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SYSTEM OF VECTOR QUASI-EQUILIBRIUM PROBLEMS AND ITS APPLICATIONS *

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Abstract: A new system of vector quasi-equilibrium problems is introduced and its existence of solution is proved. As applications, some existence results of weak Pareto equilibrium for both constrained multicriteria games and multicriteria games without constrained correspondences are also shown.

Key words: existence; system of vector quasi-equilibrium problems; multicriteria game; C-upper semicontinuous; weak Pareto equilibrium

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1 Introduction and Preliminaries

Firstly, we introduce a new system of vector quasi-equilibrium problems, that is, a family of quasi-equilibrium problems with vector-valued bifunctions defined on a product set.

Let I be an index set. For each $i \in I$, let E_i be a Hausdorff topological vector space. Consider a family of nonempty convex subsets $\{X_i\}_{i\in I}$ with $X_i \subset E_i$. Let

$$X = \prod_{i\in I} X_i, \quad K = \prod_{i\in I} K_i.$$

An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i, therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$.

For each $i \in I$, let Y_i be a Hausdorff topological vector space, C_i be a pointed, closed and convex cone in Y_i with $\mathrm{int}C_i \neq \emptyset$, where $\mathrm{int}C_i$ denotes the interior of C_i, and let φ_i is a map from $X \times X_i$ into Y_i, $A_i : X \to 2^{X_i}$ be a set-valued map. Then the system of vector quasi-equilibrium problem (in short, SVQEP) is to find $\bar{x}$ in X such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $\varphi_i(\bar{x}, y_i) \notin -\mathrm{int}C_i$, $\forall y_i \in A_i(\bar{x})$.

If $Y_i = \mathbb{R}$, $C_i = \{y \in \mathbb{R} \mid y \leq 0\}$ for each $i \in I$ and φ_i is a mapping from $X \times X_i$ into $\mathbb{R}$, then the SVQEP reduces to the model of generalized game in Ref.[1, p.286] and quasi-variational inequalities in Ref.[2, pp.152–153].

If $Y_i = Y$, $C_i = C$ for each $i \in I$, φ_i is a mapping from $X \times X_i$ into Y and $A_i(x) = X_i$ for all $x \in X$, then the SVQEP reduces to the system of vector equilibrium problems in Ref.[3].

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Definition 1.1 *Letting $C_i \subset Y_i$ be a convex cone with* $\mathrm{int}C_i \neq \emptyset$*, the mapping $\varphi : X \times X_i \to Y_i$ is called to be C_i-0-partially diagonally quasiconvex if, for any finite set $\{y_{i_1}, y_{i_2}, \cdots, y_{i_n}\}$ in X_i, and for all $x = (x^i, x_i) \in X$ with $x_i \in \mathrm{co}\{y_{i_1}, y_{i_2}, \cdots, y_{i_n}\}$, there exists some j $(j = 1, 2, \cdots, n)$ such that $\varphi(x, y_{i_j}) \notin -\mathrm{int}C_i$.*

Remark 1.1 If $Y_i = \mathbb{R}$ and $Y_i = \{y \in \mathbb{R} : y \geq 0\}$, then the C_i-0-partially diagonal quasiconvexity reduces to Definition 3 in Ref.[4]. Furthermore, let $I = \{1\}$, then Definition 3 in Ref.[4] reduces to the γ-diagonal quasiconvexity in Refs.[5] and [6], here $\gamma = 0$.

As a special case of the SVQEP, we consider a constrained multicriteria game in its strategic form $\Gamma = (I, \{X_i\}, \{A_i\}, \{F_i\}, \{C_i\})_{i\in I}$. Here, I is an infinite set of players; for each $i \in I$, X_i is the strategy set of ith player; $A_i : X \to 2^{X_i}$ is the constrained correspondence, and $F_i : X \to Y_i$ is the payoff function (or say, loss function or multicriteria) of ith player, where Y_i is a real topological space with a partial ordering induced by a pointed closed convex cone C_i with $\mathrm{int}C_i \neq \emptyset$.

