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DYNAMIC ANALYSIS OF FLEXIBLE-LINK AND FLEXIBLE-JOINT ROBOTS *

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Abstract: The dynamic modeling and simulation of an N-flexible-link and N-flexible-joint robot is reported. Each flexible joint is modeled as a linearly elastic torsional spring and the approach of assumed modes is adopted to describe the deformation of the flexible-link. The complete governing equations of motion of the flexible-link-joint robots are derived *via* Kane's method. An illustrative example is given to validate the algorithm presented and to show the effects of flexibility on the dynamics of robots.

Key words: flexible robot; dynamics; numerical simulation; modeling

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Introduction

The multi-rigid-body models with rigid-links and rigid-joints are usually used in the dynamic analysis of robots[1−2]. However, along with the development of robots towards the high speed and the use of lightweight materials with distributed flexibility, the flexibility of structure has led to a new and challenging problem in the area of dynamic modeling and control. The traditional multi-rigid-body dynamic models are no longer adequate to describe the dynamical characteristics of these flexible mechanical systems precisely and it can even lead to completely wrong conclusions at some critical applications. Increasing interest has arisen to properly account for the inherent flexibility of flexible robots. The flexibility of robots includes link flexibility and joint flexibility. References [3−10] studied the dynamics of robots with flexible-links and rigid-joints, and Refs.[11−15] for the dynamics of robots with rigid-links and flexible-joints. Few works[16−18] focused on the dynamics of robots with both flexible-links and flexible-joints, and only one-link models were given as simulation examples in the works. This paper focuses on the dynamics of flexible robots with N links and N joints. The complete governing equations of motion of the flexible-link-joint robots are derived *via* Kane's method. And the corresponding software package is developed. An example is also given to validate the algorithm presented in this paper and to show the effects of flexibility on the dynamics of robots.

1 Physical Model of Flexible Robots

The system considered here is an N-flexible-link robot driven by N DC-motors through N revolute flexible-joints.

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1.1 Simplified model of flexible-joint

Consider a robot with revolute flexible-joint and model the elasticity of the ith joint as a linear torsional spring with stiffness k_i. k_i accounts for the stiffness of driver and gearing. Let q_i be the theoretical rotational angle of link i, ε_i be the torsional angle of the ith joint, θ_i be the real rotational angle of link i, ϕ_i be the angular displacement of rotor i and N_i be the gear ratio, respectively. The relationships of $\theta_i = q_i + \varepsilon_i$ and $\phi_i = N_i q_i$ hold.

1.2 Simplified model of flexible-link

Assume that the links are slender beams. So the rotary inertia and shear effects can be neglected. Therefore, the present analysis is based on the Euler-Bernoulli beam theory. Also assume that the links can undergo large overall rigid motion, however the elastic displacements are small.

2 Kinematics of Flexible Robots

2.1 Coordinate systems and transformation matrices

To express the transformation between different coordinate systems clearly, we establish four coordinate systems for link i. Fix coordinate system $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ at the proximal end of link i (oriented so that the X_b is coincide with the neutral axis of link i in its undeformed condition). This will be referred to as the base reference of link i. Fix coordinate system $(X_\mathrm{d}Y_\mathrm{d}Z_\mathrm{d})_i$ at the distal end of link i. This is the distal frame of link i. When link i is in its undeformed state, the distal frame can be located by a pure translation of the base reference $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ along the length L_i of the link. Let $(H_xH_yH_z)'_i$ and $(H_xH_yH_z)_i$ be two Denavit-Hartenberg[1] frames fixed at the proximal end (at joint i) and at the distal end (at joint $i+1$) of link i, respectively. When joint i is motionless, $(H_xH_yH_z)'_i$ is coincident with $(H_xH_yH_z)_{i-1}$, and matrix $\boldsymbol{HH}_i^{i-1}$, the transformation matrix between them, is the function of θ_i. Matrix $\boldsymbol{dH}_i$, the 4×4 homogeneous transformation matrix between frames $(X_\mathrm{d}Y_\mathrm{d}Z_\mathrm{d})_i$ and $(H_xH_yH_z)_i$, is a constant matrix. The transformation matrix $\boldsymbol{Hb}_i$ of $(H_xH_yH_z)'_i$ and $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ is also a constant matrix. Define the joint-transformation matrix $\boldsymbol{T}_i$ of joint i be the transformation matrix from $(X_\mathrm{d}Y_\mathrm{d}Z_\mathrm{d})_{i-1}$ to $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$, then

