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Analytical solution for functionally graded anisotropic cantilever beam subjected to linearly distributed load *

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Abstract The bending problem of a functionally graded anisotropic cantilever beam subjected to a linearly distributed load is investigated. The analysis is based on the exact elasticity equations for the plane stress problem. The stress function is introduced and assumed in the form of a polynomial of the longitudinal coordinate. The expressions for stress components are then educed from the stress function by simple differentiation. The stress function is determined from the compatibility equation as well as the boundary conditions by a skilful deduction. The analytical solution is compared with FEM calculation, indicating a good agreement.

Key words functionally graded, plane stress problem, stress function, linearly distributed load, analytical solution

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Introduction

Functionally graded material (FGM) is a kind of material in which the individual material composition varies continuously along certain directions in a controllable way. Therefore, the way of change of material properties in these composites may be determined to produce an optimal elastic response. The use of FGMs has been increasing in various engineering applications; these inhomogeneous solids also have received considerable scientific interest and numerous research papers have been published. In the following, however, only works related to functionally graded beams will be referenced.

The problem of simply supported FGM orthotropic beams subjected to arbitrary normal stresses was studied by Sankar[1], who assumed that all the elastic compliance parameters are proportional to $e^{\lambda z}$, where λ is a constant and z is the thickness coordinate. Sankar and Tzeng[2] considered the thermal stress problem of FGM orthotropic beams, of which the elastic compliance parameters are proportional to $e^{\lambda z}$, the thermo-mechanical coupling parameters are proportional to $e^{\gamma z}$, and the temperature increment is proportional to $e^{\kappa z}\sin\xi x$. Under these restrictions, exact solutions have been found. For a simply supported orthotropic beam subjected to arbitrary normal stresses, Zhu and Sankar[3] assumed that the elastic compliance parameters are proportional to a polynomial of z, for which exact solution cannot be obtained by Fourier series expansion method. Thus, they sought for an approximate solution using Galerkin method. If the simply supported beam is anisotropic, Sankar's method would be invalid to obtain any exact solution, even for a homogeneous beam. Using the trial-and-error method, Lekhnitskii[4] investigated nonhomogeneous orthotropic cantilever beams subjected to

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a transverse force and a bending moment at the free end. He assumed that the elastic compliance parameters are functions of the thickness coordinate with no restriction on the form of these functions. It is remarkable that the stress expressions are still very simple and usable.

In this paper, the stress function approach is employed to study the problem of a functionally graded anisotropic cantilever beam subjected to a linearly distributed load. The elastic compliance parameters can be arbitrary functions of the thickness coordinate. The analytical solution can be easily degenerated to that of a homogeneous beam.

1 Basic formulations

The basic equations for plane stress static problem include the equations of equilibrium, strain-displacement relations and stress-strain relations as

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{1}$$

$$\varepsilon_x=\frac{\partial u}{\partial x}, \quad \varepsilon_y=\frac{\partial v}{\partial y}, \quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \tag{2}$$

$$\varepsilon_x=s_{11}\sigma_x+s_{12}\sigma_y+s_{16}\tau_{xy}, \quad \varepsilon_y=s_{12}\sigma_x+s_{22}\sigma_y+s_{26}\tau_{xy}, \quad \gamma_{xy}=s_{16}\sigma_x+s_{26}\sigma_y+s_{66}\tau_{xy}, \tag{3}$$

where σ_x, σ_y and τ_{xy} denote the stress components; ε_x, ε_y and γ_{xy} the strain components; u and v the displacement components. In this paper, we consider a functionally graded cantilever beam whose elastic compliance parameters are functions of $y, s_{ij}=s_{ij}(y), i,j=1,2,6$.

The compatibility equation expressed in strain can be derived from Eq. (2)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}=0. \tag{4}$$

2 Analytical solution

Consider a cantilever beam of unit width, with span l and height h, subjected to a linearly distributed load q as shown in Fig. 1. The boundary conditions are

$$\sigma_y=-q_0 x/l, \quad \tau_{xy}=0, \quad \text{at} \;\; y=-h/2; \tag{5}$$

$$\sigma_y=0, \quad \tau_{xy}=0, \quad \text{at} \;\; y=h/2; \tag{6}$$

$$N_0=0, \quad M_0=0, \quad Q_0=0, \quad \text{at} \;\; x=0; \tag{7}$$

$$u=v=0, \quad \frac{\partial v}{\partial x}=0, \;\; \text{or} \;\; \frac{\partial u}{\partial y}=0, \quad \text{at } (x,y)=(l,0), \tag{8}$$

where q_0 is the load amplitude, and N_0, M_0 and Q_0 are the axial force, bending moment, and shear force at $x=0$, respectively.

