs. (1–5) reduce to

$$f'''(\eta) + f(\eta) f''(\eta) - nf'(\eta)^2 + nd^2 = 0, \tag{12}$$

$$\Theta''(\eta) + Prf(\eta)\Theta'(\eta) = 0, \quad \theta''(\eta) + Pr[f(\eta)\theta'(\eta) - 2f'(\eta)\theta(\eta)] = -PrE(f''(\eta))^2, \tag{13}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = d, \quad \Theta(0) = 1, \quad \Theta(\infty) = 0, \quad \theta(0) = 0, \quad \theta(\infty) = 0, \tag{14}$$

where $n = \frac{1}{k+1}$, $d = \frac{a}{c}$, $Pr = \frac{\mu c_p}{k}$ is the Prandtl number, and $E = \frac{\mu}{c_p(T_{\mathrm{w}} - T_\infty)}$ is the Eckert number.

2 HAM solution for $f(\eta)$, $\Theta(\eta)$ and $\theta(\eta)$

2.1 Zeroth-order deformation equation

In order to solve the governing nonlinear equations, we employ the homotopy analysis method. According to the boundary conditions of (14), it is reasonable to assume that $f(\eta)$, $\Theta(\eta)$, and $\theta(\eta)$ can be expressed by the set of base functions $\{\eta^i \exp(-k\eta) \mid i \geq 0, k \geq 0\}$ in the following forms:

$$\begin{cases} f(\eta) = \displaystyle\sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty} a_{i,k}\eta^i \exp(-k\eta), \\ \Theta(\eta) = \displaystyle\sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty} w_{i,k}\eta^i \exp(-k\eta), \\ \theta(\eta) = \displaystyle\sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty} b_{i,k}\eta^i \exp(-k\eta), \end{cases} \tag{15}$$

where $a_{i,k}, b_{i,k}, w_{i,k} (i,\ k = 0, 1, 2, \cdots)$ are coefficients.

Under the first rule of solution expression, the initial guess approximations are

$$f_0(\eta) = 1 + d\eta - d\eta\exp(-\eta) - \exp(-\eta), \quad \Theta_0(\eta) = \exp(-\eta), \quad \theta_0(\eta) = \eta\exp(-\eta), \tag{16}$$

and the auxiliary linear operators are

$$\mathcal{L}_f(f) = f''' + f'', \quad \mathcal{L}_\Theta(\Theta) = \Theta'' - \Theta, \quad \mathcal{L}_\theta(\theta) = \theta'' + \theta'. \tag{17}$$

The operators in the above equations satisfy

$$\begin{cases} \mathcal{L}_f[C_1 + C_2\eta + C_3\exp(-\eta)] = 0, \\ \mathcal{L}_\Theta[C_4\exp(\eta) + C_5\exp(-\eta)] = 0, \\ \mathcal{L}_\theta[C_6 + C_7\exp(-\eta)] = 0, \end{cases} \tag{18}$$

in which $C_i (i = 1, 2, 3, 4, 5, 6, 7)$ are arbitrary constants.

The zeroth order deformation problems are

$$(1-q)\mathcal{L}_f[F(\eta,q)-f_0] = qh_f\mathcal{N}_f[F(\eta,q)], \quad F(0,q)=0, \quad F'(0,q)=1, \quad F'(\infty,q)=d; \tag{19}$$

$$(1-q)\mathcal{L}_\Theta[\Theta^*(\eta,q)-\Theta_0] = qh_\Theta\mathcal{N}_\Theta[\Theta^*(\eta,q)], \quad \Theta^*(0,q)=1, \quad \Theta^*(\infty,q)=0; \tag{20}$$

$$(1-q)\mathcal{L}_\theta[\theta^*(\eta,q)-\theta_0] = qh_\theta\mathcal{N}_\theta[\theta^*(\eta,q)], \quad \theta^*(0,q)=0, \quad \theta^*(\infty,q)=0; \tag{21}$$

where the non-linear differential operators $\mathcal{N}_f$, $\mathcal{N}_\Theta$ and $\mathcal{N}_\theta$ are

$$\mathcal{N}_f[F] = \frac{\partial^3 F}{\partial\eta^3} + F\frac{\partial^2 F}{\partial\eta^2} - n\left[\frac{\partial F}{\partial\eta}\right]^2 + nd^2, \quad \mathcal{N}_\Theta[\Theta^*] = \frac{\partial^2\Theta^*}{\partial\eta^2} + PrF\frac{\partial\Theta^*}{\partial\eta}, \tag{22}$$

$$\mathcal{N}_\theta[\theta^*] = \frac{\partial^2\theta^*}{\partial\eta^2} + PrF\frac{\partial\theta^*}{\partial\eta} - 2Pr\frac{\partial F}{\partial\eta}\theta^* + PrE[\frac{\partial^2 F}{\partial\eta^2}]^2. \tag{23}$$