$$\boldsymbol{T}_i = \boldsymbol{dH}_{i-1}\boldsymbol{HH}_i^{i-1}\boldsymbol{Hb}_i. \tag{1}$$

Obviously $\boldsymbol{T}_i$ is the function of θ_i. Define $\boldsymbol{E}_i$ be the link-transformation matrix of link i from $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ to $(X_\mathrm{d}Y_\mathrm{d}Z_\mathrm{d})_i$. According to the assumption of small deformation of the links, the small deformable angles can be added vectorally. $\boldsymbol{E}_i$ can then be written as

$$\boldsymbol{E}_i = \boldsymbol{E}_{i1} + \boldsymbol{E}_{i2}, \tag{2}$$

in which

$$\boldsymbol{E}_{i1} = \begin{bmatrix} 1 & 0 & 0 & L_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{i2} = \begin{bmatrix} 0 & -\varphi_3^i & \varphi_2^i & \delta_1^i \\ \varphi_3^i & 0 & -\varphi_1^i & \delta_2^i \\ -\varphi_2^i & \varphi_1^i & 0 & \delta_3^i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

where $\boldsymbol{\varphi}^i = (\varphi_1^i,\ \varphi_2^i,\ \varphi_3^i)^\mathrm{T}$ and $\boldsymbol{\delta}^i = (\delta_1^i,\ \delta_2^i,\ \delta_3^i)^\mathrm{T}$ are the elastic angular and linear displacements of link i at the origin of the coordinate $(X_\mathrm{d}Y_\mathrm{d}Z_\mathrm{d})_i$, respectively.

Define $\boldsymbol{A}_i^{i-1}$ be the 4×4 homogeneous transformation matrix from $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_{i-1}$ to $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$, then we have

$$\boldsymbol{A}_i^{i-1} = \boldsymbol{E}_{i-1}\boldsymbol{T}_i = \begin{bmatrix} \boldsymbol{R}_i^{i-1} & \boldsymbol{p}_i^{i-1} \\ \boldsymbol{0} & 1 \end{bmatrix}, \tag{4}$$

where $\boldsymbol{R}_i^{i-1}$ is the 3×3 direction cosine matrix, $\boldsymbol{0}$ is the 1×3 zero matrix, $\boldsymbol{p}_i^{i-1}$ is the vector from the origin of frame $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_{i-1}$ to the origin of frame $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ which is expressed in the frame $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_{i-1}$.

2.2 Generalized speeds selection

Let $\boldsymbol{\varphi}^i$ and $\boldsymbol{\delta}^i$ be the angular and linear displacements resulted from the deformation of link i at position x, respectively. Then $\boldsymbol{\varphi}^i$ and $\boldsymbol{\delta}^i$ can be approximated as

$$\boldsymbol{\varphi}^i = \sum_{k=1}^{n_i} \boldsymbol{\varphi}_k^i(x) e_k^i(t), \tag{5}$$

$$\boldsymbol{\delta}^i = \sum_{k=1}^{n_i} \boldsymbol{\delta}_k^i(x) e_k^i(t), \tag{6}$$

where $e_k^i(t)$, the function of time t, is the kth modal coordinate of link i. n_i is the number of modes of link i. $\boldsymbol{\varphi}_k^i, \boldsymbol{\delta}_k^i$, the functions of position x, are the kth modal vectors of link i. They include three components $\varphi_{kj}^i, \delta_{kj}^i (j\!=\!1,2,3)$, respectively.