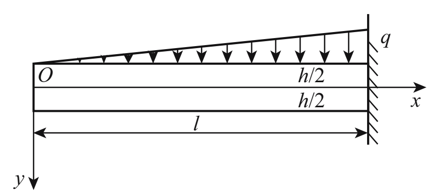

Fig. 1 A cantilever beam, coordinate and loading

The stress components can be expressed with stress function ϕ as

$$\sigma_x=\frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y=\frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy}=-\frac{\partial^2 \phi}{\partial x \partial y}. \tag{9}$$

Then Eq. (1) is satisfied. Further, we assume a polynomial form of the stress function

$$\phi = \phi_0(y) + x\phi_1(y) + x^2\phi_2(y) + x^3\phi_3(y), \tag{10}$$

where ϕ_0, ϕ_1 , ϕ_2 and ϕ_3 are undetermined functions. Substitution of Eq. (10) into Eq. (9) yields

$$\sigma_x = \phi_0'' + x\phi_1'' + x^2\phi_2'' + x^3\phi_3'', \qquad \sigma_y = 2\phi_2 + 6x\phi_3, \qquad \tau_{xy} = -(\phi_1' + 2x\phi_2' + 3x^2\phi_3'), \tag{11}$$

where a prime denotes the derivative with respect to y. From Eq. (11), we find that the linear term in ϕ_0 and the constant term in ϕ_1 make no contribution to the stress field, they will be omitted hereafter. Substitution of Eq. (11) into Eqs. (5) and (6) gives

$$\phi_1'(-h/2) = 0, \qquad \phi_1'(h/2) = 0, \tag{12a}$$

$$\phi_2(-h/2) = 0, \qquad \phi_2'(-h/2) = 0, \tag{12b}$$

$$\phi_2(h/2) = 0, \qquad \phi_2'(h/2) = 0, \tag{12c}$$

$$\phi_3(-h/2) = -q_0/(6l), \qquad \phi_3'(-h/2) = 0, \tag{12d}$$

$$\phi_3(h/2) = 0, \qquad \phi_3'(h/2) = 0. \tag{12e}$$

By using Eq. (11), the expressions for N_0, M_0, Q_0 are

$$N_0 = \int_{-h/2}^{h/2} \sigma_x(0,y)dy = \phi_0'(h/2) - \phi_0'(-h/2), \tag{13a}$$

$$M_0 = \int_{-h/2}^{h/2} \sigma_x(0,y)ydy = \frac{h}{2}[\phi_0'(h/2) + \phi_0'(-h/2)] - \phi_0(h/2) + \phi_0(-h/2), \tag{13b}$$

$$Q_0 = \int_{-h/2}^{h/2} \tau_{xy}(0,y)dy = \phi_1(-h/2) - \phi_1(h/2). \tag{13c}$$

Substituting Eq. (11) into Eq. (3), and in turn into Eq. (4), gives rise to

$$(s_{11}\phi_3'')'' = 0, \qquad (s_{11}\phi_2'' - 3s_{16}\phi_3')'' - 3(s_{16}\phi_3'')' = 0, \tag{14a,14b}$$

$$(s_{11}\phi_1'' - 2s_{16}\phi_2' + 6s_{12}\phi_3)'' - (2s_{16}\phi_2'' - 6s_{66}\phi_3')' + 6s_{12}\phi_3'' = 0, \tag{14c}$$

$$(s_{11}\phi_0'' - s_{16}\phi_1' + 2s_{12}\phi_2)'' - (s_{16}\phi_1'' - 2s_{66}\phi_2' + 6s_{26}\phi_3)' + 2s_{12}\phi_2'' - 6s_{26}\phi_3' = 0. \tag{14d}$$

From Eqs. (14a) and (12d), we obtain

$$\phi_3'' = a_1A_1 + a_2A_2, \qquad \phi_3' = a_1A_1^0 + a_2A_2^0, \qquad \phi_3 = a_1A_1^1 + a_2A_2^1 - q_0/(6l), \tag{15,16,17}$$

where and hereafter a_i $(i = 1, 2, ..., 8)$ are integral constants, a_1 and a_2 can be obtained by substituting Eqs. (16) and (17) into Eq. (12e), and

$$A_1 = y/s_{11}, \quad A_2 = 1/s_{11}, \quad X_i^0 = \int_{-h/2}^{y} X_i(\xi)d\xi, \quad X_i^1 = \int_{-h/2}^{y} X_i^0(\xi)d\xi \quad (X = A, B, C, D).$$

