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Limit analysis of viscoplastic thick-walled cylinder and spherical shell under internal pressure using a strain gradient plasticity theory *

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Abstract Plastic limit load of viscoplastic thick-walled cylinder and spherical shell subjected to internal pressure is investigated analytically using a strain gradient plasticity theory. As a result, the current solutions can capture the size effect at the micron scale. Numerical results show that the smaller the inner radius of the cylinder or spherical shell, the more significant the scale effects. Results also show that the size effect is more evident with increasing strain or strain-rate sensitivity index. The classical plastic-based solutions of the same problems are shown to be a special case of the present solution.

Key words thick-walled cylinder and spherical shell, strain gradient, nonlocal, viscoplasticity

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Introduction

In the context of classical plasticity, the internally pressurized thick-walled cylinder and spherical shell have been well studied. Classical plasticity theories assume that the stress state at a material point in a continuum is influenced only by the stress states of the points in the immediate neighborhood of the material point. These plasticity theories do not consider long-range interactions among material points and are therefore local in nature. Lacking an internal length parameter, classical plasticity theories cannot describe the size effect observed in numerous experiments involving a small length scale. This has motivated the development of strain gradient plasticity theories. The simplest version of this strain gradient plasticity theory may be suggested by Mühlhaus and Aifantis[1]. The strain gradient plasticity theory introduces the second-order gradients of the effective plastic strain into the constitutive equation for the flow stress, while leaving all other features of classical plasticity unaltered. This theory has been widely used[2–5].

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The objective of the present paper is to analyze the plastic limit loads of viscoplastic thick-walled cylinders and spherical shells subjected to internal pressures using a strain gradient plasticity theory.

1 Theoretical analysis for thick-walled cylinder

We consider a problem of widening deformation with an internal pressure $P_{\rm i}$ being the driving load. The undeformed inner and outer radii of the cylinder are denoted by a_0 and b_0, and the deformed radii by a and b, respectively.

The plastic limit load of a viscoplastic plane strain cylinder is investigated analytically using a strain gradient plasticity theory.

The viscoplastic behavior including the strain gradient can be described by[2]

$$\sigma_{\rm i} = K\dot{\varepsilon}_{\rm i}^m - c\nabla^2\varepsilon_{\rm i}, \tag{1}$$

where K and m are two material constants; $\sigma_{\rm i}$ and $\dot{\varepsilon}_{\rm i}$ are the effective stress and the effective strain rate, respectively; c is the strain gradient coefficient, and ∇^2 is the Laplacian operator.

In the cylindrical coordinate system, the incompressibility condition requires that

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{2}$$

where v is the radial velocity at a point (r, θ). Furthermore, if we consider a velocity control problem with the innermost edge expanding uniformly at a constant speed V. Then we can rewrite the radial velocity as

$$v = \frac{Va}{r}. \tag{3}$$

Accordingly, we can express the strain rates as

$$\dot{\varepsilon}_r = \frac{\partial v}{\partial r} = -\frac{Va}{r^2}, \tag{4}$$

$$\dot{\varepsilon}_\theta = \frac{v}{r} = \frac{Va}{r^2}, \tag{5}$$

$$\dot{\varepsilon}_z = 0. \tag{6}$$

The equivalent strain rate can be expressed as

$$\dot{\varepsilon}_{\rm i} = \sqrt{\frac{2}{3}(\dot{\varepsilon}_r^2 + \dot{\varepsilon}_\theta^2 + \dot{\varepsilon}_z^2)}. \tag{7}$$

Considering the flow rule, we obtain

$$S_{ij} = \frac{2}{3}\frac{\sigma_{\rm i}}{\dot{\varepsilon}_{\rm i}}\dot{\varepsilon}_{ij}, \tag{8}$$

where S_{ij} is the deviatoric part of the stress tensor, and $\dot{\varepsilon}_{ij}$ is the strain rate tensor. From Eq. (8), we get

$$\dot{\varepsilon}_r = \frac{3}{4}\frac{\dot{\varepsilon}_{\rm i}}{\sigma_{\rm i}}(\sigma_r - \sigma_\theta) = -\dot{\varepsilon}_\theta. \tag{9}$$

The equivalent stress is

$$\sigma_{\rm i} = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r). \tag{10}$$

The equation of equilibrium is

$$r\frac{d\sigma_r}{dr} = \sigma_\theta - \sigma_r. \tag{11}$$

Substituting Eq. (10) into Eq. (11) results in

$$r\frac{d\sigma_r}{dr} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{i}}. \tag{12}$$