According to Ref.[18], we select generalized speeds $U_j(j = 1, \cdots, 2N + \sum\limits_{i=1}^{N} n_i)$ as follows:

$$U_j = \dot{q}_j, \quad j = 1, \cdots, N, \tag{7}$$

$$U_j = \dot{\theta}_{j-N}, \quad j = N+1, \cdots, 2N, \tag{8}$$

$$U_j = \dot{e}_k^i(t), \quad j = 2N + \sum_{r=1}^{i-1} n_r + k, \quad 1 \le k \le n_i, \quad i = 1, \cdots, N. \tag{9}$$

2.3 Kinematical analysis

We do kinematical analysis according to Refs.[10,18]. Let $\boldsymbol{\omega}^{i-1}$ and ${}^*\boldsymbol{\omega}^{i-1}$ be the angular velocities of frame $(X_{\rm b}Y_{\rm b}Z_{\rm b})_{i-1}$ and frame $(X_{\rm d}Y_{\rm d}Z_{\rm d})_{i-1}$, respectively. Upon utilizing the additional theorem for angular velocities, the angular velocity of the frame $(X_{\rm d}Y_{\rm d}Z_{\rm d})_{i-1}$ is obtained as

$${}^*\boldsymbol{\omega}^{i-1} = \boldsymbol{\omega}^{i-1} + \dot{\boldsymbol{\varphi}}^{i-1}. \tag{10}$$

The initial condition for recurrence is $\boldsymbol{\omega}^0 = \mathbf{0}$, $\dot{\boldsymbol{\varphi}}^0 = \mathbf{0}$, ${}^*\boldsymbol{\omega}^0 = \mathbf{0}$. And the angular velocity of the frame of $(X_{\rm b}Y_{\rm b}Z_{\rm b})_i$ is

$$\boldsymbol{\omega}^i = \boldsymbol{R}_{i-1}^i\, {}^*\boldsymbol{\omega}^{i-1} + \boldsymbol{H}_z^i \dot{\theta}_i = \boldsymbol{R}_{i-1}^i\, {}^*\boldsymbol{\omega}^{i-1} + \boldsymbol{H}_z^i U_{N+i}, \quad i = 1, 2, \cdots, N. \tag{11}$$

Consequently, the jth partial angular velocity of $\boldsymbol{\omega}^i$, $\boldsymbol{\omega}_j^i$, can be obtained as

$$\boldsymbol{\omega}_j^i = \begin{cases} \mathbf{0} & (1 \le j \le N), \\ \boldsymbol{R}_{i-1}^i \boldsymbol{\omega}_j^{i-1} = \boldsymbol{R}_{j-N}^i \boldsymbol{H}_z^{j-N} & (N < j < N+i), \\ \boldsymbol{H}_z^i & (j = N+i), \\ \mathbf{0} & (N+i < j \le 2N), \\ \boldsymbol{R}_m^i \boldsymbol{\varphi}_k^m & (j = 2N + \sum\limits_{r=1}^{m-1} n_r + k, 1 \le k \le n_m, 1 \le m \le i-1), \\ \mathbf{0} & (j > 2N + \sum\limits_{r=1}^{m-1} n_r, m \ge i). \end{cases} \tag{12}$$

The angular acceleration $\boldsymbol{\varepsilon}^i$ of coordinate system $(X_{\rm b}Y_{\rm b}Z_{\rm b})_i$, the time derivative of $\boldsymbol{\omega}^i$, is

$$\boldsymbol{\varepsilon}^i = \boldsymbol{R}_{i-1}^i \boldsymbol{\varepsilon}^{i-1} + (\boldsymbol{R}_{i-1}^i\, {}^*\boldsymbol{\omega}^{i-1}) \times (U_{N+i} \boldsymbol{H}_z^i) + \dot{U}_{N+i} \boldsymbol{H}_z^i + \boldsymbol{R}_{i-1}^i ({}^*\boldsymbol{\omega}^{i-1} \times \dot{\boldsymbol{\varphi}}^{i-1} + \ddot{\boldsymbol{\varphi}}^{i-1}). \tag{13}$$