Substituting Eqs. (15) and (16) into Eq. (14b), and integrating, in view of Eq. (12b), we obtain

$$\phi_2'' = \sum_{i=1}^{4} a_iB_i, \qquad \phi_2' = \sum_{i=1}^{4} a_iB_i^0, \qquad \phi_2 = \sum_{i=1}^{4} a_iB_i^1, \tag{18,19,20}$$

where a_3 and a_4 can be calculated by substituting Eqs. (19) and (20) into Eq. (12c), and

$$B_1 = \frac{3s_{16}}{s_{11}}A_1^0 + \frac{3}{s_{11}}\int_{-h/2}^{y} s_{16}A_1d\xi, \quad B_2 = \frac{3s_{16}}{s_{11}}A_2^0 + \frac{3}{s_{11}}\int_{-h/2}^{y} s_{16}A_2d\xi, \quad B_3 = A_1, \quad B_4 = A_2.$$

Substituting Eqs. (15) and (19) into Eq. (14c), and integrating it, in view of Eq. (12a), we obtain

$$\phi_1'' = \sum_{i=1}^{6} a_i C_i + \frac{s_{12}}{s_{11}}\frac{q}{l}, \quad \phi_1' = \sum_{i=1}^{6} a_i C_i^0 + \int_{-h/2}^{y} \frac{s_{12}}{s_{11}}\frac{q}{l} d\xi, \tag{21,22}$$

$$\phi_1 = \sum_{i=1}^{6} a_i C_i^1 + \int_{-h/2}^{y} \frac{s_{12}}{s_{11}}\frac{q}{l}(y-\xi) d\xi, \tag{23}$$

where the constant term has been omitted in ϕ_1 , and

$$C_1 = \frac{2s_{16}}{s_{11}} B_1^0 - \frac{6s_{12}}{s_{11}} A_1^1 + \frac{1}{s_{11}} \int_{-h/2}^{y} [2s_{16}B_1 - 6s_{66}A_1^0 - 6s_{12}A_1(y-\xi)] d\xi,$$

$$C_2 = \frac{2s_{16}}{s_{11}} B_2^0 - \frac{6s_{12}}{s_{11}} A_2^1 + \frac{1}{s_{11}} \int_{-h/2}^{y} [2s_{16}B_2 - 6s_{66}A_2^0 - 6s_{12}A_2(y-\xi)] d\xi,$$

$$C_3 = \frac{2s_{16}}{s_{11}} B_3^0 + \frac{2}{s_{11}} \int_{-h/2}^{y} s_{16}B_3 d\xi, \quad C_4 = \frac{2s_{16}}{s_{11}} B_4^0 + \frac{2}{s_{11}} \int_{-h/2}^{y} s_{16}B_4 d\xi, \quad C_5 = A_1, \quad C_6 = A_2.$$

From Eqs. (22) and (12a), (13c) and (23), and noticing $\phi_1(-h/2) = 0$, we have

$$\sum_{i=1}^{6} a_i C_i^0(h/2) + \int_{-h/2}^{h/2} \frac{s_{12}}{s_{11}}\frac{q}{l} d\xi = 0, \quad \sum_{i=1}^{6} a_i C_i^1(h/2) + \int_{-h/2}^{h/2} \frac{s_{12}}{s_{11}}\frac{q}{l}(y-\xi) d\xi = 0, \tag{24,25}$$

where a_5 and a_6 can be obtained from Eqs. (24) and (25). Substituting Eqs. (16)–(22) into Eq. (14d) and integrating, gives

$$\phi_0'' = \sum_{i=1}^{8} a_i D_i + D_9, \quad \phi_0' = \sum_{i=1}^{8} a_i D_i^0 + D_9^0, \quad \phi_0' = \sum_{i=1}^{8} a_i D_i^1 + D_9^1, \tag{26,27,28}$$

where the linear and constant terms are omitted in ϕ_0 , and

$$D_1 = -\frac{2s_{12}}{s_{11}} B_1^1 + \frac{s_{16}}{s_{11}} C_1^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} [6s_{26}A_1^1 - 2s_{66}B_1^0 + s_{16}C_1 + (6s_{26}A_1^0 - 2s_{12}B_1)(y-\xi)] d\xi,$$

$$D_2 = -\frac{2s_{12}}{s_{11}} B_2^1 + \frac{s_{16}}{s_{11}} C_2^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} [6s_{26}A_2^1 - 2s_{66}B_2^0 + s_{16}C_2 + (6s_{26}A_2^0 - 2s_{12}B_2)(y-\xi)] d\xi,$$

