Applied Mathematics and Mechanics (English Edition), 2007, **28**(9):1173–1180

Solution and its application of transient stream/groundwater model subjected to time-dependent vertical seepage *

TAO Yue-zan (陶月赞), YAO Mei (姚梅), ZHANG Bing-feng (张炳峰)

(School of Civil Engineering, Hefei University of Technology, Hefei 230009, P. R. China)

(Communicated by LI Jia-chun)

Abstract Based on the first linearized Boussinesq equation, the analytical solution of the transient groundwater model, which is used for describing phreatic flow in a semi-infinite aquifer bounded by a linear stream and subjected to time-dependent vertical seepage, is derived out by Laplace transform and the convolution integral. According to the mathematical characteristics of the solution, different methods for estimating aquifer parameters are constructed to satisfy different hydrological conditions. Then, the equation for estimating water exchange between stream and aquifer is proposed, and a recursion equation or estimating the intensity of phreatic evaporation is also proposed. A phreatic aquifer stream system located in Huaibei Plain, Anhui Province, China, is taken as an example to demonstrate the estimation process of the methods stated herein.

Key words stream/groundwater aquifer, time-dependent vertical seepage, parameters of aquifer, water quantity exchange, phreatic evaporation

Chinese Library Classification O357.3
2000 Mathematics Subject Classification 76S05
Digital Object Identifier(DOI) 10.1007/s10483-007-0905-y

Introduction

Transient stream/groundwater flow model, which is used for describing phreatic flow in a semi-infinite aquifer bounded by liner stream, and its analytical solution are interesting problems in the field of seepage-mechanics. They are the theoretical base for analyzing the varying process and the variation regulation of phreatic level bounded by a permeable boundary[1−4], evaluating water quantity exchange between stream and aquifer[5,6], and estimating some aquifer parameters[7]. In recent years, some problems of stream/groundwater flow under the hydro geological conditions such as non horizontal aquifer[8−10] and vertical seepage[11] (includes precipitation or irrigation-water and phreatic evaporation) and so on, have received the widespread attention.

As to the solution of the transient stream/groundwater flow model, even if the vertical seepage is supposed as a constant variable, published solutions[1,2] are so complex that they are difficult for application[11]. If the vertical seepage is a time-dependent variable, the solving methods mentioned herein may provide an approximate solution or a more complex analytical solution. Combing the solving method of Ref. [11] with the convolution integral, a simple analytical solution of the transient stream/groundwater flow subjected to time-dependent vertical seepage is derived. Then, application of the analytical solution is demonstrated by an actual study.

* Received May 31, 2006; Revised May 31, 2007
Project supported by National Natural Science Foundation of China (No. 40474065) and the National TCM Project in the 11th Five-Year Plan Period of China (No. 2006BAB01B01)
Corresponding author TAO Yue-zan, Professor, Doctor, E-mail: taoyuezan@sohu.com

1 Solution of the model subjected to constant vertical seepage

In terms of transient stream/groundwater flow problem (i.e., the J. G. Ferris model, 1950)[11], its hydrogeological conditions can be described as follows. The horizontal phreatic aquifer, which is isotropic and infinite in a two-dimensional plane, is incised completely by a linear stream. In the aquifer, phreatic level is horizontal at the start time and groundwater flow can be treated as one-dimensional flow. Water stage of the stream rises rapidly to a fixed altitude. Under the aforementioned conditions, the model of transient stream/groundwater flow subjected to constant vertical seepage can be written as follows:

$$\begin{cases} \mu\dfrac{\partial h}{\partial t} = k\dfrac{\partial}{\partial x}\left(h\dfrac{\partial h}{\partial x}\right) + \varepsilon & (0 < x < +\infty, t > 0), \\ h(x,t)\,|_{t=0} = h(x,0) & (x > 0), \\ h(x,t)\,|_{x=0} = \Delta H & (t \geq 0), \\ h(x,t)\,|_{x\to\infty} = h(x,0) + \frac{\varepsilon}{\mu}t & (t \geq 0), \end{cases}$$

where μ is specific yield, k (m/d) is the coefficient of permeability, h (m) is groundwater level, ΔH (m) is the fluctuation range of stream stage, ε (m/d) is the vertical seepage intensity (replenishment such as rainfall infiltration is positive, discharge such as phreatic evaporation is negative), x (m) is the distance from the calculating point to the stream boundary, and t (d) is time.