Definition 1.2 *A vector $\bar{x} \in X$ is said to be a weak Pareto equilibrium of a constrained multicriteria game $\Gamma = (I, \{X_i\}, \{A_i\}, \{F_i\}, \{C_i\})_{i\in I}$ if, for each $i \in I$,*

$$\bar{x_i} \in A_i(\bar{x}), \quad F_i(\bar{x}) - F_i(\bar{x^i}, y_i) \notin -\mathrm{int}C_i, \quad \forall y_i \in A_i(\bar{x}).$$

Let $I = \mathbb{N}$, $Y_i = \mathbb{R}$, $X^i = \prod_{j\in\mathbb{N}, j\neq i} X_j$ and $X = \prod_{j\in\mathbb{N}} X_j = X^i \times X_i$. For each $i \in \mathbb{N}$, if F_i is replaced by a real function $U_i : \prod_{j\in\mathbb{N}} X_j \to \mathbb{R}$, then the constrained multicriteria game reduces to an abstract economy or generalized N-person game $\Gamma = (\{X_i\}, \{A_i\}, \{U_i\})_{i\in\mathbb{N}}$ in Refs.[7, p.345] and [8, p.145]. $\bar{x} \in X$ is called to be an equilibrium point of Γ, if for each $i \in \mathbb{N}$, $\bar{x_i} \in A_i(\bar{x})$ and $U_i(\bar{x}^i, \bar{x_i}) \geq U_i(\bar{x}^i, y_i)$, for all $y_i \in A_i(\bar{x})$. Note that this model of abstract economy is a variant of the model in Ref.[9] and the constrained multicriteria game is a variant of the model in Ref.[10]. There exists an important existence result of the equilibrium for this abstract economy (see Ref.[8, p.145, Theorem 5.2.5]) as follows:

Theorem A *Assume that the abstract economy $\Gamma = (\{X_i\}, \{A_i\}, \{U_i\})_{i\in\mathbb{N}}$ satisfies the following conditions*:

(i) *for all $i \in \mathbb{N}$, the set $X_i \subset R^i$ is nonempty convex and compact*;

(ii) *for each $i \in \mathbb{N}$, $A_i : X \to 2^{X_i}$ is a nonempty closed convex valued and continuous correspondence*;

(iii) *for each $i \in \mathbb{N}$, $U_i : X \to \mathbb{R}$ is continuous, and for all $x^i \in X^i$, $U_i(x^i, .)$ is quasiconcave.*

Then Γ at least has an equilibrium point.

Theorem A is a generalization of the Nash equilibrium theorem in Ref.[11].

Let $A_i(x) = X_i$ for each $i \in I$ and for all $x \in X$, then the constrained multicriteria game reduces to a multicriteria game without constrained correspondences (*i.e.*, non-constrained multicriteria game) in its strategic form $\Sigma = (I, \{X_i\}, \{F_i\}, \{C_i\})_{i\in I}$.

If I be a finite set, $Y_i = R^{k_i}$ and $C_i = R_+^{k_i}$ for each $i \in I$, then the above model of the non-constrained multicriteria game reduces to the model discussed by several authors (see Refs.[12–15] and the references therein).

Definition 1.3 *A vector $\bar{x} \in X$ is said to be a weak Pareto equilibrium of a non-constrained multicriteria game $\Sigma = (I, \{X_i\}, \{F_i\}, \{C_i\})_{i\in I}$ if, for each $i \in I$,*

$$\bar{x}_i \in X_i, \quad F_i(\bar{x}) - F_i(\bar{x}^i, y_i) \notin -\mathrm{int}C_i, \quad \forall y_i \in X_i.$$

Definition 1.4[3,16,17] *Let M be a nonempty convex subset of a topological vector space E and Y a real topological space with an ordering cone C. The function $\Phi : M \to Y$ is said to be C-quasiconvex if, for all $z \in Y$, the set $\{x \in M : \Phi(x) \in z - C\}$ is convex. $\Phi : M \to Y$ is said to be C-quasiconcave if $-\Phi$ is C-quasiconvex.*