$$D_3 = -\frac{2s_{12}}{s_{11}} B_3^1 + \frac{s_{16}}{s_{11}} C_3^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} [s_{16}C_3 - 2s_{66}B_3^0 - 2s_{12}B_3(y-\xi)] d\xi,$$

$$D_4 = -\frac{2s_{12}}{s_{11}} B_4^1 + \frac{s_{16}}{s_{11}} C_4^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} \left[s_{16}C_4 - 2s_{66}B_4^0 - 2s_{12}B_4(y-\xi)\right] d\xi,$$

$$D_5 = \frac{s_{16}}{s_{11}} C_5^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} s_{16}C_5 d\xi, \quad D_6 = \frac{s_{16}}{s_{11}} C_6^0 + \frac{1}{s_{11}} \int_{-h/2}^{y} s_{16}C_6 d\xi,$$

$$D_7 = A_1, \quad D_8 = A_2, \quad D_9 = \frac{s_{16}}{s_{11}} \int_{-h/2}^{y} \frac{s_{12}}{s_{11}}\frac{q}{l} d\xi + \frac{1}{s_{11}} \int_{-h/2}^{y} \left(\frac{s_{12}s_{16}}{s_{11}} - s_{26}\right)\frac{q}{l} d\xi.$$

Substituting Eqs. (13a) and (13b) into Eq. (7), by virtue of $\phi_0'(-h/2) = 0$ and $\phi_0(-h/2) = 0$, we have

$$\sum_{i=1}^{8} a_i D_i^0(h/2) + D_9^0(h/2) = 0, \quad \sum_{i=1}^{8} a_i D_i^1(h/2) + D_9^1(h/2) = 0, \tag{29,30}$$

where a_7 and a_8 can be obtained from Eqs. (29) and (30). Thus all integral constants in the stress function have been determined, and hence the expressions for the stress components in Eq. (11) have been derived completely. Integrating Eq. (3) gives

$$u = x(s_{11}\phi_0'' - s_{16}\phi_1' + 2s_{12}\phi_2) + \frac{1}{2}x^2(s_{11}\phi_1'' - 2s_{16}\phi_2' + 6s_{12}\phi_3) + \frac{1}{3}x^3(s_{11}\phi_2'' - 3s_{16}\phi_3') + \frac{1}{4}x^4 s_{11}\phi_3'' - \int_{-h/2}^{y} F(\xi)d\xi + \omega y + u_0, \tag{31}$$

$$v = \sum_{k=0}^{3} x^k \int_{-h/2}^{y} s_{12}\phi_k'' d\xi - \frac{a_7}{2}x^2 - \frac{a_5}{6}x^3 - \frac{a_3}{12}x^4 - \frac{a_1}{20}x^5 + \int_{-h/2}^{y} [s_{22}(2\phi_2 + 6x\phi_3) - s_{26}(\phi_1' + 2x\phi_2' + 3x^2\phi_3')]d\xi - \omega x + v_0, \tag{32}$$

where

$$F(y) = -s_{16}\phi_0'' + s_{66}\phi_1' - 2s_{26}\phi_2 + \int_{-h/2}^{y} (s_{12}\phi_1'' - 2s_{26}\phi_2' + 6s_{22}\phi_3)d\xi,$$

and u_0, v_0 and ω can be determined from the boundary conditions in Eq. (8). The complete expressions for displacement components can be obtained by substituting those for $\phi_0 \sim \phi_3$ into Eqs. (31) and (32).

3 A numerical example

In the above derivation, we have not imposed any constraints on the elastic compliance constants, and naturally, analytical solutions could be obtained for any particular set of elastic compliance constants. Let us consider, for example, an FGM orthotropic cantilever beam with only one material constant, s_{11} , varying in the thickness direction in being an exponential law, namely $s_{11}(y) = s_{11}^0 \mathrm{e}^{\lambda(y/h+1/2)}$, where s_{11}^0 is the value at $y = -h/2$. Since all other elastic compliance coefficients do not vary with y, we have $s_{ij} = s_{ij}^0$, whose values are listed in Table 1. The length and height of the beam are 1 m and 0.1 m, respectively, and $q_0 = 10^6$ N/m. The boundary conditions at the fixed end are taken as $u = v = 0$, $\frac{\partial v}{\partial x} = 0$, at the point $(x, y) = (0, l)$.

Table 1 Material properties (unit: $\mathrm{m^2/N}$)

s_{11}^0	s_{12}^0	s_{16}^0	s_{22}^0	s_{26}^0	s_{66}^0
0.150 0E–10	–0.025 9E–10	0	0.103 2E–10	0	0.146 4E–10

The displacement v at $y = 0$, i.e., the deflection curve of the beam, is shown in Fig. 2. It is seen that the deflection of the beam increases with λ. At the point $(x, y) = (0, 0)$, the displacement $v = 0.099\,2h$ when $\lambda = 1$, and $v = 0.036\,17h$ when $\lambda = -1$. The former is about 2.743 times larger than the latter. Thus a smaller value of λ shall be taken for the beam designed with a larger rigidity, and vice versa.

