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Analytic solutions to problem of elliptic hole with two straight cracks in one-dimensional hexagonal quasicrystals *

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Abstract By means of the complex variable function method and the technique of conformal mapping, the anti-plane shear problem of an elliptic hole with two straight cracks in one-dimensional hexagonal quasicrystals is investigated. The solution of the stress intensity factor (SIF) for mode III problem has been found. Under the condition of limitation, both the known results and the SIF solution at the crack tip of a circular hole with two straight cracks and cross crack in one-dimensional hexagonal quasicrystals can be obtained.

Key words quasicrystal, elliptic hole with two straight cracks, cross crack, SIF, complex variable function method

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Introduction

Quasicrystals (QCs) were discovered as a new solid structure and material in 1984. The depiction of elastic problem of QCs requires not only phonon fields which describe the vibration of crystal system, but also phason fields which describe the atom arrangement along the direction of quasiperiodic, and they are coupling each other. Since their discovery, many important results have been obtained in all kinds of problems about QCs[1−10]. It is well known that defects greatly influence the application to solid material including QCs. So the study of defects (holes, cracks, dislocations etc.) has attracted the extensive attention of researchers in both experimental and theoretical work. At present, there are some research papers about single defect problem of QCs. A moving screw dislocation in one-dimensional (1D) hexagonal QCs was investigated[11]; a semi-infinite crack in a strip of 1D hexagonal QCs[12] and two semi-infinite collinear cracks in a strip of 1D hexagonal QCs[13] were investigated; the interaction between dislocations and cracks in 1D hexagonal QCs[14,15] were also investigated, etc. The analytic solution of crack problems is very important to fracture mechanics and engineering. In this paper, we consider the problem of an elliptic hole with two straight cracks in one-dimensional (1D) hexagonal QCs.

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1 Formulations

In 1D hexagonal QCs, let $u_i(i=1,2,3)$ and v denote the phonon and phason displacements, respectively,

$$[\varepsilon_{ij}, w_j] = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}, w_3, w_1, w_2],$$
$$[\sigma_{ij}, H_j] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}, H_3, H_1, H_2],$$

where ε_{ij} and $w_j(i,j=1,2,3)$ denote the phonon and phason strains, σ_{ij} and $H_j(i,j=1,2,3)$ denote the phonon and phason stresses, respectively. According to the quasicrystal elasticity theory[12], we have the generalized Hooke's law:

$$\begin{cases} \sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1 w_3, \quad \sigma_{22} = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1 w_3, \\ \sigma_{33} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33} + R_2 w_3, \quad \sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} + R_3 w_2, \\ \sigma_{31} = \sigma_{13} = 2C_{44}\varepsilon_{31} + R_3 w_1, \quad \sigma_{12} = \sigma_{21} = 2C_{66}\varepsilon_{12}, \quad H_1 = 2R_3\varepsilon_{31} + K_2 w_1, \\ H_2 = 2R_3\varepsilon_{23} + K_2 w_2, \quad H_3 = R_1(\varepsilon_{11} + \varepsilon_{22}) + R_2\varepsilon_{33} + K_1 w_3, \end{cases} \tag{1}$$

where C_{ij} and K_i are elastic constants in phonon and phason fields, respectively; R_i is the phonon-phason coupling elastic constant, and $C_{66} = (C_{11} - C_{12})/2$.

The equations of deformation geometry are

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad w_j = \partial_j v, \tag{2}$$

where $\partial_j u_i = \frac{\partial u_i}{\partial x_j}, i,j=1,2,3$, the same notation hereafter.

The equilibrium equations (if the body force is neglected) are given by

$$\begin{cases} \partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} = 0, \quad \partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23} = 0, \\ \partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33} = 0, \quad \partial_1 H_1 + \partial_2 H_2 + \partial_3 H_3 = 0. \end{cases} \tag{3}$$

Equations (1)–(3) are the fundamental equations of elasticity theory of 1D hexagonal QCs.