When phreatic fluctuation range $h(x,t) - h(x,0) \leq 0.1h_m$ (where h_m is the average thickness of phreatic flows. This condition can be satisfied normally in actual study[1,3−5,11]). Based on the first linearized Boussinesq equation, the analytical solution of the model can be derived by Laplace transform and the convolution integral and is shown as

$$h(x,t) = h(x,0) + \Delta H \cdot \mathrm{erfc}(\frac{x}{2\sqrt{at}}) + \frac{\varepsilon}{\mu}\int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right)dt, \tag{1}$$

where a (m^2/d) is the coefficient of pressure conductivity and $a = kh_m/\mu$, erf(z) is the error function, and erfc(z) is complementary error function, $z = x/(2\sqrt{at})$.

Compared with the analytical solution provided by published conferences, Eq. (1) will be convenient for application due to its simple expression.

2 Solution of the model subjected to time-dependent vertical seepage

In the groundwater model stated in the earlier paragraph, intensity of vertical seepage ε is treated as a constant variable. However, during a longer evaluation period, phreatic aquifer may be influenced by several seepages, such as phreatic evaporation, precipitation or irrigation-water infiltration. Evaporation is the opposite seepage of infiltration, so ε is not a constant variable. Even if that aquifer is influenced by the same seepage, ε should be treated as a time-dependent variable when its intensity varies obviously.

When ε can not be treated as a constant variable, the evaluation period, as shown in Fig. 1, should be divided into several calculating time-zones according to the actual variation process of ε. In each time zone, the variation range of ε should be small so as that ε can be treated as a stepped variable. In time zone T_m ($T_m = t_{m+1} - t_m$), the average value of ε is ε_m.

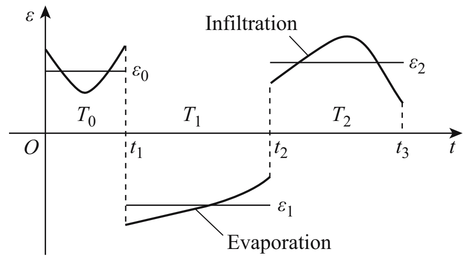

Fig. 1 Variation process of ε and its time zone

In the case of $t > t_m$, Eq. (2) can be obtained from Eq. (1) by convolution integral:

$$h(x,t) = \sum_{i=0}^{m} h(x,t_i),$$

$$h(x,t) = h(x,0) + \Delta H \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \sum_{i=0}^{m} \frac{(\varepsilon_{i+1} - \varepsilon_i)}{\mu} \int_{t_i}^{t} \operatorname{erf} \frac{x}{2\sqrt{a(t-t_i)}} dt. \tag{2}$$

Equation (2) is the analytical solution of the transient stream/groundwater flow subjected to time-dependent vertical seepage. It is the theoretical base for analyzing the varying process and regulation of phreatic level bounded by a permeable boundary.

3 Application of the solution

Transient stream/groundwater flow model and its analytical solution can be used for not only analyzing the variation of phreatic level but also estimating the aquifer parameters of the model, evaluating water quantity exchange between stream and aquifer and estimating the vertical seepage intensity of aquifer.

3.1 Method for estimating aquifer parameters

One important aim of the research on transient stream/groundwater flow problem is to evaluate groundwater-level variation responses to aquifer parameter. Corresponding to the mathematical characteristics of the analytical solution subjected to constant vertical seepage, different methods on aquifer parameter estimation are constructed to satisfy to different hydrological conditions. There are three parameters (a, μ and k) in the model mentioned earlier, and two of them are needed to be estimated based on their relationship. Under normal conditions, the value μ can be determined easily by a simple wild experiment, and a is usually estimated by analyzing groundwater-level variation.