Inserting Eq. (1) into Eq. (12) gives

$$r\frac{d\sigma_r}{dr} = \frac{2}{\sqrt{3}}(K\dot{\varepsilon}_{\mathrm{i}}^m - c\nabla^2\varepsilon_{\mathrm{i}}), \tag{13}$$

where

$$\dot{\varepsilon}_{\mathrm{i}} = \frac{2}{\sqrt{3}}\dot{\varepsilon}_\theta = \frac{2}{\sqrt{3}}\frac{Va}{r^2}, \tag{14}$$

$$\varepsilon_{\mathrm{i}} = \frac{2}{\sqrt{3}}\varepsilon_\theta = -\frac{1}{\sqrt{3}}\ln(1+\frac{a_0^2-a^2}{r^2}). \tag{15}$$

The boundary conditions of this problem are

$$\sigma_r|_{r=a} = -P_{\mathrm{i}}, \tag{16}$$

$$\sigma_r|_{r=b} = 0. \tag{17}$$

From Eqs. (13)–(17), we obtain

$$\frac{P_{\mathrm{i}}}{K} = \left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2V}{a}\right)^m\frac{1}{m}\left(1-\frac{a^{2m}}{b^{2m}}\right) + \frac{8c}{3K}\int_a^b\frac{1}{r}\frac{a_0^2-a^2}{(r^2+a_0^2-a^2)^2}dr. \tag{18}$$

Furthermore, we get

$$\begin{aligned}\frac{P_{\mathrm{i}}}{K} = {}& \left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2V}{a}\right)^m\frac{1}{m}\left(1-\frac{a^{2m}}{b^{2m}}\right) - \frac{4c}{3K}\frac{1}{a_0^2-a^2}\\ &\cdot\left[\ln\frac{a^2(b^2+a_0^2-a^2)}{a_0^2b^2} + \frac{b^2}{a_0^2-a^2+b^2} - \frac{a^2}{a_0^2}\right].\end{aligned} \tag{19}$$

From Eq. (19) with $c=0$, we obtain

$$\frac{P_{\mathrm{ilocal}}}{K} = \left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2V}{a}\right)^m\frac{1}{m}\left(1-\frac{a^{2m}}{b^{2m}}\right). \tag{20}$$

For the case with $m\to 0$, we reduce the material to the perfectly plastic one such that[6–8]

$$\frac{P_{\mathrm{ilocal}}}{K} = \frac{1}{\sqrt{3}}\ln\frac{b^2}{a^2}. \tag{21}$$

2 Theoretical analysis for thick-walled spherical shell

The analysis for this case is very similar to that for the thick-walled cylinder. The undeformed inner and outer radii of the spherical shell are denoted by a_0 and b_0, and the deformed radii by a and b, respectively.

In the spherical coordinate system, the incompressibility condition requires that

$$\frac{\partial v}{\partial r} + \frac{2v}{r} = 0, \tag{22}$$

where v is the radial velocity at a point (r,θ,φ). Furthermore, if we consider a velocity control problem with the innermost edge expanding uniformly at a constant speed V. Then we can rewrite the radial velocity as

$$v=\frac{Va^2}{r^2}. \tag{23}$$

Accordingly, we can express the strain rates as

$$\dot{\varepsilon}_r=\frac{\partial v}{\partial r}=-\frac{2Va^2}{r^3}, \tag{24}$$

$$\dot{\varepsilon}_\theta=\frac{v}{r}=\frac{Va^2}{r^3}, \tag{25}$$

$$\dot{\varepsilon}_\varphi=\frac{v}{r}=\frac{Va^2}{r^3}. \tag{26}$$

The equivalent strain rate can be expressed as

$$\dot{\varepsilon}_\mathrm{i}=\sqrt{\frac{2}{3}(\dot{\varepsilon}_r^2+\dot{\varepsilon}_\theta^2+\dot{\varepsilon}_\varphi^2)}. \tag{27}$$

And from Eq. (8), we get

$$\dot{\varepsilon}_r=-\frac{\dot{\varepsilon}_\mathrm{i}}{\sigma_\mathrm{i}}(\sigma_\theta-\sigma_r),\quad \dot{\varepsilon}_\theta=\dot{\varepsilon}_\varphi=\frac{\dot{\varepsilon}_\mathrm{i}}{2\sigma_\mathrm{i}}(\sigma_\theta-\sigma_r). \tag{28}$$