In the above equations $q \in [0,1]$ is the embedding parameter, h_f, h_Θ, and h_θ are auxiliary non-zero parameters. As q increases from 0 to 1, $F(\eta,q)$ varies from the initial guess $f_0(\eta)$ to the exact solution $f(\eta)$, $\Theta^*(\eta,q)$ varies from the initial guess $\Theta_0(\eta)$ to the exact solution $\Theta(\eta)$, $\theta^*(\eta,q)$ varies from the initial guess $\theta_0(\eta)$ to the exact solution $\theta(\eta)$. Due to Taylor's theorem, one can write

$$F(\eta,q) = F(\eta,0) + \sum_{m=1}^{+\infty} f_m(\eta)q^m, \quad f_m(\eta) = \frac{1}{m!}\frac{\partial^m F(\eta,q)}{\partial q^m}|_{q=0}; \tag{24}$$

$$\Theta^*(\eta,q) = \Theta^*(\eta,0) + \sum_{m=1}^{+\infty} \Theta_m(\eta)q^m, \quad \Theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \Theta^*(\eta,q)}{\partial q^m}|_{q=0}; \tag{25}$$

$$\theta^*(\eta,q) = \theta^*(\eta,0) + \sum_{m=1}^{+\infty} \theta_m(\eta)q^m, \quad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta^*(\eta,q)}{\partial q^m}|_{q=0}. \tag{26}$$

The convergence of the series (24–26) strongly depends upon h_f, h_Θ, and h_θ. Assuming that h_f, h_Θ, and h_θ are so properly selected that the series (24–26) are convergent at $q = 1$, then we have

$$f(\eta) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta), \quad \Theta(\eta) = \Theta_0(\eta) + \sum_{k=1}^{+\infty} \Theta_k(\eta), \quad \theta(\eta) = \theta_0(\eta) + \sum_{k=1}^{+\infty} \theta_k(\eta).$$

2.2 High-order deformation equations

For conciseness, define the vectors

$$\boldsymbol{f}_n(\eta) = \{f_0(\eta), f_1(\eta), f_2(\eta), \cdots, f_n(\eta)\}, \quad \boldsymbol{\Theta}_n(\eta) = \{\Theta_0(\eta), \Theta_1(\eta), \Theta_2(\eta), \cdots, \Theta_n(\eta)\},$$
$$\boldsymbol{\theta}_n(\eta) = \{\theta_0(\eta), \theta_1(\eta), \theta_2(\eta), \cdots, \theta_n(\eta)\}.$$

Differentiating the zeroth order deformation equations (19–21) k times with respect to q, then dividing by $k!$, and finally setting $q = 0$, we get the following Kth-order deformation equations:

$$\mathcal{L}_f[f_k(\eta) - \chi_k f_{k-1}(\eta)] = h_f H_f(\eta) R_k^f(\boldsymbol{f}_{k-1}), \tag{27}$$

$$\mathcal{L}_\Theta[\Theta_k(\eta) - \chi_k \Theta_{k-1}(\eta)] = h_\Theta H_\Theta(\eta) R_k^\Theta(\boldsymbol{\Theta}_{k-1}), \tag{28}$$

$$\mathcal{L}_\theta[\theta_k(\eta) - \chi_k \theta_{k-1}(\eta)] = h_\theta H_\theta(\eta) R_k^\theta(\boldsymbol{\theta}_{k-1}), \tag{29}$$

subject to the boundary conditions:

$$f_k(0) = f'_k(0) = f'_k(+\infty) = \theta_k(0) = \theta_k(+\infty) = \Theta_k(0) = \Theta_k(+\infty) = 0. \tag{30}$$

Here,

$$R_m^{\Theta}(\boldsymbol{\Theta}_{m-1}) = \Theta''_{m-1} + Pr\sum_{s=0}^{m-1} f_s\Theta'_{m-1-s}, \tag{31}$$

$$R_m^{f}(\boldsymbol{f}_{m-1}) = f'''_{m-1} + \sum_{s=0}^{m-1} f_s f''_{m-1-s} - n\sum_{s=0}^{m-1} f'_s f'_{m-1-s} + n(1-\chi_m)d^2, \tag{32}$$

$$R_m^{\theta}(\boldsymbol{\theta}_{m-1}) = \theta''_{m-1} + Pr\sum_{s=0}^{m-1}[f_s\theta'_{m-1-s} - 2f'_s\theta_{m-1-s} + Ef''_s f''_{m-1-s}], \tag{33}$$

and

$$\chi_m = \begin{cases} 0, & m > 1, \\ 1, & m = 1. \end{cases} \tag{34}$$

2.3 Recursive formulae

It is easy to solve the set of linear differential equations (27–30), especially by the symbolic software such as Maple, Mathematic and so on. Maple is used to solve the linear equations (27–30) up to the first few order of approximations and we have solutions of problems as

$$f_m(\eta) = a^0_{m,\,0} + \sum_{k=1}^{m+1}\sum_{i=0}^{2m+2-k} a^i_{m,k}\eta^i\exp(-k\eta), \tag{35}$$

$$\Theta_m(\eta) = w^0_{m,\,0} + \sum_{k=1}^{m+1}\sum_{i=0}^{2m+2-k} w^i_{m,k}\eta^i\exp(-k\eta), \tag{36}$$

$$\theta_m(\eta) = b^0_{m,\,0} + \sum_{k=1}^{m+1}\sum_{i=0}^{2m+2-k} b^i_{m,k}\eta^i\exp(-k\eta). \tag{37}$$