Assume that cracks penetrate through the solid along the quasiperiodic direction (x_3 direction). In this case, it is evident that all the field variables are independent of x_3. Considering the situation, we have

$$\partial_3 u_i = 0, \quad \partial_3 v = 0, \quad \partial_3\sigma_{ij} = 0, \quad \partial_3 H_j = 0, \quad i,j=1,2,3. \tag{4}$$

Substitution of Eq. (4) in Eqs. (1)–(3) leads to two separate problems. One can be solved by the classical elasticity theory which is not discussed here, and the other is an anti-plane elasticity problem for coupling phonon-phason fields as follows:

$$\sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} + R_3 w_2, \quad \sigma_{31} = \sigma_{13} = 2C_{44}\varepsilon_{31} + R_3 w_1, \tag{5}$$

$$H_1 = 2R_3\varepsilon_{31} + K_2 w_1, \quad H_2 = 2R_3\varepsilon_{23} + K_2 w_2, \tag{6}$$

$$\partial_1\sigma_{31} + \partial_2\sigma_{23} = 0, \quad \partial_1 H_1 + \partial_2 H_2 = 0, \tag{7}$$

$$\varepsilon_{3j} = \varepsilon_{j3} = \frac{1}{2}\partial_j u_3, \quad w_j = \partial_j v, \quad j=1,2. \tag{8}$$

Substituting Eq. (8) into Eqs. (5) and (6) then into Eq. (7), we give

$$C_{44}\nabla^2 u_3 + R_3\nabla^2 v = 0, \quad R_3\nabla^2 u_3 + K_2\nabla^2 v = 0,$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

When $C_{44}K_2 - R_3^2 \neq 0$, we have

$$\nabla^2 u_3 = 0, \quad \nabla^2 v = 0. \tag{9}$$

Equation (9) shows that the terminal governing equations of the anti-plane elasticity problem for coupling phonon-phason fields are two harmonic equations. According to the theory of complex variable function, u_3 and v can be represented as the real part or imaginary part of two analytic functions. We assume that

$$u_3 = \mathrm{Re}\,\varphi_1(z), \quad v = \mathrm{Re}\,\psi_1(z), \tag{10}$$

where $z = x_1 + \mathrm{i}x_2$, Re stands for the real part of complex number.

2 Stress fields to problem of elliptic hole with two straight cracks in 1D hexagonal QCs

In 1D hexagonal QCs, a and b denote the ellipse's semimajor axis and semiminor axis, respectively; $c - a$ is the crack length. Assume that cracks penetrate through the solid along the quasiperiodic direction(x_3 direction). We set the appropriate coordinate system as shown in Fig. 1. From the geometry formation of QCs side, this is a two-dimensional elasticity problem of QCs. Equation (9) is the governing equation of this problem. By Eq. (4), Eq. (3) is simplified as

$$\partial_1\sigma_{11} + \partial_2\sigma_{12} = 0, \quad \partial_2\sigma_{21} + \partial_2\sigma_{22} = 0, \quad \partial_1\sigma_{31} + \partial_2\sigma_{32} = 0, \quad \partial_1 H_1 + \partial_2 H_2 = 0. \tag{11}$$

The anti-plane problem about an elliptic hole with two straight cracks (as shown in Fig. 1) has the boundary conditions as follows:

$$\begin{cases} \sqrt{x_1^2 + x_2^2} \to \infty: \ \sigma_{32} = p, \quad \sigma_{31} = H_2 = H_1 = 0, \\ \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1: \ \sigma_{32} = 0, \quad H_2 = 0, \\ -c < x_1 < -a, \quad x_2 = 0: \ \sigma_{32} = 0, \quad H_2 = 0, \\ a < x_1 < c, \quad x_2 = 0: \ \sigma_{32} = 0, \quad H_2 = 0. \end{cases} \tag{12}$$