The velocity of the origin of frame $(X_{\rm b}Y_{\rm b}Z_{\rm b})_i$, $\boldsymbol{v}^i$, is

$$\boldsymbol{v}^i = \boldsymbol{R}_{i-1}^i (\boldsymbol{v}^{i-1} + \boldsymbol{\omega}^{i-1} \times \boldsymbol{p}_i^{i-1} + \dot{\boldsymbol{\delta}}^{i-1}). \tag{14}$$

And the jth partial velocity of $\boldsymbol{v}^i, \boldsymbol{v}_j^i$, can be obtained as

$$\boldsymbol{v}_j^i = \begin{cases} \mathbf{0} & (1 \le j \le N), \\ \boldsymbol{R}_{i-1}^i(\boldsymbol{v}_j^{i-1} + \boldsymbol{\omega}_j^{i-1} \times \boldsymbol{p}_i^{i-1}) & (N < j < N+i), \\ \mathbf{0} & (N+i \le j \le 2N), \\ \boldsymbol{R}_{i-1}^i(\boldsymbol{v}_j^{i-1} + \boldsymbol{\omega}_j^{i-1} \times \boldsymbol{p}_i^{i-1}) & (2N < j \le 2N + \sum\limits_{r=1}^{i-2} n_r), \\ \boldsymbol{R}_{i-1}^i \boldsymbol{\delta}_k^{i-1} & (j = 2N + \sum\limits_{r=1}^{i-2} n_r + k,\ 1 \le k \le n_{i-1}), \\ \mathbf{0} & (j > 2N + \sum\limits_{r=1}^{i-1} n_r). \end{cases} \tag{15}$$

The acceleration $\boldsymbol{a}^i$ of the origin of frame $(X_{\rm b}Y_{\rm b}Z_{\rm b})_i$ is the time derivative of $\boldsymbol{v}^i$, it is

$$\boldsymbol{a}^i = \boldsymbol{R}_{i-1}^i[\boldsymbol{a}^{i-1} + \boldsymbol{\varepsilon}^{i-1} \times \boldsymbol{p}_i^{i-1} + \boldsymbol{\omega}^{i-1} \times (\boldsymbol{\omega}^{i-1} \times \boldsymbol{p}_i^{i-1}) + 2\boldsymbol{\omega}^{i-1} \times \dot{\boldsymbol{\delta}}^{i-1} + \ddot{\boldsymbol{\delta}}^{i-1}]. \tag{16}$$

The initial condition for recurrence is $\boldsymbol{a}^0 = -\boldsymbol{g}^0$, where $\boldsymbol{g}^0$ is the acceleration of gravity.

Let $\boldsymbol{r}_p^i$ be the position vector of point p on link i in its undeformed state, and $\boldsymbol{\delta}_p^i$ be the deformation vector. Then the velocity of the point p is

$$\boldsymbol{v}_p^i = \boldsymbol{v}^i + \boldsymbol{\omega}^i \times (\boldsymbol{r}_p^i + \boldsymbol{\delta}_p^i) + \sum_{k=1}^{n_i} \boldsymbol{\delta}_{pk}^i \dot{e}_k^i(t). \tag{17}$$

And its jth partial velocity $\boldsymbol{v}_{pj}^i$ is

$$\boldsymbol{v}_{pj}^i = \begin{cases} \boldsymbol{v}_j^i + \boldsymbol{\omega}_j^i \times (\boldsymbol{r}_p^i + \boldsymbol{\delta}_p^i) & (j \le 2N + \sum\limits_{r=1}^{i-1} n_r), \\ \boldsymbol{\delta}_{pk}^i & (j = 2N + \sum\limits_{r=1}^{i-1} n_r + k,\ 1 \le k \le n_i), \\ \mathbf{0} & (j > 2N + \sum\limits_{r=1}^{i} n_r). \end{cases} \tag{18}$$

The acceleration of the point p is

$$\boldsymbol{a}_p^i = \boldsymbol{a}^i + \boldsymbol{\varepsilon}^i \times (\boldsymbol{r}_p^i + \boldsymbol{\delta}_p^i) + \boldsymbol{\omega}^i \times [\boldsymbol{\omega}^i \times (\boldsymbol{r}_p^i + \boldsymbol{\delta}_p^i)] + 2\boldsymbol{\omega}^i \times \dot{\boldsymbol{\delta}}_p^i + \ddot{\boldsymbol{\delta}}_p^i. \tag{19}$$

3 Dynamic Equations of Flexible Robots

We will use the Kane's method to analysis the dynamics of system and accord with the method of Refs.[10,18].