3.1.1 When $\varepsilon \neq 0 \cap \Delta H \neq 0$

If transient flow is subjected to ΔH and ε, fluctuation velocity of phreatic level is $v_t(x,t)$, i.e., $\frac{\partial h(x,t)}{\partial t}$. According to Eq. (1), curve $v_t(x,t)$ has an inflexion-point, and t_k is used for donating the corresponding time. Let $R = \varepsilon/(\mu \cdot \Delta H)$, t_k can be estimated as

$$t_k = \begin{cases} \dfrac{1}{2R}\left[\sqrt{\left(\dfrac{3}{2}\right)^2 + \dfrac{Rx^2}{a}} - \dfrac{3}{2}\right] & (R > 0), \\[2ex] \dfrac{1}{2R}\left[-\sqrt{\left(\dfrac{3}{2}\right)^2 + \dfrac{Rx^2}{a}} - \dfrac{3}{2}\right] & (R < 0). \end{cases} \tag{3}$$

In Eq. (3), ε is positive seepage (such as precipitation infiltration and irrigation-water infiltration) when $R > 0$, ε is negative seepage (such as phreatic evaporation) when $R < 0$. Under the condition of that μ and ε are known, a can be estimated by Eq. (3) based on the value t_k. During the estimation process, the period when ε varies in a small range should be chosen, and value ε should be determined by other methods before estimation.

3.1.2 When $\varepsilon = 0 \cap \Delta H \neq 0$

Under this condition, Eq. (1) is changed into the solution of the J. G. Ferris model, curve $v_t(x,t)/t$ has also an inflexion-point and t_j is used for donating the corresponding time. Then,

$$t_j = \frac{x^2}{6a} \quad (\varepsilon = 0). \tag{4}$$

Equation (4) has been discussed or applied by many references, and it only needs the inflexion-point time in estimating a because of the fact that it has no relationship with specific yield μ.

3.1.3 When $\varepsilon \neq 0 \cap \Delta H = 0$

Under this condition, curve $v_t(x, t)/t$ has no inflexion-point, and so a new type-curve method is proposed for estimating aquifer parameters. When $\Delta H = 0$, Eq. (5) is derived from Eq. (1):

$$\frac{\partial h}{\partial t} = \frac{\varepsilon}{\mu} \operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right). \tag{5}$$

In each actual study, x that denotes the distance between each groundwater well and stream is a known variable, and ε/μ can be treated as a constant variable if ε varies in a small range. Let $\varphi(v) = (\mu/\varepsilon)\cdot\frac{\partial h}{\partial t}$, curve $\varphi(v)/t$ can be plotted out according to the calculation on the actual groundwater level. Under the condition that x is a known value, the type-curve $\operatorname{erf}(z)/t$ can be made corresponding to different a. Fitting the curve $\varphi(v)/t$ with the type-curve $\operatorname{erf}(z)/t$, the coefficient of pressure conductivity a of the actual aquifer is determined.

3.2 Evaluation on water quality exchange between stream and aquifer

In a phreatic aquifer stream system, $q(t)$ ($\mathrm{m}^3/(\mathrm{d}\cdot\mathrm{km})$) denotes the flow rate per unit width and expresses the horizontal water exchange rate from stream to aquifer. By Darcy's law, estimation equation of $q(t)$ can be derived from Eq. (1):

$$q(t) = -kh_m \left.\frac{\partial h}{\partial x}\right|_{x=0},$$
$$q(t) = \mu\Delta H\sqrt{\frac{a}{\pi t}} - \varepsilon t\left(a - 2\sqrt{\frac{a}{\pi t}}\right). \tag{6}$$

In the right side of Eq. (6), the first item and the second clause denote the flow per unit width caused, respectively, only by ΔH and only by ε, and are expressed by $q_j(t)$ and $q_\varepsilon(t)$ respectively. Obviously, $q_j(t)$ is also the flow per unit width of the J. G. Ferris model.

From start to node t, the accumulated flow volume per unit width is $Q(T)$ (m^3/km). By the integral of Eq. (6), $Q(T)$ can be estimated as

$$Q(T) = 2\mu\Delta H\sqrt{\frac{at}{\pi}} - \varepsilon t\left(\frac{at}{2} - \frac{4}{3}\sqrt{\frac{at}{\pi}}\right). \tag{7}$$

$Q_j(T)$ and $Q_\varepsilon(T)$ are, respectively, used for denoting the first and the second clause of the right side of Eq. (7), and moreover, $Q_j(T)$ is the total flow volume per unit width of the J G Ferris model.