The dimensionless stress components σ_x/q_0, σ_y/q_0 and τ_{xy}/q_0 at $x = 0.5l$ are shown in Figs. 3, 4 and 5, respectively. We can find the obvious influence of the inhomogeneity parameter λ on the distributions of these stress components.

We also compare our analytical solution with the FEM calculation by MSC. Nastran code. The Quad4 element of 0.005 m × 0.005 m is employed, i.e., there are 4000 elements in total for the whole beam. Since the beam is inhomogeneous, the material property of each element is set equal to that at the center of the element. The boundary conditions in the FEM model are $u = v = 0$ at $x = l$, $-h/2 \leq y \leq h/2$, and a linearly distributed load $q = q_0 x/l$ is applied on the edge of $y = -h/2$. The FEM solution is simultaneously presented in Figs. 2–5, where a good agreement can be observed between the two solutions.

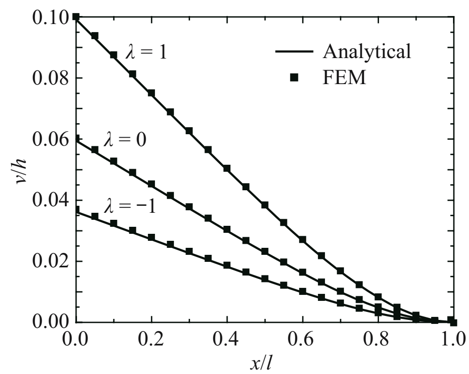

Fig. 2 The deflection curves for different λ

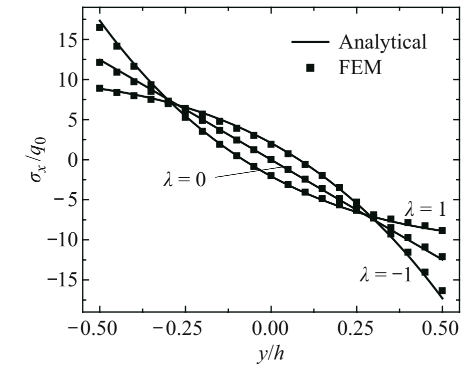

Fig. 3 Dimensionless stress σ_x/q_0 at $x = 0.5l$

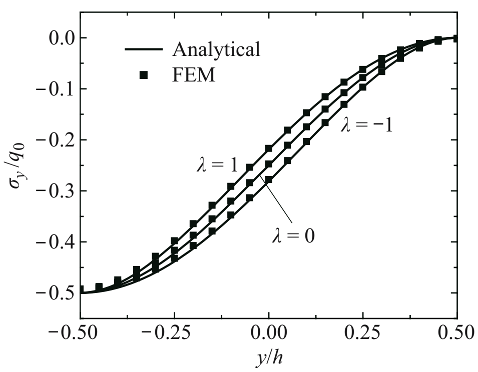

Fig. 4 Dimensionless stress σ_y/q_0 at $x = 0.5l$

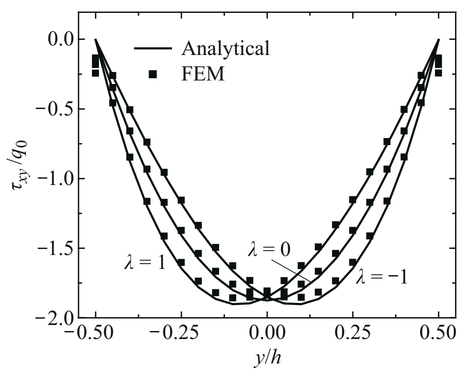

Fig. 5 Dimensionless stress τ_{xy}/q_0 at $x = 0.5l$

4 Conclusions

The stress function in the form of Eq. (10), with ϕ_3, ϕ_2, ϕ_1 and ϕ_0 expressed in Eqs. (17), (20), (23) and (28), can not only be used to solve the aforementioned bending problem of cantilever beams, but also to solve problems of fixed-fixed beams, simply supported-simply supported beams as well as simply supported-fixed beams, subjected to the same linear load. If the stress function is taken to be a polynomial of higher order of x, it can be used to solve problems of beams subjected to distributed loads of higher orders. The method presented in this paper is very convenient for solving problems of beam under a specific load, providing a useful supplement to Ref. [5].

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