3.1 Generalized active forces

When using Kane's method, we must consider not only the generalized active forces caused by all active forces but also the generalized active forces caused by internal forces of flexible links and flexible joints.

①Perfect constraints in the joints do not produce any generalized active force.

②Generalized active forces due to gravity are considered in Eq.(16).

③Let τ_i be the torque acted at joint i. The jth generalized active force due to torque τ_i is

$$(F_A)_j^i = \begin{cases} \tau_i, & i = j, \\ 0, & i \ne j. \end{cases} \tag{20}$$

Then the jth generalized active force due to all torques $\tau_i\,(i=1,2,\cdots,N)$ is

$$(F_A)_j=\sum_{i=1}^{N}(F_A)_j^i=\begin{cases}\tau_j, & 1\le j\le N,\\ 0, & j>N.\end{cases}\tag{21}$$

④Generalized active forces produced by internal forces of flexible links

The jth generalized active force produced by bend of the link i is (ignore its pull-compression and torsion)

$$(F_L)_j^i=-\int_0^{L_i}E_iI_i\frac{\partial^2\boldsymbol{\delta}_p^i}{\partial x^2}\cdot\frac{\partial^2}{\partial x^2}(\boldsymbol{\delta}_{pj}^i)dx,\tag{22}$$

where x is the distance from the origin of the frame $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ to the differential element at point p before the link i deforms. E_iI_i is the flexural rigidity of link i.

Finally, the jth generalized active force produced by internal forces of all links is

$$(F_L)_j=\sum_{i=1}^{N}(F_L)_j^i=\begin{cases}0, & j\le N,\\ 0, & N<j\le 2N,\\ \sum\limits_{i=m}^{N}(F_L)_j^i, & 2N+\sum\limits_{r=1}^{m-1}n_r<j\le 2N+\sum\limits_{r=1}^{m}n_r,\ 1\le m\le N.\end{cases}\tag{23}$$

⑤Generalized active forces produced by internal forces of flexible joints

According to Ref.[18], we can obtain the jth generalized active force produced by internal forces of the ith flexible joint as

$$(F_J)_j^i=\begin{cases}k_i\varepsilon_i, & j=i,\\ -k_i\varepsilon_i, & j=N+i,\\ 0, & j\ne i,\ j\ne N+i.\end{cases}\tag{24}$$

The jth generalized active force produced by internal forces of all flexible joints is

$$(F_J)_j=\sum_{i=1}^{N}(F_J)_j^i=\begin{cases}k_j\varepsilon_j, & j\le N,\\ -k_{j-N}\varepsilon_{j-N}, & N<j\le 2N,\\ 0, & j>2N.\end{cases}\tag{25}$$

3.2 Generalized inertia forces

Generalized inertia forces of system are made up of generalized inertia forces of flexible links and generalized inertia forces of rotors. We will calculate these forces as follows:

①Generalized inertia forces produced by the flexible links

Let the mass of differential element at point p of the ith link be dm, then its jth generalized inertia force is

$$d(F_L^*)_j^i=-\boldsymbol{v}_{pj}^i\cdot\boldsymbol{a}_p^i dm.\tag{26}$$

Thus the jth generalized inertia force due to link i is

$$(F_L^*)_j^i=\iiint\limits_i d(F_L^*)_j^i=-\iiint\limits_i \boldsymbol{v}_{pj}^i\cdot\boldsymbol{a}_p^i dm.\tag{27}$$