Substituting Eqs. (35–37) into Eqs. (27–30), the recurrence formulaes for the coefficients $a^i_{m,k}$ of $f_m(\eta)$, $b^i_{m,k}$ of $\theta_m(\eta)$, and $w^i_{m,k}$ of $\Theta_m(\eta)$ are obtained for $m \geq 1$,

$$\begin{aligned} a^0_{m,0} = {} & \chi_m a^0_{m-1,0} - \sum_{u=0}^{2m-1}\mu^u_{1,1}(h_f r^u_{m-1,1} + \delta^u_{m,1}) - \sum_{k=2}^{m+1} k\sum_{u=1}^{2m+2-k}\mu^u_{k,1}(\delta^u_{m,k} - \Gamma^u_{m,k}) \\ & + \sum_{k=2}^{m}(k-1)(\sum_{u=1}^{2m-k}\mu^u_{k,0}(h_f r^u_{m-1,k} + \Delta^u_{m,k}) + \frac{1}{k^2(1-k)}(\Delta^0_{m,k} + h_f r^0_{m-1,k})) \\ & - \sum_{k=2}^{m}\sum_{u=1}^{2m-k}\mu^u_{k,1}(h_f r^u_{m-1,k} + \Delta^u_{m,k}) + \sum_{k=2}^{m+1}(k-1)\sum_{u=0}^{2m+2-k}\mu^u_{k,0}(\delta^u_{m,k} - \Gamma^u_{m,k}); \\ a^0_{m,1} = {} & \chi_m a^0_{m-1,1} + \sum_{u=0}^{2m-1}\mu^u_{1,1}(h_f r^u_{m-1,1} + \delta^u_{m,1}) + \sum_{k=2}^{m+1} k\sum_{u=1}^{2m+2-k}\mu^u_{k,1}(\delta^u_{m,k} - \Gamma^u_{m,k}) \\ & + \sum_{k=2}^{m}(-k)(\sum_{u=1}^{2m-k}\mu^u_{k,0}(h_f r^u_{m-1,k} + \Delta^u_{m,k}) + \frac{1}{k^2(1-k)}(\Delta^0_{m,k} + h_f r^0_{m-1,k})) \\ & + \sum_{k=2}^{m}\sum_{u=1}^{2m-k}\mu^u_{k,1}(h_f r^u_{m-1,k} + \Delta^u_{m,k}) - \sum_{k=2}^{m+1} k\sum_{u=0}^{2m+2-k}\mu^u_{k,0}(\delta^u_{m,k} - \Gamma^u_{m,k}); \end{aligned}$$

$$
\begin{aligned}
b^0_{m,0} &= \chi_m b^0_{m-1,0} = 0;\\
b^0_{m,1} &= \chi_m b^0_{m-1,1} - \sum_{k=2}^{m} \frac{1}{k(k-1)}(h_\theta t^0_{m-1,k} - \Lambda^0_{m,k} + \Xi^0_{m,k})\\
&\quad - \sum_{k=2}^{m}\sum_{u=1}^{2m-k} M^u_{k,0}(h_\theta t^u_{m-1,k} + \Xi^u_{m,k} - \Lambda^u_{m,k}) + \sum_{k=2}^{m+1}\sum_{u=0}^{2m+2-k} M^u_{k,0}(\Upsilon^u_{m,k} + \Pi^u_{m,k});\\
w^0_{m,1} &= \chi_m \chi_{2m} w^0_{m-1,1} - \sum_{n=2}^{m+1}\sum_{q=0}^{2m+1-n} \xi^q_{m,n}\mathcal{P}^q_{n,0};\\
w^i_{m,0} &= \chi_m \chi_{2m+1-i} w^i_{m-1,0}, \quad 0 \le i \le 2m+1;\\
w^i_{m,1} &= \chi_m \chi_{2m-i} b^i_{m-1,1} + \sum_{q=i-1}^{2m} \xi^q_{m,1}\mathcal{P}^q_{1,k-1}, \quad 0 \le i \le 2m;\\
w^i_{m,k} &= \chi_m \chi_{2m+1-i-k} b^i_{m-1,k} + \sum_{q=i}^{2m+1-k} \xi^q_{m,k}\mathcal{P}^q_{k,i}, \quad 2 \le k \le m+1, \quad 0 \le i \le 2m+1-k;\\
a^i_{m,1} &= \chi_m a^i_{m-1,1} + \sum_{u=i-1}^{2m-1} \mu^u_{1,i}(h_f r^u_{m-1,1} + \delta^u_{m,1}), \quad 1 \le i \le 2m-1;\\
a^{2m}_{m,1} &= \mu^{2m-1}_{1,2m}(h_f r^{2m-1}_{m-1,1} + \delta^{2m-1}_{m,1}); \quad a^{2m+1}_{m,1} = 0;\\
b^i_{m,1} &= \chi_m b^i_{m-1,1} + \sum_{u=i-1}^{2m-1} M^u_{1,i}(h_\theta t^u_{m-1,1} + \Pi^u_{m,1} - \Upsilon^u_{m,1}), \quad 1 \le i \le 2m-1;\\
b^{2m}_{m,1} &= M^{2m-1}_{1,2m}(h_\theta t^{2m-1}_{m-1,1} + \Pi^{2m-1}_{m,1} - \Upsilon^{2m-1}_{m,1}), \quad b^{2m+1}_{m,1} = 0.
\end{aligned}
$$