Subjected to a time-dependent ε and in the case of $t > t_m$, $Q(T)$ can be derived from Eq. (6) by the same principle that was used to derive Eq. (7) from Eq. (1):

$$Q(T) = \frac{2\mu\Delta H}{\sqrt{\pi}}\sqrt{at} - \sum_{i=0}^{m}(\varepsilon_{i+1} - \varepsilon_i)\cdot(t - t_i)\left[\frac{1}{2}a(t - t_i) - \frac{4}{3\sqrt{\pi}}\sqrt{a(t - t_i)}\right]. \tag{8}$$

In the process of evaluating water quantity exchange or phreatic evaporation (stated as follows), the value of aquifer parameters such as specific yield μ and coefficient of pressure conductivity a must be known.

3.3 Calculation on vertical intensity

Fluctuation range of phreatic level, which is written as $h(x,t) - h(x, 0)$ in Eq. (1), is mainly influenced by three such factors as the fluctuation range of stream-water stage ΔH, the intensity of phreatic evaporation ε and the stream boundary. So, the ΔH influence and boundary condition should be considered in the process of calculation of phreatic evaporation, otherwise, the calculation result would be distorted.

The phreatic fluctuation range only formed by phreatic evaporation is $h_\varepsilon(x,\ t)$ and its corresponding fluctuation velocity is $v_\varepsilon(x,\ t)$. From Eq. (1),

$$v_\varepsilon(x,t) = \frac{\partial h(x,t)}{\partial t} - \frac{\Delta H\cdot x}{2\sqrt{\pi a}}t^{-\frac{3}{2}}\exp\left(-\frac{x^2}{4at}\right). \tag{9}$$

$\frac{\partial h(x,t)}{\partial t}$ can be calculated directly by the wild observation data of phreatic level, and $v_\varepsilon(x,\ t)$ in each time-zone can be calculated by Eq. (9) if value a is known. Under the condition of that $v_\varepsilon(x,\ t)$ of each time-zone is known, based on Eq. (7), the following recursion equation for estimating phreatic evaporation intensity ε_i in time-zone i is obtained:

$$\begin{cases} v_\varepsilon(x,t) = \dfrac{\varepsilon_0}{\mu}\mathrm{erf}\left(\dfrac{x}{2\sqrt{at}}\right) & (t_1 \geq t > 0), \\ v_\varepsilon(x,t) = \dfrac{\varepsilon_0}{\mu}\mathrm{erf}\left(\dfrac{x}{2\sqrt{at}}\right) + \dfrac{\varepsilon_1 - \varepsilon_0}{\mu}\mathrm{erf}\left(\dfrac{x}{2\sqrt{a(t-t_1)}}\right) & (t_2 \geq t > t_1), \\ \cdots & \\ v_\varepsilon(x,t) = \dfrac{\varepsilon_0}{\mu}\mathrm{erf}\left(\dfrac{x}{2\sqrt{at}}\right) + \sum\limits_{i=1}^{m}\dfrac{\varepsilon_i - \varepsilon_{i-1}}{\mu}\mathrm{erf}\left(\dfrac{x}{2\sqrt{a(t-t_i)}}\right) & (t > t_{m-1}). \end{cases} \tag{10}$$

Based on Eq. (10), ε_0 will be estimated at first, then ε_1. Proceeding step by step, ε_m will be estimated at last.

4 Actual study

In the middle part of the Huaibei Plain, Anhui Province, P. R. China, phreatic aquifer is mainly constructed by silt and fine sand, and its thickness is about 8.0 m. The regional buried depth of phreatic level (BDPH) is about 2.5 m. For fairly high permeability and shallow BDPH, phreatic level responds evidently to precipitation, phreatic evaporation, and stream water stage. There is a large channel, which is controlled by a sluice gate in the study area. A national representative auto-log well for observing phreatic level variation is located at 3 km of upper reaches of the sluice. The straight distance of the well aparte from the channel is 65 km. The elevation of ground surface nearby the well is 30.72 m.

The temporal period from July 17 to July 23 in 1994 is selected as the evaluation period. On the afternoon of July 16 in 1994, channel-water stage was raised to 1.46 m quickly due to the sluice gate closure. The evaluation period is divided into two time zones. The two days of July 17 and July 18 are treated as precipitation time zone, and the following five days as evaporation time zone. Based on the wild experiment, specific yield μ of phreatic aquifer is 0.035.

4.1 Estimation of value a

By use of the actual regime of groundwater level in the precipitation time zone, a is estimated first. During the period from July 17 to July 18, the temporal process of rainfall is some what even, and the mean value is 18.0 mm/d. According to the concerning research results, rainfall infiltration coefficient in the area is 0.2. So, the infiltration intensity is 3.6 mm/d in the precipitation time zone.