Substitution of Eq.(18) into Eq.(27) gives

$$(F_L^*)_j^i=\begin{cases}-\iiint\limits_i(\boldsymbol{v}_j^i+\boldsymbol{\omega}_j^i\times^*\boldsymbol{r}_p^i)\cdot\boldsymbol{a}_p^i dm & (j\le 2N+\sum\limits_{r=1}^{i-1}n_r),\\ -\iiint\limits_i\boldsymbol{\delta}_{pk}^i\cdot\boldsymbol{a}_p^i dm & (j=2N+\sum\limits_{r=1}^{i-1}n_r+k,\ \ 1\le k\le n_i),\\ \boldsymbol{0} & (j>2N+\sum\limits_{r=1}^{i}n_r),\end{cases}\tag{28}$$

where

$$ {}^{*}\boldsymbol{r}_p^i = \boldsymbol{r}_p^i + \boldsymbol{\delta}_p^i = \boldsymbol{r}_c^i + \boldsymbol{r}_{cp}^i + \boldsymbol{\delta}_p^i. \tag{29} $$

$\boldsymbol{r}_c^i$ in Eq.(29) is the position vector of the mass center of link i. $\boldsymbol{r}_{cp}^i$ is the vector from the mass center to point p.

The first line of Eq.(28) can be expressed as

$$ \begin{aligned} (F_L^*)_j^i &= -\boldsymbol{v}_j^i \cdot \iiint\limits_i \boldsymbol{a}_p^i dm - \boldsymbol{\omega}_j^i \cdot \iiint\limits_i ({}^{*}\boldsymbol{r}_p^i \times \boldsymbol{a}_p^i) dm \\ &= -\boldsymbol{v}_j^i \cdot \boldsymbol{F}_i^* - \boldsymbol{\omega}_j^i \cdot \boldsymbol{T}_i^*. \end{aligned} \tag{30} $$

$\boldsymbol{F}_i^*$ and $\boldsymbol{T}_i^*$ in Eq.(30) have the following details, respectively,

$$ \begin{aligned} \boldsymbol{F}_i^* &= \iiint\limits_i \boldsymbol{a}_p^i dm \\ &= M_i[\boldsymbol{a}^i + \boldsymbol{\varepsilon}^i \times \boldsymbol{r}_c^i + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times \boldsymbol{r}_c^i)] + \boldsymbol{\varepsilon}^i \times \boldsymbol{E}^i + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times \boldsymbol{E}^i) + 2\boldsymbol{\omega}^i \times \dot{\boldsymbol{E}}^i + \ddot{\boldsymbol{E}}^i, \end{aligned} \tag{31} $$

where

$$ \begin{aligned} \boldsymbol{E}^i &= \iiint\limits_i \boldsymbol{\delta}_p^i dm = \iiint\limits_i \sum_{k=1}^{n_i} \boldsymbol{\delta}_{pk}^i e_k^i(t) dm = \sum_{k=1}^{n_i} \boldsymbol{E}_k^i e_k^i(t), \\ \boldsymbol{E}_k^i &= \iiint\limits_i \boldsymbol{\delta}_{pk}^i dm, \end{aligned} $$

and

$$ \begin{aligned} \boldsymbol{T}_i^* &= \iiint\limits_i ({}^{*}\boldsymbol{r}_p^i \times \boldsymbol{a}_p^i) dm \\ &= (M_i \boldsymbol{r}_c^i + \boldsymbol{E}^i) \times \boldsymbol{a}^i + {}^{*}\boldsymbol{I}^i \cdot \varepsilon^i + \boldsymbol{\omega}^i \times ({}^{*}\boldsymbol{I}^i \cdot \boldsymbol{\omega}^i) + 2\boldsymbol{A}^i + \boldsymbol{B}^i, \end{aligned} \tag{32} $$