Lemma 1.1 *If $\Phi : M \to Y$ is C-quasiconvex, then $\forall z \in Y$, $\Phi^{+1}(z - \mathrm{int}C) = \{x \in X : \Phi(x) \in z - \mathrm{int}C\}$ is convex.*

Proof Let $x_1, x_2 \in \Phi^{+1}(z - \text{int}C)$. Then, $x_1, x_2 \in M$, and $\Phi(x_1) \in z - \text{int}C$, $\Phi(x_2) \in z - \text{int}C$. Taking $z_0 \in -\text{int}C$, there exists $\alpha \in (0,1)$ which close to 0, such that $c = \alpha \cdot z_0 \in -\text{int}C$ and $\Phi(x_i) - c - z \in -C$, $i = 1, 2$.

This implies that

$$x_1, x_2 \in \Phi^{+1}(z + c - C).$$

Since Φ is C-quasiconvex, $\forall \lambda \in (0,1)$,

$$\lambda x_1 + (1-\lambda)x_2 \in \Phi^{+1}(z + c - C).$$

Since $c - C \subset -\text{int}C$, we have that

$$\lambda x_1 + (1-\lambda)x_2 \in \Phi^{+1}(z - \text{int}C).$$

Definition 1.5[18] *Let X be a topological space and Y a real topological space with an ordering cone C. A vector-valued function $\Phi : X \to Y$ is said to be C-lower semicontinuous on X if for all $y \in Y$, the set $\{x \in X : \Phi(x) \in y + \text{int}C\}$ is open. $\Phi : X \to Y$ is said to be C-upper semicontinuous on X if $-\Phi$ is C-lower semicontinuous on X, that is, for all $y \in Y$, the set $\{x \in X : \Phi(x) \in y - \text{int}C\}$ is open.*

Remark 1.2 If $\Phi : X \to Y$ is continuous, then Φ is both C-upper semicontinuous and C-lower semicontinuous.

Similar to Theorem 2.1 in Ref.[18], we have

Lemma 1.2 *If f and g are C-upper semicontinuous on X, then $f + g$ is C-upper semicontinuous on X.*

Definition 1.6[19] *Let X and Y be two topological spaces. A correspondence $T : X \to 2^Y$ is said to be lower semicontinuous if the set $\{x \in X : T(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A set-valued map $T : X \to 2^Y$ is said to be upper semicontinuous if the set $\{x \in X : T(x) \subset V\}$ is open in X for every open subset V of Y. A set-valued map $T : X \to 2^Y$ is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.*

In Section 2 of this paper, we prove the existence of solutions for the SVQEP. As applications, in Section 3 of this paper, some existence results of weak Pareto equilibrium for both constrained multicriteria games and multicriteria games without constrained correspondences are also shown.

2 Existence of SVQEP

Let co$A$ denote the convex hull of the set A.

Theorem 2.1 *Let the index set I be countable. For each $i \in I$, let Y_i be a Hausdorff topological vector space, C_i be a nonempty, pointed, closed and convex cone in Y_i with $\text{int}C_i \neq \emptyset$, and X_i be a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space E_i. Let $A_i : X \to 2^{X_i}$ be a set-valued map and φ_i be a map from $X \times X_i$ into Y_i. Assume that the following conditions are satisfied:*

(i) *for each $i \in I$, $\varphi_i(x, y_i)$ is C_i-0-partially diagonally quasiconvex;*

(ii) *for each $i \in I$, for all $y_i \in X_i$, $x \mapsto \varphi_i(x, y_i)$ is C_i-upper semicontinuous on X;*

(iii) *for each $i \in I$, $A_i : X \to 2^{X_i}$ is upper semicontinuous with nonempty convex closed values and open lower sections.*

Then, there exists $\bar{x} \in X$, such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$, and $\varphi_i(\bar{x}, y_i) \notin -\text{int}C_i$, for all $y_i \in A_i(\bar{x})$.