The analytic result of linear elasticity theory shows that the above problem is equivalent to the one that the state of stress at infinity is free from external stresses and the edge of the elliptic hole with two straight cracks is the shear stress $\sigma_{32} = -p$. Therefore, we consider the latter problem which has the following boundary conditions:

$$\begin{cases} \sqrt{x_1^2 + x_2^2} \to \infty: \sigma_{32} = \sigma_{31} = H_2 = H_1 = 0, \\ \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1: \sigma_{32} = -p, \quad H_2 = -q, \\ -c < x_1 < -a, \quad x_2 = 0: \sigma_{32} = -p, \quad H_2 = -q, \\ a < x_1 < c, \quad x_2 = 0: \sigma_{32} = -p, \quad H_2 = -q. \end{cases} \tag{13}$$

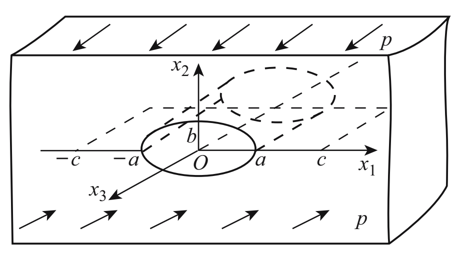

Fig. 1 An elliptic hole with two straight cracks in one-dimensional hexagonal quasicrystals

We solve the boundary value problems (9) and (13) of the above anti-plane problem by complex variable function method as follows.

Denoting $z = x_1 + \mathrm{i}x_2, \mathrm{i} = \sqrt{-1}$, let $f(z)$ is an analytic function, we get

$$\frac{\partial f}{\partial x_1} = \frac{df}{dz} \quad \frac{\partial f}{\partial x_2} = \mathrm{i}\frac{df}{dz}. \tag{14}$$

Let

$$f(z) = P(x_1, x_2) + \mathrm{i}Q(x_1, x_2) = \mathrm{Re}f(z) + \mathrm{i}\mathrm{Im}f(z). \tag{15}$$

By Cauchy-Riemann relation, we have

$$\frac{\partial P}{\partial x_1} = \frac{\partial Q}{\partial x_2}, \quad \frac{\partial P}{\partial x_2} = -\frac{\partial Q}{\partial x_1}. \tag{16}$$

Substituting Eq. (8) into Eqs. (5) and (6), by Eqs. (10) and (14), we have

$$\begin{cases} \sigma_{23} = \sigma_{32} = C_{44}\dfrac{\partial}{\partial x_2}\mathrm{Re}\varphi_1 + R_3\dfrac{\partial}{\partial x_2}\mathrm{Re}\psi_1, \quad \sigma_{31} = \sigma_{13} = C_{44}\dfrac{\partial}{\partial x_1}\mathrm{Re}\varphi_1 + R_3\dfrac{\partial}{\partial x_1}\mathrm{Re}\psi_1, \\ H_1 = K_2\dfrac{\partial}{\partial x_1}\mathrm{Re}\psi_1 + R_3\dfrac{\partial}{\partial x_1}\mathrm{Re}\varphi_1, \quad H_2 = K_2\dfrac{\partial}{\partial x_2}\mathrm{Re}\psi_1 + R_3\dfrac{\partial}{\partial x_2}\mathrm{Re}\varphi_1. \end{cases} \tag{17}$$

Substitution of Eq. (15) in Eqs. (16) and (17), we obtain

$$\sigma_{31} - \mathrm{i}\sigma_{32} = C_{44}\varphi_1' + R_3\psi_1', \quad H_1 - \mathrm{i}H_2 = K_2\psi_1' + R_3\varphi_1', \tag{18}$$

where $\varphi_1' = \frac{d\varphi_1}{dz}$, $\psi_1' = \frac{d\psi_1}{dz}$. By Eq. (18), we have $\sigma_{23} = \sigma_{32} = -\mathrm{Im}(C_{44}\varphi_1' + R_3\psi_1')$, $H_2 = -\mathrm{Im}(K_2\psi_1' + R_3\varphi_1')$, for arbitrary complex function $f(z)$, we have $\mathrm{Im}f(z) = \dfrac{1}{2\mathrm{i}}(f - \overline{f})$, further,

$$\sigma_{23} = \sigma_{32} = -\frac{1}{2\mathrm{i}}[C_{44}(\varphi_1' - \overline{\varphi_1'}) + R_3(\psi_1' - \overline{\psi_1'})], \quad H_2 = \frac{1}{2\mathrm{i}}[K_2(\psi_1' - \overline{\psi_1'}) + R_3(\varphi_1' - \overline{\varphi_1'})]. \tag{19}$$