The equivalent stress is

$$\sigma_\mathrm{i}=\sigma_\theta-\sigma_r. \tag{29}$$

The equation of equilibrium is

$$\frac{r}{2}\frac{d\sigma_r}{dr}=\sigma_\theta-\sigma_r. \tag{30}$$

Substituting Eq. (29) into Eq. (30) results in

$$\frac{r}{2}\frac{d\sigma_r}{dr}=\sigma_\mathrm{i}. \tag{31}$$

Inserting Eq. (1) into Eq. (31) gives

$$r\frac{d\sigma_r}{dr}=2(K\dot{\varepsilon}_\mathrm{i}^m-c\nabla^2\varepsilon_\mathrm{i}), \tag{32}$$

where

$$\dot{\varepsilon}_\mathrm{i}=2\dot{\varepsilon}_\theta=\frac{2Va^2}{r^3}, \tag{33}$$

$$\varepsilon_\mathrm{i}=2\varepsilon_\theta=\frac{2}{3}\ln\left(\frac{r^3}{r^3-a^3+a_0^3}\right). \tag{34}$$

The boundary conditions of this problem are

$$\sigma_r|_{r=a}=-P_\mathrm{i}, \tag{35}$$

$$\sigma_r|_{r=b}=0. \tag{36}$$

From Eqs. (32)–(36), we obtain

$$\frac{P_\mathrm{i}}{K}=\frac{2}{3}\left(\frac{2V}{a}\right)^m\frac{1}{m}\left(1-\frac{a^{3m}}{b^{3m}}\right)-\frac{2c}{K}\int_a^b\frac{1}{r}\left[\frac{2}{r^2}-\frac{8r(a_0^3-a^3)+2r^4}{(r^3+a_0^3-a^3)^2}\right]dr. \tag{37}$$

Furthermore, we get

$$\begin{aligned}\frac{P_{\rm i}}{K} =& \frac{2}{3}\left(\frac{2V}{a}\right)^{m}\frac{1}{m}\left(1-\frac{a^{3m}}{b^{3m}}\right)-\frac{2c}{K}\Big\{\frac{1}{a^2}-\frac{1}{b^2}-2\Big\{\frac{b}{b^3+a_0^3-a^3}-\frac{a}{a_0^3}\\ &+(a_0^3-a^3)^{-\frac{2}{3}}\ln\frac{b+(a_0^3-a^3)^{\frac{1}{3}}}{a+(a_0^3-a^3)^{\frac{1}{3}}}-\frac{1}{2}(a_0^3-a^3)^{-\frac{2}{3}}\ln\frac{b^2-(a_0^3-a^3)^{\frac{1}{3}}b+(a_0^3-a^3)^{\frac{2}{3}}}{a^2-(a_0^3-a^3)^{\frac{1}{3}}a+(a_0^3-a^3)^{\frac{2}{3}}}\\ &+\sqrt{3}(a_0^3-a^3)^{-\frac{2}{3}}\left[\arctan\frac{2b-(a_0^3-a^3)^{\frac{1}{3}}}{\sqrt{3}(a_0^3-a^3)^{\frac{1}{3}}}-\arctan\frac{2a-(a_0^3-a^3)^{\frac{1}{3}}}{\sqrt{3}(a_0^3-a^3)^{\frac{1}{3}}}\right]\Big\}\Big\}.\end{aligned}\tag{38}$$

From Eq. (38) with $c=0$, we obtain

$$\frac{P_{\rm ilocal}}{K}=\frac{2}{3}\left(\frac{2V}{a}\right)^{m}\frac{1}{m}\left(1-\frac{a^{3m}}{b^{3m}}\right).\tag{39}$$

For the case with $m\to 0$, we reduce the material to the perfectly plastic one such that[7]