For $2 \le k \le m$,

$$
\begin{aligned}
a^0_{m,k} &= \chi_m a^0_{m-1,k} + \sum_{u=0}^{2m+2-k} \mu^u_{k,0}(\delta^u_{m,k} - \Gamma^u_{m,k}) + \sum_{u=1}^{2m-k} \mu^u_{k,0}(h_f r^u_{m-1,k} + \Delta^u_{m,k}).\\
&\quad + \frac{1}{k^2(1-k)}(\Delta^0_{m,k} + h_f r^0_{m-1,k});\\
b^0_{m,k} &= \chi_m b^0_{m-1,k} - \sum_{u=0}^{2m+2-k} M^u_{k,0}(\Upsilon^u_{m,k} + \Pi^u_{m,k} + F^u_{m,k})\\
&\quad + \frac{1}{k(1-k)}(\Xi^0_{m,k} + h_\theta t^0_{m-1,k} - \Lambda^0_{m,k}) + \sum_{u=1}^{2m-k} M^u_{k,0}(h_\theta t^u_{m-1,k} + \Xi^u_{m,k} - \Lambda^u_{m,k}).
\end{aligned}
$$

For $2 \le k \le m, \quad 2 \le i \le 2m-k$,

$$
\begin{aligned}
a^i_{m,k} &= \chi_m a^i_{m-1,k} + \sum_{u=i}^{2m-k} \mu^u_{k,i}(h_f r^u_{m-1,k} + \Delta^u_{m,k}) + \sum_{u=i}^{2m+2-k} \mu^u_{k,i}(\delta^u_{m,k} - \Gamma^u_{m,k});\\
b^i_{m,k} &= \chi_m b^i_{m-1,k} + \sum_{u=i}^{2m-k} M^u_{k,i}(h_\theta t^u_{m-1,k} + \Xi^u_{m,k} - \Lambda^u_{m,k}) + \sum_{u=i}^{2m+2-k} M^u_{k,i}(\Upsilon^u_{m,k} + \Pi^u_{m,k} + F^u_{m,k}).
\end{aligned}
$$

For $2 \le k \le m, \quad 2m+1-k \le i \le 2m+2-k$,

$$a^i_{m,k} = \sum_{u=i}^{2m+2-k} \mu^u_{k,i}(\delta^u_{m,k} - \Gamma^u_{m,k}); \quad a^i_{m,m+1} = \sum_{u=i}^{m+1} \mu^u_{m+1,i}(\delta^u_{m,m+1} - \Gamma^u_{m,m+1});$$

$$b^i_{m,k} = \sum_{u=i}^{2m+2-k} M^u_{k,i}(\Upsilon^u_{m,k} + \Pi^u_{m,k} + \mathcal{F}^u_{m,k});$$

$$b^i_{m,m+1} = \sum_{u=i}^{m+1} M^u_{m+1,i}(\Upsilon^u_{m,m+1} + \Pi^u_{m,m+1} + \mathcal{F}^u_{m,m+1}).$$

Here,

$$M^u_{k,i} = \frac{u!}{i!}, \quad \mu^u_{k,i} = \frac{i!(u-i+2)}{u!}, \quad k=1, \quad 0 \le i \le u+1;$$

$$M^u_{k,i} = \frac{u!}{i!(k-1)^{u-i+1}}\{1-(\frac{1}{n})^{u-i+1}\}, \quad k \ge 2, \quad 0 \le i \le u;$$

$$\mathcal{P}^u_{1,i} = \frac{u!}{(i+1)!2^{u-i+1}}, \quad u \ge 0, \quad 0 \le i \le u;$$

$$\mathcal{P}^u_{k,i} = \frac{u!}{i!(k-1)^{u-i+1}}\{1-(\frac{1}{n})^{u-i+1}\}, \quad k \ge 2, \quad 0 \le i \le u;$$

$$\mu^u_{k,i} = \sum_{j=i}^{u} \frac{u!}{i!(k+1)^{u-j+1}(k-1)^{j-i+1}}, \quad k \ge 2, \quad 0 \le i \le u, \quad u \ge 0.$$