where

$$ \begin{aligned} \boldsymbol{A}^i &= \iiint\limits_i {}^{*}\boldsymbol{r}_p^i \times (\boldsymbol{\omega}^i \times \dot{\boldsymbol{\delta}}_p^i) dm \\ &= \iiint\limits_i [({}^{*}\boldsymbol{r}_p^i \cdot \dot{\boldsymbol{\delta}}_p^i)\boldsymbol{\omega}^i - ({}^{*}\boldsymbol{r}_p^i \cdot \boldsymbol{\omega}^i)\dot{\boldsymbol{\delta}}_p^i] dm \\ &= \iiint\limits_i \Big({}^{*}\boldsymbol{r}_p^i \cdot \sum_{k=1}^{n_i} \boldsymbol{\delta}_{pk}^i \dot{e}_k^i(t)\Big)\boldsymbol{\omega}^i dm - \iiint\limits_i ({}^{*}\boldsymbol{r}_p^i \cdot \boldsymbol{\omega}^i)\dot{\boldsymbol{\delta}}_p^i dm \\ &= \left[\sum_{k=1}^{n_i} h_k^i \dot{e}_k^i(t)\right] \boldsymbol{\omega}^i - \iiint\limits_i (\boldsymbol{\omega}^i \cdot {}^{*}\boldsymbol{r}_p^i)\dot{\boldsymbol{\delta}}_p^i dm, \\ h_k^i &= \iiint\limits_i ({}^{*}\boldsymbol{r}_p^i \cdot \boldsymbol{\delta}_{pk}^i) dm, \\ \boldsymbol{B}^i &= \iiint\limits_i {}^{*}\boldsymbol{r}_p^i \times \ddot{\boldsymbol{\delta}}_p^i dm = \iiint\limits_i {}^{*}\boldsymbol{r}_p^i \times \sum_{k=1}^{n_i} \boldsymbol{\delta}_{pk}^i \ddot{e}_k^i(t) dm = \sum_{k=1}^{n_i} \boldsymbol{H}_k^i \ddot{e}_k^i(t), \\ \boldsymbol{H}_k^i &= \iiint\limits_i ({}^{*}\boldsymbol{r}_p^i \times \boldsymbol{\delta}_{pk}^i) dm, \end{aligned} $$

M_i is the mass of link i, ${}^*\boldsymbol{I}^i$ is the inertia tensor of link i about the origin of the coordinate system $(X_\mathrm{b}Y_\mathrm{b}Z_\mathrm{b})_i$ after link i deforms. Because of the assumption of small deflections, ${}^*\boldsymbol{I}^i$ can be approximated as that of link i in its undeformed state.

The second line of Eq.(28) can be expressed as

$$(F_L^*)_j^i = -\left[\boldsymbol{a}^i \cdot \boldsymbol{E}_k^i + \boldsymbol{\varepsilon}^i \cdot \boldsymbol{H}_k^i + \boldsymbol{\omega}^i \cdot (\boldsymbol{Q}_k^i - h_k^i \boldsymbol{\omega}^i) + 2\boldsymbol{\omega}^i \cdot \sum_{j=1}^{n_i} \boldsymbol{R}_{jk}^i \dot{e}_j^i(t) + \sum_{j=1}^{n_i} r_{jk}^i \ddot{e}_j^i(t)\right], \quad (33)$$

where $\boldsymbol{r}_{jk}^i = \iiint\limits_i (\boldsymbol{\delta}_{pj}^i \cdot \boldsymbol{\delta}_{pk}^i) dm$, $\boldsymbol{R}_{jk}^i = \iiint\limits_i (\boldsymbol{\delta}_{pj}^i \times \boldsymbol{\delta}_{pk}^i) dm$, $\boldsymbol{Q}_k^i = \iiint\limits_i (\boldsymbol{\delta}_{pk}^i \cdot \boldsymbol{\omega}^i)^* \boldsymbol{r}_p^i dm$.