If L stands for the elliptic hole with two straight cracks, from Eq. (13), we obtain

$$C_{44}(\varphi_1' - \overline{\varphi_1'}) + R_3(\psi_1' - \overline{\psi_1'}) = 2p\,\mathrm{i}, \quad K_2(\psi_1' - \overline{\psi_1'}) + R_3(\varphi_1' - \overline{\varphi_1'}) = 2q\,\mathrm{i}\ , \ z \in L. \tag{20}$$

We find that the function

$$\begin{aligned} z &= \omega(\zeta) \\ &= \frac{a+b}{2}\frac{2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}{8d\zeta} \\ &\quad + \frac{a-b}{2}\frac{8d\zeta}{2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}, \end{aligned} \tag{21}$$

provides a conformal map from the unit disk in the ζ plane to the exterior region of an elliptic hole with two straight cracks in the z plane, and the unit circle τ is transformed into the elliptic hole with two straight cracks L, where we take

$$d = \frac{c + \sqrt{c^2 - a^2 + b^2}}{a+b}, \quad k = \frac{d^4 + 6d^2 + 1}{(d^2-1)^2}. \tag{22}$$

The above conformal mapping maps the upper edge of a onto $A_2(\frac{2d}{d^2+1}, \frac{-(d^2-1)}{d^2+1})$, the lower edge of a onto $A_1(\frac{2d}{d^2+1}, \frac{d^2-1}{d^2+1})$, and $\omega^{-1}(c) \to 1$, $\omega^{-1}(b\mathrm{i}) \to -\mathrm{i}$, $\omega^{-1}(-c) \to -1$, $\omega^{-1}(-b\mathrm{i}) \to \mathrm{i}$, as shown in Fig. 2.

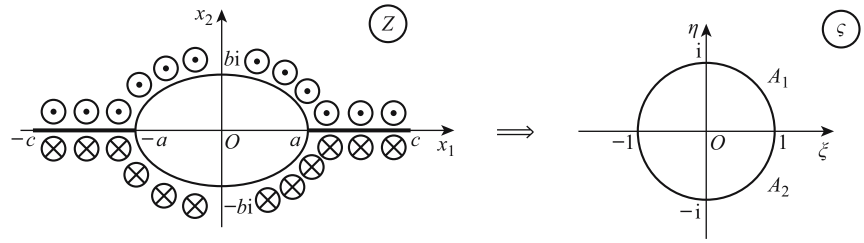

Fig. 2 Conformal mapping from exterior of elliptic hole with two straight cracks onto unit disk

We introduce the notation

$$\varphi(\zeta)=\varphi_1(z)=\varphi_1(\omega(\zeta)),\quad \psi(\zeta)=\psi_1(z)=\psi_1(\omega(\zeta)).$$

By using the chain rule we obtain

$$\varphi_1'(z)=\varphi'(\zeta)/\omega'(\zeta),\quad \psi_1'(z)=\psi'(\zeta)/\omega'(\zeta). \tag{23}$$

Substituting Eq. (23) into (20), then $\zeta\equiv\sigma=\mathrm{e}^{\mathrm{i}\theta}$ into it, we have

$$\begin{aligned}
&\varphi'(\sigma)-\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\frac{R_3}{C_{44}}\left[\psi'(\sigma)-\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\psi'(\sigma)}\right]=\frac{2p\,\mathrm{i}}{C_{44}}w'(\sigma),\\
&\psi'(\sigma)-\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\psi'(\sigma)}+\frac{R_3}{K_2}\left[\varphi'(\sigma)-\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}\right]=\frac{2q\,\mathrm{i}}{K_2}w'(\sigma).
\end{aligned}$$

Multiplying both sides of the above two equations by $\frac{d\sigma}{2\pi\mathrm{i}(\sigma-\zeta)}$, and integrating around the unit circle τ, we have