The coefficients $\delta^i_{m,k}, \Pi^i_{m,k}, \Upsilon^i_{m,k}, \Delta^i_{m,k}, \Xi^i_{m,k}, \Gamma^i_{m,k}, \mathcal{F}^i_{m,k}, \mathcal{J}^i_{m,k}$, and $\xi^i_{m,k}$ for $m \ge 1$ are

$$\delta^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} h_f a^{i-t}_{m-1-s,k-r} d^t_{s,r}, \quad 2 \le k \le m+1;$$

$$\delta^i_{m,1} = \sum_{t=0}^{m-1} h_f a^0_{m-t-1,0} d^0_{t,1}, \quad \Pi^i_{m,1} = \sum_{t=0}^{m-1} Prh_\theta a^0_{m-t-1,0} s^0_{t,1};$$

$$\Pi^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} Prh_\theta a^{i-t}_{m-1-s,k-r} s^t_{s,r}, \quad 2 \le k \le m+1;$$

$$\Upsilon^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} 2Prh_\theta b^{i-t}_{m-1-s,k-r} c^t_{s,r}, \quad 2 \le k \le m+1,$$

$$\Upsilon^i_{m,k} = \sum_{t=0}^{m-k} 2Prh_\theta b^0_{m-t-k,0} c^0_{t+k-1,k}, \quad k=1;$$

$$\Delta^0_{m,k} = \sum_{t=0}^{m-k} h_f a^0_{m-t-k,0} d^0_{t+k-1,k}, \quad \Delta^i_{m,k} = \sum_{t=[\frac{i-k+1}{2}]}^{m-k} h_f a^0_{m-t-k,0} d^i_{t+k-1,k}, \quad 1 \le i;$$

$$\Xi^0_{m,k} = \sum_{t=0}^{m-k} 2Prh_\theta a^0_{m-t-k,0} s^0_{t+k-1,k}, \quad \Xi^i_{m,k} = \sum_{t=[\frac{i-k+1}{2}]}^{m-k} 2Prh_\theta a^0_{m-t-k,0} s^i_{t+k-1,k}, \quad 1 \le i;$$

$$\Gamma^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} h_f n c^{i-t}_{m-1-s,k-r} c^t_{s,r};$$

$$\digamma^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} PrEh_f d^{i-t}_{m-1-s,k-r} d^t_{s,r};$$

$$\mathcal{J}^i_{m,k} = \sum_{s=0}^{m-1} \sum_{r=\max\{1,k+s-m\}}^{\min\{s+1,k-1\}} \sum_{t=\max\{0,i+2s+k-r-2m\}}^{\min\{2s+2-r,i\}} PrEh_\Theta a^{i-t}_{m-1-s,k-r} g^t_{s,r};$$

$$\xi^i_{m,k} = h_\Theta (q^i_{m-1,k} + Pr\mathcal{J}^i_{m,k});$$

where the coefficients $c^i_{m,k}$, $d^i_{m,k}$, $r^i_{m,k}$, $s^i_{m,k}$, $t^i_{m,k}$, $g^i_{m,k}$, and $q^i_{m,k}$ are

$$\begin{aligned}
c^i_{m,k} &= (i+1)a^{i+1}_{m,k} - ka^i_{m,k}, \quad 0 \le i \le 2m; \quad c^{2m+1}_{m,k} = -ka^{2m+1}_{m,k};\\
d^i_{m,k} &= (i+1)(i+2)a^{i+2}_{m,k} - 2k(i+1)a^{i+1}_{m,k} + k^2 a^i_{m,k}, \quad 0 \le i \le 2m-1;\\
d^{2m}_{m,k} &= -2k(2m+1)a^{2m+1}_{m,k} + k^2 a^{2m}_{m,k}, \quad d^{2m+1}_{m,k} = k^2 a^{2m+1}_{m,k};\\
r^i_{m,k} &= (i+1)d^{i+1}_{m,k} - kd^i_{m,k}, \quad 0 \le i \le 2m, \quad r^{2m+1}_{m,k} = -kd^{2m+1}_{m,k};\\
s^i_{m,k} &= (i+1)b^{i+1}_{m,k} - kb^i_{m,k}, \quad 0 \le i \le 2m; \quad s^{2m+1}_{m,k} = -kb^{2m+1}_{m,k};\\
t^i_{m,k} &= (i+1)(i+2)b^{i+2}_{m,k} - 2k(i+1)b^{i+1}_{m,k} + k^2 b^i_{m,k}, \quad 0 \le i \le 2m-1;\\
t^{2m}_{m,k} &= -2k(2m+1)b^{2m+1}_{m,k} + k^2 b^{2m}_{m,k}, \quad t^{2m+1}_{m,k} = k^2 b^{2m+1}_{m,k};\\
g^i_{m,k} &= (i+1)w^{i+1}_{m,k} - kw^i_{m,k}, \quad 0 \le i \le 2m; \quad g^{2m+1}_{m,k} = -kw^{2m+1}_{m,k};\\
q^i_{m,k} &= -(i+1)g^{i+1}_{m,k} - kg^i_{m,k}, \quad 0 \le i \le 2m; \quad q^{2m+1}_{m,k} = -kg^{2m+1}_{m,k}.
\end{aligned}$$