Proof For each $i \in I$, define a correspondence $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i : \varphi_i(x, y_i) \in -\text{int}C_i\}, \quad \forall x = (x^i, x_i) \in X.$$

By Hypothesis (i), we know that $x_i \notin \text{co}(P_i(x))$ for each $i \in I$ and for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some $i \in I$ and some points $\bar{x} = (\bar{x}^i, \bar{x_i}) \in X$ such that $\bar{x_i} \in \text{co}(P_i(\bar{x}))$. Then there exists finite points $y_{i_1}, y_{i_2}, \cdots, y_{i_n}$ in X_i, and $\alpha_j \geq 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x_i} = \sum_{j=1}^n \alpha_j y_{i_j}$ and $y_{i_j} \in P_i(\bar{x})$ for all $j = 1, 2, \cdots, n$. That is, $\varphi_i(\bar{x}, y_{i_j}) \in -\text{int}C_i(\bar{x_i})$ for $j = 1, 2, \cdots, n$, which contradict to the hypothesis that $\varphi_i(x, z_i)$ is C_i-0-partially diagonally quasiconvex.

By Hypothesis (ii), we know that for each $i \in I$, and for each $y_i \in X_i$, the set ${P_i}^{-1}(y_i) = \{x \in X : \varphi_i(x, y_i) \in -\text{int}C_i\}$ is open, *i.e.*, P_i has open lower sections.

For each $i \in I$, also define another correspondence, $G_i : X \to 2^{X_i}$ by $G_i(x) = A_i(x) \cap \text{co}(P_i(x))$, $\forall x \in X$. Let the set $W_i = \{x \in X : G_i(x) \neq \emptyset\}$. Since A_i and P_i have open lower sections in X, and by Lemma 5 and Lemma 4 in Ref.[20], we know that coP_i and G_i also have open lower sections in X. Hence, $W_i = \cup_{y_i \in X_i} {G_i}^{-1}(y_i)$ is an open set in X. Then, the correspondence $G_i|_{W_i}: W_i \to 2^{X_i}$ has open lower sections in W_i, and for all $x \in W_i$, $G_i(x)$ is nonempty and convex. Also, since X is a metrizable space[21,p.50], W_i is paracompact[22,p.831], hence, by Lemma 6[20], there is a continuous function $f_i : W_i \to X_i$ such that $f_i(x) \in G_i(x) \subset A_i(x)$ for all $x \in W_i$. Define $T_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} f_i(x) & \text{if} \quad x \in W_i, \\ A_i(x) & \text{if} \quad x \notin W_i. \end{cases}$$

Now, we prove that T_i is upper semicontinuous. In fact, for each open set V_i in X_i, the set

$$\begin{aligned}\{x \in X : T_i(x) \subset V_i\} =& \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X \setminus W_i : A_i(x) \subset V_i\} \\ &\subset \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : A_i(x) \subset V_i\}.\end{aligned}$$

On the other hand, when $x \in W_i$, and $f_i(x) \in V_i$, we have $T_i(x) = f_i(x) \in V_i$. When $x \in X$ and $A_i(x) \subset V_i$, since $f_i(x) \in A_i(x)$ if $x \in W_i$, we know that $T_i(x) \subset V_i$ and so

$$\{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : A_i(x) \subset V_i\} \subset \{x \in X : T_i(x) \subset V_i\}.$$

Therefore,

$$\{x \in X : T_i(x) \subset V_i\} = \{x \in W_i : f_i(x) \in V_i\} \cup \{x \in X : A_i(x) \subset V_i\}.$$

Since f_i is continuous and D_i is upper semicontinuous, the sets $\{x \in W_i : f_i(x) \in V_i\}$ and $\{x \in X : A_i(x) \subset V_i\}$ are open. It follows that $\{x \in X : T_i(x) \subset V_i\}$ is open and so the mapping $T_i : X \to 2^{X_i}$ is upper semicontinuous. Now define $T : X \to 2^X$ by $T(x) = \prod_{i \in I} T_i(x)$, for each $x \in X$. By Lemma 3[23,p.124], T is upper semicontinuous. Since for each $x \in X$, $T(x)$ is convex, closed and nonempty, by Theorem 1[23,p.122], there is $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$. Note that for each $i \in I$, $\bar{x} \notin W_i$. Otherwise, there is some $i \in I$ such that $\bar{x} \in W_i$, then $\bar{x_i} = f_i(\bar{x}) \in \text{co}(P_i(\bar{x}))$, which contradicts to that $x_i \notin \text{co}(P_i(x))$ for each $i \in I$ and for all $x \in X$.