Groundwater level is extracted from the continuous record of the auto-log well and is used for calculating fluctuation velocity. Based on forward and backward interpolation, the velocity values are v_1 and v_2, respectively. When the time interval of extraction is 6 h, the inflexion-point will occur at approximate 18 h. So, before and after 18 h, the time interval of extraction is encrypted 3 h. The inflexion-point occurs in the time-interval from 15 h to 18 h, so the time of inflexion-point t_k is determined and $t_k = 16.5$ h. Research data and estimation results are given in Table 1 and Fig. 2.

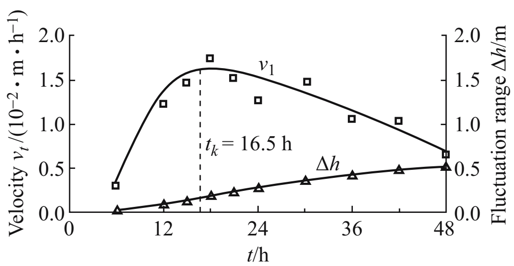

Fig. 2 Temporal variation of groundwater level

Table 1 Phreatic level and its fluctuation (from July 17 to July 18 in 1994)

t/h	6	12	15	18	21	24	30	36	42	48
h/m	27.76	27.83	27.88	27.93	27.97	28.01	28.10	28.16	28.22	28.22
$v_1/(10^{-2}\cdot\mathrm{m}\cdot\mathrm{h}^{-1})$	0.30	1.52	1.47	1.73	1.50	0.63	1.47	1.05	1.03	0.65
$v_2/(10^{-2}\cdot\mathrm{m}\cdot\mathrm{h}^{-1})$	1.22	1.47	1.73	1.50	1.27	1.47	1.05	1.03	0.65	

Note: $v_1(t_i) = (h_{i+1} - t_i)/(t_{i+1} - t_i)$, $v_2(t_i) = (h_i - t_{i-1})/(t_i - t_{i-1})$

Based on the aforementioned data, such as $x = 65$ m, $t_k = 16.5$ h, and $R = 0.0036/(1.46 \times 0.035) = 0.07$, a can be estimated by Eq. (3) and hence $a = 854.82$ m^2/d.

4.2 Evaluation on water quantity exchange

If the aquifer parameters are known, based on the daily mean groundwater level (observation date), the vertical intensity of water quantity exchange between aquifer and atmospheric air can be estimated by Eq. (8), the horizontal intensity of water quantity exchange between aquifer and channel can be estimated by Eq. (10). Concerning data and estimation results are given in Table 2 and Fig. 3.

Table 2 Evaluation on vertical and horizontal intensity of water exchange (July, 1994)

Date		17	18	19	20	21	22	23
Mean groundwater level h/m		27.90	28.19	28.27	28.28	28.28	28.27	28.27
Vertical intensity $\varepsilon/(\mathrm{m}\cdot\mathrm{d}^{-1})$		0.0038	0.0030	−0.0006	−0.0028	−0.0029	−0.0032	−0.0023
Horizontal intensity $/(\mathrm{m}^3\cdot\mathrm{km}^{-1})$	diurnal(q)	0.23	−3.77	−4.42	−2.27	−0.06	2.17	4.42
	accumulated(Q)	0.23	−3.55	−7.96	−10.2	−10.3	−8.13	−3.70

Note: "−" denotes phreatic evaporation in vertical direction or denotes that water flows from aquifer to stream in horizontal direction.

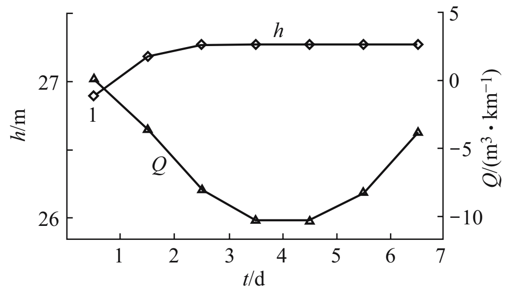

Fig. 3 Varying process of $Q(T)$ and phreatic

Based on this evaluation, the mean evaporation of groundwater is 2.4 mm/d during the period from July 19 to July 23. Within the whole evaluation period of seven days, aquifer discharges to channel as a whole and the accumulated discharging intensity is 3.70 m^3/km.