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\varphi'(\sigma)}{\sigma-\zeta}d\sigma-\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma-\zeta}d\sigma+\frac{R_3}{C_{44}}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\psi'(\sigma)}{\sigma-\zeta}d\sigma-\frac{R_3}{C_{44}}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\psi'(\sigma)}}{\sigma-\zeta}d\sigma\\
&=\frac{2p\,\mathrm{i}}{C_{44}}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma,
\end{aligned}\tag{24}$$

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\psi'(\sigma)}{\sigma-\zeta}d\sigma-\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\psi'(\sigma)}}{\sigma-\zeta}d\sigma+\frac{R_3}{K_2}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\varphi'(\sigma)}{\sigma-\zeta}d\sigma-\frac{R_3}{K_2}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma-\zeta}d\sigma\\
&=\frac{2q\,\mathrm{i}}{K_2}\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma.
\end{aligned}\tag{25}$$

We notice that $\varphi'(\zeta)$ and $\psi'(\zeta)$ are analytic in $|\zeta|<1$ and continuous in $|\zeta|\le 1$. It follows by Cauchy's integral formula in complex functions that

$$\frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\varphi'(\sigma)}{\sigma-\zeta}d\sigma=\varphi'(\zeta),\quad \frac{1}{2\pi\mathrm{i}}\int_\tau\frac{\psi'(\sigma)}{\sigma-\zeta}d\sigma=\psi'(\zeta)\ ,\ |\zeta|<1. \tag{26}$$

Equation (21) gives

$$\omega'(\zeta) = (1-\zeta^2)\left[(d^2+1) + \frac{(d^2-1)(1+k)(1+\zeta^2)}{\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}\right] \cdot \left\{-\frac{a+b}{8d\zeta^2} + \frac{8d(a-b)}{[2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}]^2}\right\}. \tag{27}$$

By Eq. (27), noting that $\overline{\sigma} = 1/\sigma$, we have $\omega'(\sigma)/\overline{\omega'(\sigma)} = -1/\sigma^2$.

Since $\varphi'(\zeta)$ and $\psi'(\zeta)$ are analytic in $|\zeta| < 1$, we can assume that $\varphi'(\zeta)$ and $\psi'(\zeta)$ have Taylor series as follows:

$$\varphi'(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \quad \psi'(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n, \quad |\zeta| < 1.$$

We notice that $\overline{\varphi'(\sigma)}$ and $\overline{\psi'(\sigma)}$ are regarded as the boundary value (the value on the unit circle τ) of $\overline{\varphi'}(\frac{1}{\zeta}) = \sum\limits_{n=0}^{\infty} \overline{a_n}(\frac{1}{\zeta})^n$ and $\overline{\psi'}(\frac{1}{\zeta}) = \sum\limits_{n=0}^{\infty} \overline{b_n}(\frac{1}{\zeta})^n$, respectively. Thus $\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}$ and $\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\overline{\psi'(\sigma)}$ in Eqs. (24) and (25) are regarded as the boundary value of $-\frac{1}{\zeta^2}\overline{\varphi'}(\frac{1}{\zeta})$ and $-\frac{1}{\zeta^2}\overline{\psi'}(\frac{1}{\zeta})$, respectively, which are analytic functions in $|\zeta| > 1$. According to Cauchy's integral formula of infinite region, for $|\zeta| < 1$, we have

$$\frac{1}{2\pi \mathrm{i}}\int_\tau \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma-\zeta}d\sigma = \frac{1}{2\pi \mathrm{i}}\int_\tau \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\psi'(\sigma)}}{\sigma-\zeta}d\sigma = 0. \tag{28}$$

Substituting Eqs. (28) and (27) into Eqs. (24) and (25), we obtain

$$\varphi'(\zeta) + \frac{R_3}{C_{44}}\psi'(\zeta) = \frac{2p\,\mathrm{i}}{C_{44}}\frac{1}{2\pi \mathrm{i}}\int_\tau \frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma, \quad \psi'(\zeta) + \frac{R_3}{K_2}\varphi'(\zeta) = \frac{2q\mathrm{i}}{K_2}\frac{1}{2\pi \mathrm{i}}\int_\tau \frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma. \tag{29}$$