Thus $\bar{x_i} \in A_i(\bar{x})$ and $G_i(\bar{x}) = \emptyset$ for each $i \in I$. That is, $\bar{x_i} \in A_i(\bar{x})$ and $A_i(\bar{x}) \cap \text{co}(P_i(\bar{x})) = \emptyset$, which implies $\bar{x_i} \in A_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for each $i \in I$. Consequently, there exists $\bar{x} \in X$, such that for each $i \in I$, $\bar{x_i} \in A_i(\bar{x})$, and $\varphi_i(\bar{x}, y_i) \notin -\text{int}C_i$, for all $y_i \in A_i(\bar{x})$. Hence, Γ has at least one weak equilibrium $\bar{x} \in X$.

Let $A_i(x) \equiv X_i$ for each $i \in I$ and for all $x \in X$, by Theorem 2.1, we have

Corollary 2.2 *For each $i \in I$, let Y_i be a real topological space with a partial ordering induced by a pointed closed convex cone C_i with* int$C_i \neq \emptyset$*, X_i be a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space E_i, and let φ_i be a map from $X \times X_i$ into Y_i. Assume that the following conditions are satisfied:*

(i) *for each $i \in I$, $\varphi_i(x, y_i)$ is C_i-0-partially diagonally quasiconvex*;

(ii) *for each $i \in I$, for all $y_i \in X_i$, $x \mapsto \varphi_i(x, y_i)$ is C_i-upper semicontinuous on X.*

Then, there exists $\bar{x} \in X$, such that for each $i \in I$, $\bar{x_i} \in X_i$, $\varphi_i(\bar{x}, y_i) \notin -\text{int}C_i$, for all $y_i \in X_i$.

Remark 2.1 If for all $x \in X$, $\varphi_i(x, x_i) = 0$ and the map $y_i \mapsto \varphi_i(x, y_i)$ is C_i-quasiconvex, then φ_i is C_i-0-partially diagonally quasiconvex (similar to the corresponding part of the proof in Theorem 2.1). Hence, Corollary 2.2 generalizes Theorem 2.1 in Ref.[3] with more general convexity and more general continuity. And so Theorem 2.1 is also a generalization of Theorem 2.1 in Ref.[3].

3 Multicriteria Games

As an application of Theorem 2.1, we will obtain an existence theorem of the weak Pareto equilibrium for the constrained multicriteria game as follows.

Theorem 3.1 *Let $\Gamma = (I, \{X_i\}, \{A_i\}, \{F_i\}, \{C_i\})_{i\in I}$ be a constrained multicriteria game, where I is countable and for each $i \in I$, Y_i is a real topological space with a partial ordering induced by a pointed closed convex cone C_i with* $\text{int}C_i \neq \emptyset$*, X_i is a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space E_i, $A_i : X \to 2^{X_i}$ is the constrained correspondence, and $F_i : X \to Y_i$ is the ith player's payoff function. Assume that the following conditions are satisfied*:

(i) *for each $i \in I$, for all $x^i \in X^i$, the map $y_i \mapsto F_i(x^i, y_i)$ is C_i-quasiconcave*;

(ii) *for each $i \in I$, for all $y_i \in X_i$, $x \mapsto F_i(x) - F_i(x^i, y_i)$ is C_i-upper semicontinuous on X*;

(iii) *for each $i \in I$, A_i is a upper semicontinuous correspondence with nonempty convex closed values and open lower sections.*

Then, Γ has at least one weak Pareto equilibrium $\bar{x} \in X$.