For the detailed procedure of deriving the above relations, the reader can refer to Ref. [11]. Using the above recurrence formulae, we can calculate all coefficients $a^i_{m,k}$, $b^i_{m,k}$, $w^i_{m,k}$ by using only the first six coefficients:

$$a^0_{0,0} = 1, \quad a^1_{0,0} = d, \quad a^0_{0,1} = -1, \quad a^1_{0,1} = -d, \quad b^1_{0,1} = 1, \quad w^0_{0,1} = 1,$$

given by the initial approximations of (16). Therefore, the following explicit, totally analytic solutions of the present flow are

$$f(\eta) = \lim_{N\to+\infty} \sum_{m=0}^{N} f_m(\eta) = \lim_{N\to+\infty} [\sum_{m=0}^{N} a^0_{m,0} + \sum_{k=1}^{N+1} \mathrm{e}^{-k\eta} (\sum_{m=k-1}^{2N} \sum_{i=0}^{2m+1-k} a^i_{m,k}\eta^i)], \tag{38}$$

$$\Theta(\eta) = \lim_{N\to+\infty} \sum_{m=0}^{N} \Theta_m(\eta) = \lim_{N\to+\infty} [\sum_{m=0}^{N} w^0_{m,0} + \sum_{k=1}^{N+1} \mathrm{e}^{-k\eta} (\sum_{m=k-1}^{2N} \sum_{i=0}^{2m+1-k} w^i_{m,k}\eta^i)], \tag{39}$$

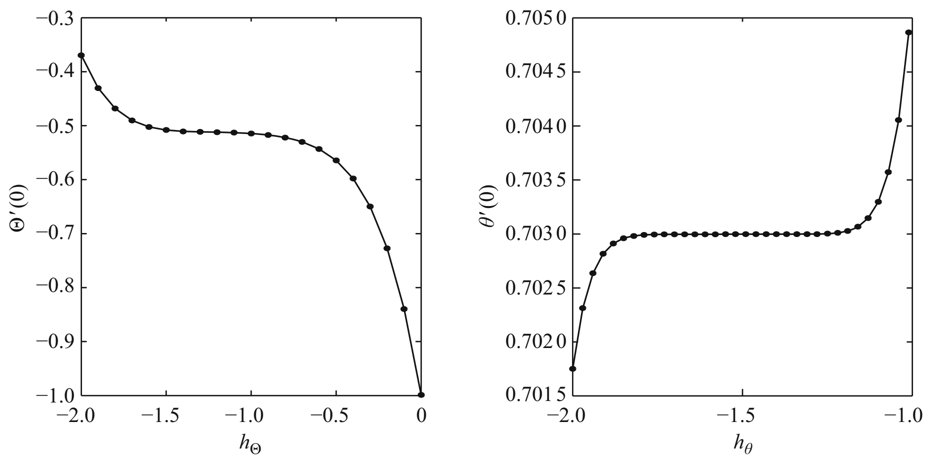
$$\theta(\eta) = \lim_{N\to+\infty} \sum_{m=0}^{N} \theta_m(\eta) = \lim_{N\to+\infty} [\sum_{m=0}^{N} b^0_{m,0} + \sum_{k=1}^{N+1} \mathrm{e}^{-k\eta} (\sum_{m=k-1}^{2N} \sum_{i=0}^{2m+1-k} b^i_{m,k}\eta^i)]. \tag{40}$$

3 Results and discussion

As pointed out by Liao[11], the convergence and rate of approximation for the HAM solution strongly depend on the values of auxiliary parameters h_f, h_θ and h_Θ. The range for the admissible values of h_f is $-2 \le h_f \le 2$ for the plane flow. Our calculations depict that the series (38) of the velocity field converges in the whole region of η for $h_f = 0.13$. It is found that our analytical approximations for $h_f = 0.13$ agree well with the numerical ones by using the shooting technique, as shown in Table 1. From Fig. 1 we observe that the ranges for the admissible values h_Θ and h_θ are $-1.5 < h_\Theta < -0.5, -1.8 < h_\theta < -1.2$.