By (27), it is not difficult to find that $\omega'(\zeta)$ is analytic in $|\zeta| > 1$ and continuous in $|\zeta| \geq 1$. According to Cauchy integral formula of infinite region, we have

$$\frac{1}{2\pi \mathrm{i}}\int_\tau \frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma = \omega'(\infty) = \frac{(d^2+1)(a+b)}{4d} \stackrel{\text{def}}{=} F(\zeta). \tag{30}$$

Substituting Eq. (30) into Eq. (29), we have

$$\varphi'(\zeta) + \frac{R_3}{C_{44}}\psi'(\zeta) = \frac{2p\,\mathrm{i}}{C_{44}}F(\zeta), \quad \psi'(\zeta) + \frac{R_3}{K_2}\varphi'(\zeta) = \frac{2q\mathrm{i}}{K_2}F(\zeta). \tag{31}$$

Equation (31) gives

$$\varphi'(\zeta) = 2\mathrm{i}\frac{pK_2 - qR_3}{C_{44}K_2 - R_3^2}F(\zeta), \quad \psi'(\zeta) = 2\mathrm{i}\frac{qC_{44} - pR_3}{C_{44}K_2 - R_3^2}F(\zeta). \tag{32}$$

By Eq. (23), we have

$$\varphi_1'(z) = \frac{\varphi'(\zeta)}{\omega'(\zeta)} = \frac{2\mathrm{i}(pK_2 - qR_3)}{C_{44}K_2 - R_3^2}\frac{F(\zeta)}{\omega'(\zeta)}, \quad \psi_1'(z) = \frac{\psi'(\zeta)}{\omega'(\zeta)} = \frac{2\mathrm{i}(qC_{44} - pR_3)}{C_{44}K_2 - R_3^2}\frac{F(\zeta)}{\omega'(\zeta)}. \tag{33}$$

Substituting Eq. (33) into Eq. (18), we get

$$\sigma_{31} - \mathrm{i}\sigma_{32} = 2p\,\mathrm{i}\frac{F(\zeta)}{\omega'(\zeta)}, \quad H_1 - \mathrm{i}H_2 = 2q\mathrm{i}\frac{F(\zeta)}{\omega'(\zeta)}. \tag{34}$$

Equation (34) indicates that stress distribution has nothing to do with the material constants, which is identical to the classical elasticity theory (linear elasticity fracture mechanics). From Eq. (34), one can obtain $\sigma_{31}, \sigma_{32}, H_1, H_2$. Because of the long calculating, we neglect them here. Substituting $\zeta = \omega^{-1}(z)$ which is given by Eq. (21) into Eq. (33), we can obtain the stress potential functions in the z plane.

3 Computations of SIFs

As we know, the crack tip always causes the destruction of materials. The SIFs at the crack tip can reflect the stress intensity around the crack tip. Therefore, computations and determination of the SIFs are very important. According to quasicrystal elasticity theory[12], the computational formulae of SIFs in ζ plane are

$$K_{\mathrm{III}}^{\parallel} = \lim_{\zeta\to\zeta_1} 2\sqrt{\pi}\,p\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}}, \quad K_{\mathrm{III}}^{\perp} = \lim_{\zeta\to\zeta_1} 2\sqrt{\pi}\;q\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}}, \tag{35}$$

where $\zeta = \zeta_1$ is the corresponding point of the crack tip.