Proof For each $i \in I$, we define a function $\varphi_i : X \times X_i \to Y_i$ as

$$\varphi_i(x, y_i) = F_i(x) - F_i(x^i, y_i), \text{ for all } (x, y_i) \in X \times X_i.$$

By Hypothesis (i), we can prove that φ_i is C_i-0-partially diagonally quasiconvex for each $i \in I$.

In fact, by the C_i-quasiconvexity of $-F_i(x^i, y_i)$ on y_i for each $i \in I$ and for all $x^i \in X^i$, and Lemma 1.1, we know that for all $z \in Y_i$, the set $\{y_i \in X_i : -F_i(x^i, y_i) \in z - \text{int}C_i\}$ is convex. Hence, for all $x = (x^i, x_i) \in X$, taking $z = -F_i(x)$, we know that for all $i \in I$, the set

$$\{y_i \in X_i : F_i(x) - F_i(x^i, y_i) \in -\text{int}C_i\} = \{y_i \in X_i : \varphi_i(x, y_i) \in -\text{int}C_i\}$$

is convex. Then, we can prove that $\varphi_i(x, y_i) = F_i(x) - F_i(x^i, y_i)$ is C_i-0-partially diagonally quasiconvex. Otherwise, there is some finite set $\{y_{i_1}, y_{i_2}, \cdots, y_{i_n}\}$ in X_i and some $x = (x^i, x_i) \in X$ with $x_i \in \text{co}\{y_{i_1}, y_{i_2}, \cdots, y_{i_n}\}$ such that $\varphi_i(x, y_{i_j}) \in -\text{int}C_i$, for all $j = 1, 2, \cdots, n$, *i.e.*, $y_{i_j} \in \{y_i \in X_i : \varphi_i(x, y_i) \in -\text{int}C_i\}$ for all $j = 1, 2, \cdots, n$. By the convexity of the set $\{y_i \in X_i : \varphi_i(x, y_i) \in -\text{int}C_i\}$, we know that $x_i \in \{y_i \in X_i : \varphi_i(x, y_i) \in -\text{int}C_i\}$. So $\varphi_i(x, x_i) = F_i(x) - F_i(x^i, x_i) = 0 \in -\text{int}C_i$, which is absurd. Hence, for each $i \in I$, $\varphi_i(x, y_i) = F_i(x) - F_i(x^i, y_i)$ is C_i-0-partially diagonally quasiconvex.

By Hypothesis (ii), we know that for each $i \in I$, for all $y_i \in X_i$, $\varphi_i(., y_i)$ is C_i-upper semicontinuous. Hence, by Theorem 2.1, we know that the conclusion holds.

By Lemma 1.2 and Theorem 3.1, we know that the following result holds.

Corollary 3.2 *If we replace, in Theorem* 3.1, *condition* (ii) *by the following conditions*:

(ii(a)) *for each $i \in I$, for all $y_i \in X_i$, $x^i \mapsto F_i(x^i, y_i)$ is C_i-lower semicontinuous on X^i*;

(ii(b)) *for each $i \in I$, $x \mapsto F_i(x)$ is C_i-upper semicontinuous on X*,

then the conclusion of Theorem 3.1 *still holds, i.e., the constrained multicreteria game Γ has at least one weak Pareto equilibrium $\bar{x} \in X$.*

By Corollary 3.2 and Remark 1.1, we have

Corollary 3.3 *If we replace, in Theorem* 3.1, *condition* (ii) *by the following conditions*:

(ii*) *for each* $i \in I$, $x \mapsto F_i(x)$ *is continuous on* X,
then the conclusion of Theorem 3.1 *still holds, i.e., the constrained multicreteria game* Γ *has at least a weak Pareto equilibrium* $\bar{x} \in X$.

By Corollary 3.2, it is easy to obtain a result as follows.