From Table 2 and Fig. 3, aquifer gets replenishment from channel in the early phase (July 17) of the evaluation period, aquifer discharges to channel in the period from July 18 to 21, and aquifer gets replenishment from channel again since July 22. However, groundwater level has been rising continuously from July 17 to July 21, and this is not synchronous with the variation of water quantity exchange obviously. This asynchronous phenomenon is formed by the precipitation infiltration (water quantity exchange in vertical direction) of the prior period. Hence, vertical infiltration in the actual study can not be neglected.

5 Discussion and conclusions

According to the study on the analytical solution and its application of transient stream/groundwater model subjected to time-dependent vertical seepage, which is combined with an actual example, the following conclusions may be carried out.

(i) Regarding the transient stream/groundwater problem, combining the analytical solution under the condition of constant vertical seepage with the convolution integral, the analytical solution under the condition of time-dependent vertical seepage is derived. This solution can be used not only for analyzing the variation of phreatic level but also for estimating the aquifer parameters of the model, evaluating water quantity exchange between stream and aquifer, and estimating the vertical seepage intensity of aquifer. To the problem of varying stream water stage, the principle of the aforementioned method can also be used for the derivation of the corresponding solution.

(ii) When the groundwater regime influenced by a stream or channel is used to estimate aquifer parameter, inflexion-point method should be used if stream water stage is raised obviously, and type-curve method should be used if stream water stage remains roughly stable.

(iii) Till now, phreatic evaporation estimation has to rely on experience or semi-experience method. It is the transient stream/groundwater flow model and its solution that provides a theoretical and practical method for evaluating the intensity of phreatic evaporation.

(iv) In the actual study, the anaphase process of water quantity exchange between stream and aquifer is influenced by the precipitation infiltration of the prior period. This phenomenon not only shows the importance of vertical seepage in this evaluation, but also reveals that some transformation relationship exists between vertical and horizontal water quantity exchange.

Additionally, in practical research, ε is not only a time-dependent variable but also a space-dependent variable. Therefore, it is very necessary to develop a numerical model.

References

[1] Zheng W Z. Transient flow simulation and groundwater resources evaluation[M]. Beijing: Science Press, 1983 (in Chinese).

[2] Mao C X. Analysis and control of seepage problems[M]. Beijing: Science Press, 1988 (in Chinese).

[3] Sergio E S. Modeling groundwater flow under transient nonlinear free surface[J]. *Journal of Hydrologic Engineering*, 2003, **8**(3):123–132.

[4] Srivastava R. Aquifer response to linearly varying stream stage[J]. *Journal of Hydrologic Engineering*, 2003, **8**(6):361–364.

[5] Edenhofer J, Schmitz G H. Pressure distribution in a semi-infinite horizontal aquifer with steep gradients due to steady recharge and/or drainage: the exact explicit solution[J]. *Transport in Porous Media*, 2001, **45**(3):345–364.

[6] Mishra A, Hata T, Abdelhadi A W. Models for recession flows in the upper Blue Nile river[J]. *Hydrological Processes*, 2004, **18**(15):2773–2786.

[7] Szilagyi J, Parlange M B, Albertson J D. Recession flow analysis for aquifer parameter determination[J]. *Water Resources Research*, 1998, **34**(7):1851–1857.

[8] Basha H A, Maalouf S F. Theoretical and conceptual models of subsurface hillslope flows[J]. *Water Resources Research*, 2005, **41**(7). DOI:10.1029/2004WR003769.

[9] Troch P A, Van Loon A H, Hilberts A G J. Analytical solution of the linearized hillslope-storage Boussinesq equation for exponential hillslope width functions[J]. *Water Resources Research*, 2004, **40**(8). DOI:10.1029/2003 WR002850.

[10] Woo S B, Philip L F L. Water table profiles and discharges for an inclined ditch-drained aquifer under temporally variable recharge[J]. *Journal of Irrigation and Drainage Engineering*, 2003, **129**(2):93–99.

[11] Tao Yuezan, Xi Daoying. Rule of transient phreatic flow subjected to vertical and horizontal seepage[J]. *Applied Mathematics and Mechanics* (*English Edition*), 2006, **27**(1):59–65.