From Eq. (27), one obtain

$$\sqrt{\omega''(\zeta)} \to \frac{\sqrt{(d^2+1)[(d^2-1)a+(d^2+1)b]}}{\sqrt{2d(d^2-1)}}, \quad \zeta \to 1, \tag{36}$$

and by Eq. (30), we have

$$F(\zeta) \to \frac{(d^2+1)(a+b)}{4d}, \quad \zeta \to 1. \tag{37}$$

Further, it is obvious that

$$\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} \to \frac{(a+b)\sqrt{2(d^4-1)}}{4\sqrt{d[(d^2-1)a+(d^2+1)b]}}, \quad \zeta \to 1. \tag{38}$$

Substituting Eq. (38) into Eq. (35), we obtain the SIFs of mode III of phonon and phason fields at $\zeta = 1$,

$$K_{\mathrm{III}}^{\parallel} = \lim_{\zeta\to 1} 2\sqrt{\pi}\,p\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} = \frac{p(a+b)\sqrt{\pi(d^4-1)}}{\sqrt{2d[(d^2-1)a+(d^2+1)b]}}, \tag{39}$$

$$K_{\mathrm{III}}^{\perp} = \lim_{\zeta\to 1} 2\sqrt{\pi}\;q\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} = \frac{q(a+b)\sqrt{\pi(d^4-1)}}{\sqrt{2d[(d^2-1)a+(d^2+1)b]}}. \tag{40}$$

Equations (39) and (40) give the SIFs at the crack tip of an elliptic hole with two straight cracks in 1D hexagonal QCs.

4 Discussion and conclusion about SIFs

We have obtained the SIFs at the crack tip of an elliptic hole with two straight cracks in 1D hexagonal QCs above. These results are discussed further as follows:

(i) From Eqs. (39) and (40), if $b \to 0$, we have

$$K_{\mathrm{III}}^{\parallel} = p\sqrt{\pi c}, \quad K_{\mathrm{III}}^{\perp} = q\sqrt{\pi c}.$$

These are the SIFs at the crack tip of Griffith crack in 1D hexagonal QCs, which are identical to the results in Ref. [12].

(ii) From (39) and (40), if $a \to 0$, we obtain

$$K_{\mathrm{III}}^{\parallel} = p\sqrt{\pi c}, \quad K_{\mathrm{III}}^{\perp} = q\sqrt{\pi c}.$$

These are the SIFs at the crack tip of a cross crack in 1D hexagonal QCs, which are obtained first by us. The results show that if the solid is shear loading, the SIFs at the crack tip of the crack parallel to x_1 axis are not affected by the crack parallel to x_2 axis. Because of its symmetry, the results are the same as the state of Griffith crack, as shown in Fig. 3.

(iii) From (39) and (40), if $a \to b$, we give

$$K_{\mathrm{III}}^{\parallel} = p\sqrt{\pi c\left(1 - \frac{b^4}{c^4}\right)}, \quad K_{\mathrm{III}}^{\perp} = q\sqrt{\pi c\left(1 - \frac{b^4}{c^4}\right)}.$$

These are the SIFs at the crack tip of a circular hole with two straight cracks in 1D hexagonal QCs for infinite region, which are obtained first in this paper, as shown in Fig. 4. Seen from the result, when the circular hole with two straight cracks in QCs is the shear loading, the SIFs at the crack tip are not only related to the radius of a circle, but also the crack length. Especially, if $b \to 0$, we get $K_{\mathrm{III}}^{\parallel} = p\sqrt{\pi c}, K_{\mathrm{III}}^{\perp} = q\sqrt{\pi c}$. The same results obtained as Griffith crack[12].

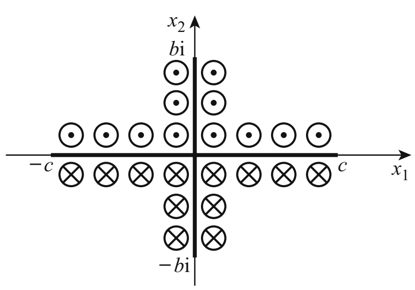

Fig. 3 Cross crack

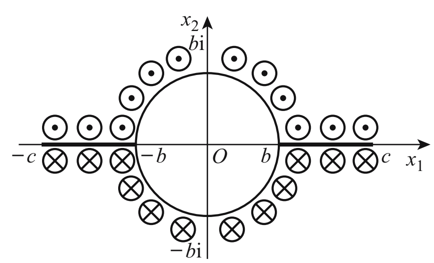

Fig. 4 Circular hole with two straight cracks

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