Corollary 3.4 *Let* $\Gamma = (I, \{X_i\}, \{A_i\}, \{U_i\})_{i\in I}$ *be an abstract economy, where* I *is countable and for each* $i \in I$, X_i *is a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space* E_i, $A_i : X \to 2^{X_i}$ *is the constrained correspondence, and* $U_i : X \to \mathbb{R}$ *is the ith player's payoff function. Assume that the following conditions are satisfied*:

(i) *for each* $i \in I$, *for all* $x^i \in X^i$, *the map* $y_i \mapsto U_i(x^i, y_i)$ *is quasiconcave*;

(ii) *for each* $i \in I$, *for all* $y_i \in X_i$, $x^i \mapsto U_i(x^i, y_i)$ *is lower semicontinuous on* X^i;

(iii) *for each* $i \in I$, $x \mapsto U_i(x)$ *is upper semicontinuous on* X;

(iv) *for each* $i \in I$, A_i *is an upper semicontinuous correspondence with nonempty convex closed values and open lower sections.*

Then, Γ *has at least one equilibrium point* $\bar{x} \in X$. *That is, for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ *and* $U_i(\bar{x}^i, \bar{x}_i) \geq U_i(\bar{x}^i, y_i)$, *for all* $y_i \in A_i(\bar{x})$.

Remark 3.1 Corollary 3.4 generalizes Theorem A from finite players to infinite players. And Theorem 3.1, Corollary 3.2 and Corollary 3.3 generalize Theorem A from scalar case to vector-valued case.

Let $A_i(x) = X_i$ for each $i \in I$ and for all $x \in X$, by Corollary 3.2, we have an existence result for multicriteria game without constraint correspondences as follows:

Corollary 3.5 *Let* $\Sigma = (I, \{X_i\}, \{F_i\}, \{C_i\})_{i\in I}$ *be a non-constrainted multicriteria game, where* I *is countable and for each* $i \in I$, Y_i *is a real topological space with a partial ordering induced by a pointed closed convex cone* C_i *with* $\mathrm{int}C_i \neq \emptyset$, X_i *is a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space* E_i, $F_i : X \to Y_i$ *is the ith player's payoff function. Assume that the following conditions are satisfied*:

(i) *for each* $i \in I$, *for all* $x^i \in X^i$, *the map* $y_i \mapsto F_i(x^i, y_i)$ *is* C_i*-quasiconcave*;

(ii) *for each* $i \in I$, *for all* $y_i \in X_i$, $x^i \mapsto F_i(x^i, y_i)$ *is* C_i*-lower semicontinuous on* X^i;

(iii) *for each* $i \in I$, $x \mapsto F_i(x)$ *is* C_i*-upper semicontinuous on* X.

Then, Σ *has at least one weak Pareto equilibrium* $\bar{x} \in X$.

In case X_i is not compact, we have

Theorem 3.6 *Let* $\Sigma = (I, \{X_i\}, \{F_i\}, \{C_i\})_{i\in I}$ *be a non-constrainted multicriteria game, where* I *is countable and for each* $i \in I$, Y_i *is a real topological space with a partial ordering induced by a pointed closed convex cone* C_i *with* $\mathrm{int}C_i \neq \emptyset$, X_i *is a nonempty convex and metrizable set in a locally convex Hausdorff topological vector space* E_i, $F_i : X \to Y_i$ *is the ith player's payoff function. Assume that the following conditions are satisfied*:

(i) *for each* $i \in I$, *for all* $x^i \in X^i$, *the map* $y_i \mapsto F_i(x^i, y_i)$ *is* C_i*-quasiconcave*;

(ii) *for each* $i \in I$, *for all* $y_i \in X_i$, $x^i \mapsto F_i(x^i, y_i)$ *is* C_i*-lower semicontinuous on* X^i;

(iii) *for each* $i \in I$, $x \mapsto F_i(x)$ *is* C_i*-upper semicontinuous on* X;

(iv) *for each* $i \in I$, *there exists a nonempty compact subset* $A_i \subset X_i$ *and a compact convex set* $B_i \subset X_i$; *let* $A = \prod_{i\in I} A_i \subset X$ *and* $B = \prod_{i\in I} B_i \subset X$ *such that, for each* $x \in X \setminus A$, *there exists* $y_i{}^* \in B_i$ *such that* $F_i(x) - F_i(x^i, y_i{}^*) \in -\mathrm{int}C_i$.