Table 1 Values of $f''(0)$ for several values of d and n

d	HAM solution $n=1$	Gupta[9] $n=1$	HAM solution $n=\frac{1}{2}$	Numerical solution $n=\frac{1}{2}$
0.1	−0.969 5	−0.969 4	−0.806 3	−0.793 3
0.2	−0.918 4	−0.918 1	−0.750 2	−0.746 8
0.5	−0.667 4	−0.667 3	−0.542 0	−0.533 4
0.8	−0.293 9		−0.223 9	−0.235 5
2.0	2.017 8	2.017 5	1.580 6	1.561 3
3.0	4.729 5	4.729 3	3.652 6	3.633 4
5.0	11.751 4		8.972 1	8.960 8



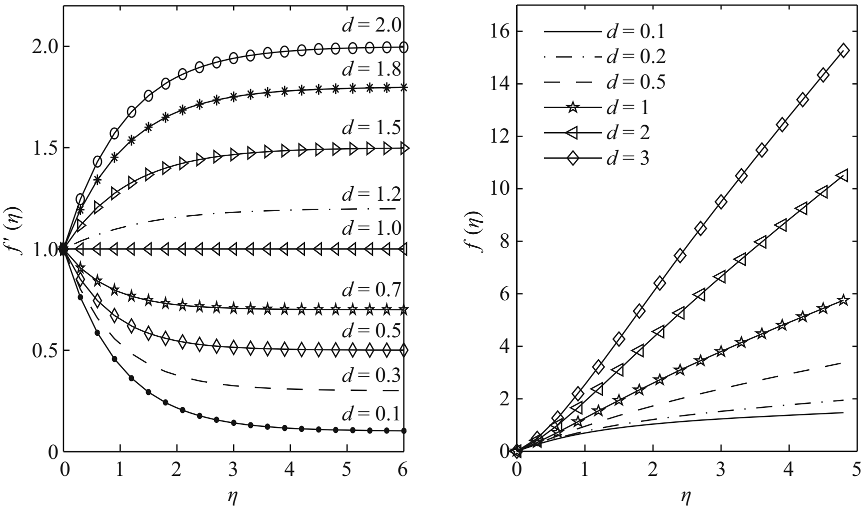
(a) The h_Θ-curve for 9th-order approximation of $\Theta'(0)$ when $Pr = 0.7$, $d = 0.2$

(b) The h_θ-curve for 9th-order approximation of $\theta'(0)$ when $Pr = 0.5$, $d = 0.2$, $E = 5.0$

Fig. 1 h_Θ-curve and h_θ-curve

In this section, the attention has been focused to the variations of $d = a/c$, Pr, n, and E. The horizontal velocity distributions for the plane flow are shown in Fig. 2(a). It is noted that the variation of $f'(\eta)$ depends on the ratio $d = a/c$ of the velocity of the stretching surface to that of the frictionless potential flow in the neighbourhood of the stagnation point and increases with increase of parameter $d = a/c$ for each fixed value of parameter η. It is also observed that the flow has a boundary layer structure when $a/c > 1$ while an inverted boundary layer is formed for $a/c < 1$. The variation of $f'(\eta)$ increases with the decrease of η for each fixed value of parameter $a/c < 1$. Further, the thickness of the boundary layer decreases with increase in $a/c > 1$. The vertical components of velocity $f(\eta)$ for several values of a/c are shown in Fig. 2(b). It can be seen that the variation of $f(\eta)$ increases with increase in a/c at a certain location.

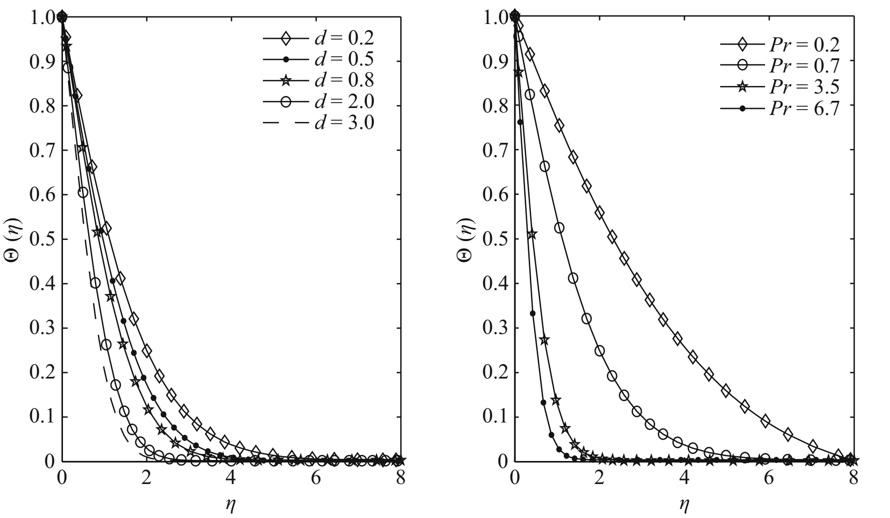
Figures 3–4 have been made in order to see the effects of Pr, d, and E on the dimensionless temperature distributions $\Theta(\eta)$ and $\theta(\eta)$. For the case of $Pr = 0.7$, Fig. 3(a) shows that the variation of $\Theta(\eta)$ for various values of d in the absence of viscous dissipation. It can be clearly seen that the variation of $\Theta(\eta)$ decreases with increase in d for each fixed value of parameter η. Figure 3(b) shows that temperature $\Theta(\eta)$ decreases with increase of Pr for $d = 0.2$. As anticipated, the thermal boundary layer thickness decreases with increasing Prandtl number. The variation of $\theta(\eta)$ for several values of d is shown in Fig. 4(a) for $Pr = 0.05$ and $E = 2$. Figure 4(b) displays in the presence of viscous dissipation, the variation of $\theta(\eta)$ for several