$$\frac{P_{\rm ilocal}}{K}=\frac{2}{3}\ln\frac{b^3}{a^3}.\tag{40}$$

3 Numerical results

In the computation, the initial inner and outer radii are denoted as a_0 and b_0, respectively. The action of the internal pressure is simulated with the innermost edge expanding uniformly at a constant speed V. In the following cases, we adopt the consistently non-dimensional parameters[8]: $a_0 = 5.0, b_0 = 10.0, V = 0.01$. The material properties[9] are for an aluminum material having $K = 145$ MPa, $c = -2.5$ N. Let $l = \sqrt{\frac{|c|}{K}} = 0.131\,3$ μm be a characteristic length, which is a material constant. From Figs. 1–3, it is clear that the limit internal pressure predicted by the gradient plasticity solution is indeed size dependent when the inner radius a_0 is very small. The smaller a_0 is, the more evident the size effect becomes. Thereby, it explains the size effect at the micron scale. On the other hand, when the inner radius a_0 is large, the prediction of the gradient plasticity solution approaches that of the classical plasticity solution. This indicates that there is no pronounced size effect if no small length scale is involved. Therefore, the use of classical plasticity to describe macroscopic behavior of an internally pressurized thick-walled cylinder is justified. Results also show that the size effect is more evident with the increasing of strain or strain-rate sensitivity index. The classical plastically-based solution of the same problem is shown as a special case of the present solution. From Figs. 4–6, the numerical result of thick-walled spherical shell is very similar to that for the thick-walled cylinder.

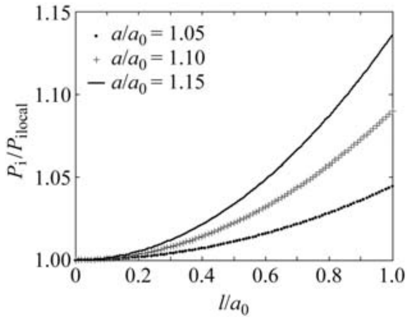

Fig. 1 Effect of deformation on limit internal pressure of the cylinder with $m = 0$

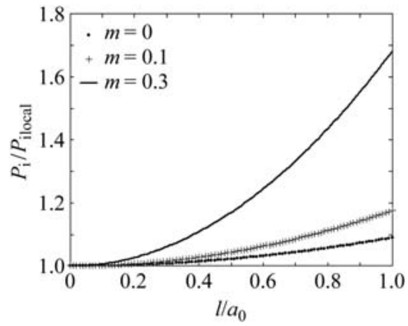

Fig. 2 Effect of strain-rate sensitivity on limit internal pressure of the cylinder with $a/a_0 = 1.1$

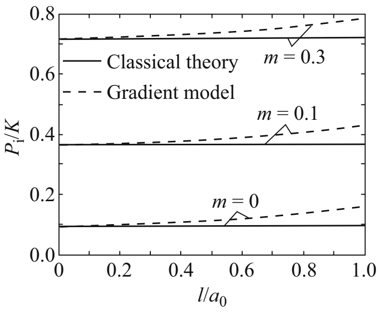

Fig. 3 Comparison of limit internal pressure of the cylinder between the gradient model and the classical theory with $a/a_0 = 1.1$

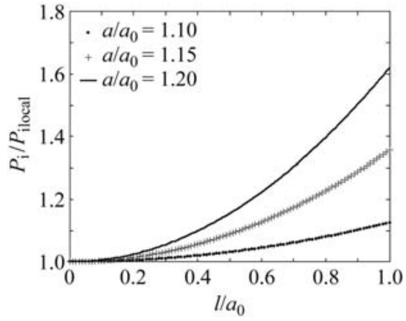

Fig. 4 Effect of deformation on limit internal pressure of the spherical shell with $m = 0$

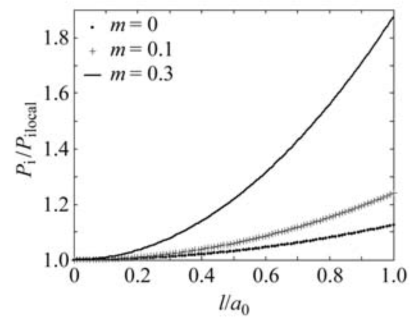

Fig. 5 Effect of strain-rate sensitivity on limit internal pressure of the spherical shell with $a/a_0 = 1.1$

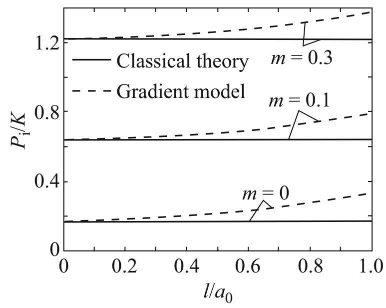

Fig. 6 Comparison of limit internal pressure of the spherical shell between the gradient model and the classical theory with $a/a_0 = 1.1$

4 Conclusions

Plastic limit loads of viscoplastic thick-walled cylinders and spherical shells subjected to internal pressures are investigated analytically using a strain gradient plasticity theory. The classical solutions of the same problems are recovered by the current gradient solution as a special case. The limit internal pressure predicted by the gradient plasticity solutions is indeed size dependent when the inner radius is very small. The size effect is more evident with the increasing of strain or strain-rate sensitivity index.

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