Then Σ *has at least one weak Pareto equilibrium* $\bar{x} = (\bar{x^i}, \bar{x_i}) \in A$.

Proof For each $i \in I$, let $\{y_{i_1}, \cdots, y_{i_k}\}$ be a finite subset of X_i. Let $Q_i = \mathrm{co}(B_i \cup \{y_{i_1}, \cdots, y_{i_k}\})$. Then, for each $i \in I$, Q_i is compact and convex. By Corollary 3.5, there exists $\bar{x} \in Q = \prod_{i\in I} Q_i$ such that, for each $i \in I$, $F_i(\bar{x}) - F_i(\bar{x^i}, y_i) \notin -\mathrm{int}C_i$, for all $y_i \in Q_i$.

It flows from $B \subseteq Q$ and Assumption (iv) that $\bar{x} \in A$. In particular, we have, $\bar{x} \in A$ such that, for each $i \in I$, $F_i(\bar{x}) - F_i(\bar{x}^i, y_{i_j}) \notin -\mathrm{int}C_i$, for all $j = 1, 2, \cdots, k$. Since A is compact, by

Assumptions (ii) and (iii) we have that, for each $i \in I$ and for all $y_i \in X_i$,

$$G(y_i) = \{x \in A : F_i(x, y_i) \notin -\text{int}C_i\}$$

is closed in A. Since every finite subfamily of closed sets $G(y_i)$ in compact set A has a nonempty intersection, for each $i \in I$, $\cap_{y_i \in X_i} G(y_i) \neq \emptyset$. Thus, there exists $\bar{x} \in A$ such that, for each $i \in I$, $\bar{x}_i \in A_i$, $F_i(\bar{x}) - F_i(\bar{x}^i, y_i) \notin -\text{int}C_i$, for all $y_i \in X_i$.

By Theorem 3.6, we have

Corollary 3.7 *Let $\Sigma = (I, \{X_i\}, \{U_i\})_{i \in I}$ be a game, where I is countable and for each $i \in I$, X_i is a nonempty convex and metrizable set in a locally convex Hausdorff topological vector space E_i, $U_i : X \to \mathbb{R}$ is the ith player's payoff function. Assume that the following conditions are satisfied*:

(i) *for each $i \in I$, for all $x^i \in X^i$, the map $y_i \mapsto U_i(x^i, y_i)$ is quasiconcave*;

(ii) *for each $i \in I$, for all $y_i \in X_i$, $x^i \mapsto U_i(x^i, y_i)$ is lower semicontinuous on X^i*;

(iii) *for each $i \in I$, $x \mapsto U_i(x)$ is upper semicontinuous on X*;

(iv) *for each $i \in I$, there exists a nonempty compact subset $A_i \subset X_i$ and a compact convex set $B_i \subset X_i$; let $A = \prod_{i \in I} A_i \subset X$ and $B = \prod_{i \in I} B_i \subset X$ such that, for each $x \in X \setminus A$, there exists ${y_i}^* \in B_i$ such that $U_i(x) < U_i(x^i, {y_i}^*)$.*

Then there exists $\bar{x} = (\bar{x^i}, \bar{x}_i) \in A$, such that for each $i \in I$, $\bar{x}_i \in X_i$, $U_i(\bar{x}) \geq U_i(\bar{x}_i, y_i)$, for all $y_i \in X_i$. That is, Σ has at least one equilibrium point $\bar{x} = (\bar{x^i}, \bar{x}_i) \in A$.

Remark 3.2 Corollary 3.7 is a generalizations of Nash equilibrium in Ref.[11] (or Theorem 13 in Ref.[24, p.335]). Corollary 3.5 and Theorem 3.6 generalize those results in Refs.[12–15].

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