(a) Horizontal velocity $f'(\eta)$ of plane flow for different values of d

(b) Vertical velocity $f(\eta)$ of plane flow for different values of d

Fig. 2 Velocity profiles of plane flow for different values of d



(a) Variation of $\Theta(\eta)$ for some values of d and $Pr = 0.7$, $n = 1$

(b) Variation of $\Theta(\eta)$ for some values of Pr and $d = 0.2$, $n = 1$

Fig. 3 Variation of $\Theta(\eta)$ for different values of d and Pr

values of E when $d = 2.0$ and $Pr = 0.05$. It can be seen that the variation of $\theta(\eta)$ increases with the increase of E.

Finally, the values of the dimensionless shear stress $f''(0)$ at the stretching surface for several values of d are given in Table 1. Our computed results agree well with the results of Gupta et al.[9]. From this table, it is clear that $d = a/c > 1$, the dimensionless shear stress $f''(0)$ increases with increase in a/c and is negative in this case. But when $a/c < 1$, $f''(0)$ decreases with increase in a/c and is negative.

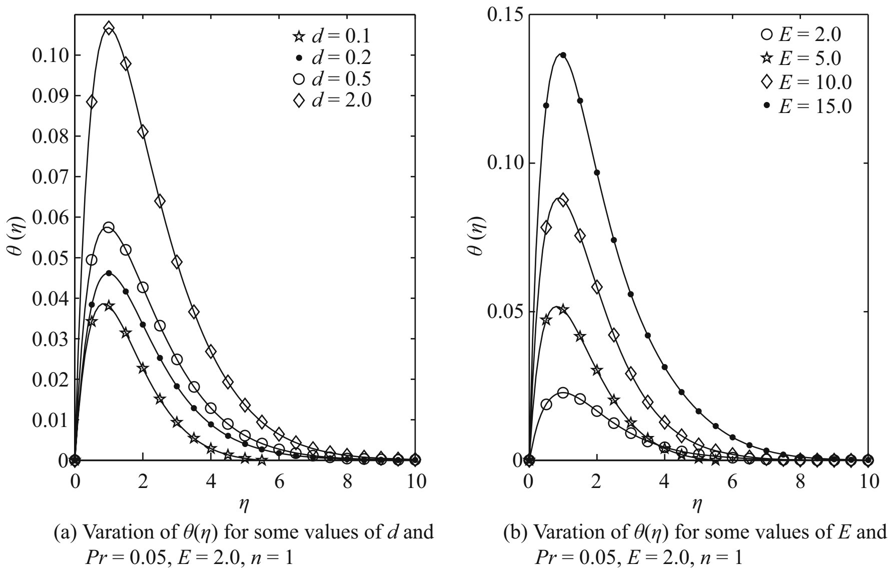


Fig. 4 Variation of $\theta(\eta)$ for different values of d and E

4 Conclusions

We have theoretically studied an exact similarity solution of the Navier-Stokes equations which represents the two-dimensional stagnation-point steady flow of an incompressible viscous fluid towards a stretching sheet and heat transfer. The sheet is stretched in its own plant with a velocity cx proportional to the distance x from the stagnation-point. Analytical solutions for the velocity and temperature distributions are obtained. It is found that the flow has a boundary layer when $a/c > 1$. Furthermore, the thickness of the boundary layer decreases with increase in a/c. On the other hand, an inverted boundary layer is formed when $a/c < 1$. In the presence of viscous dissipation, the variation of $\theta(\eta)$ increases with the increase of E while temperature $\Theta(\eta)$ decreases with increase of Pr without the viscous dissipation. Shear stress and heat flux at the surface are calculated. Shear stress at the surface decreases with increase of a/c when $a/c < 1$ but it increases with increase of a/c when $a/c > 1$.

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