The generalized inertia force $(F_L^*)_j$ produced by all flexible links is

$$(F_L^*)_j = \sum_{i=1}^{N} (F_L^*)_j^i = \begin{cases} 0 & (j \le N), \\ \sum\limits_{i=j-N}^{N} (F_L^*)_j^i & (N < j \le 2N), \\ \sum\limits_{i=m}^{N} (F_L^*)_j^i & (2N + \sum\limits_{r=1}^{m-1} n_r < j \le 2N + \sum\limits_{r=1}^{m} n_r, \quad 1 \le m \le N). \end{cases} \quad (34)$$

②Generalized inertia forces produced by rotors

According to Ref.[18], the generalized inertia forces accounting for rotors spinning can be expressed as

$$(F_r^*)_j = \sum_{i=1}^{N} (F_r^*)_j^i = \begin{cases} -I_r^j N_j^2 \ddot{q}_j, & 1 \le j \le N, \\ 0, & j > N, \end{cases} \quad (35)$$

where I_r^j is the moment of inertia of the jth rotor about its spinning axis.

3.3 Dynamic equations of flexible robots

Substitution of Eqs.(21), (23), (25), (34), (35) into Kane's dynamic equation:

$$(F_A)_j + (F_L)_j + (F_J)_j + (F_L^*)_j + (F_r^*)_j = 0, \quad (36)$$

gives the desired dynamic equations of flexible-link and flexible-joint robot as follows:

$$I_r^j N_j^2 \ddot{q}_j - k_j \varepsilon_j = \tau_j, \quad 1 \le j \le N, \quad (37)$$

$$\sum_{i=j-N}^{N} (F_L^*)_j^i - k_{j-N} \varepsilon_{j-N} = 0, \quad N < j \le 2N, \quad (38)$$

$$\sum_{i=m}^{N} (F_L^*)_j^i + \sum_{i=m}^{N} (F_L)_j^i = 0, \quad 2N + \sum_{r=1}^{m-1} n_r < j \le 2N + \sum_{r=1}^{m} n_r, \qquad 1 \le m \le N. \quad (39)$$

There are $2N + \sum\limits_{i=1}^{N} n_i$ equations and $2N + \sum\limits_{i=1}^{N} n_i$ unknowns in Eqs.(37)–(39). So the set of differential equations is closed and therefore it can be integrally solved.

4 Example of Simulation

This example is about the dynamic simulation of a flexible robot with two flexible links and one rigid link connected by three flexible joints and that of its corresponding rigid robot. We consider the bending deformation of links. Figures 1 and 2 are the front view and top view of the robot, respectively. The lengths of two flexible links are $L_1 = L_2 = 7.11$ m and the length of the rigid link is L_3=1.3589 m. The masses of three links are, respectively, M_1=314.88 kg,

M_2=279.2 kg, and M_3=455.62 kg. The flexural rigidity coefficients are $E_1I_1 = E_2I_2 = 3.8 \times 10^6$ Nm^2. The stiffness coefficients of elastic torsional springs are $k_1 = k_2 = k_3 = 1.33 \times 10^6$ Nm/rad. Let it fall from the position as shown in Fig.1.

Figures 4−10 are for the comparison of the motions of the flexible robot and its rigid one. The solid line represents the flexible and the dash line represents the rigid. From Fig.3 to Fig.5, we can see $\theta_i(i = 1, 2, 3)$ under these two conditions are approximately the same. But from Fig.6 to Fig.8, we can see $\dot{\theta}_i(i = 1, 2, 3)$ fluctuate clearly when the robot is flexible. Figures 9 and 10 represent tip deflection and tip deflection velocity of the 2nd link of the flexible one, respectively. For the rigid one the tip deflection is zero. From these results we can see the effects of flexibility on dynamic characteristics. Ignoring these effects will cause mistake in robots modeling and controlling. So the effects of flexibility must be considered in the dynamics of spatial flexible robots.

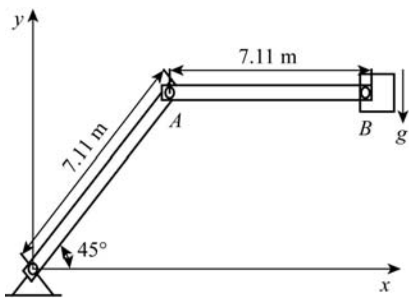

Fig.1 Front view of the flexible robot

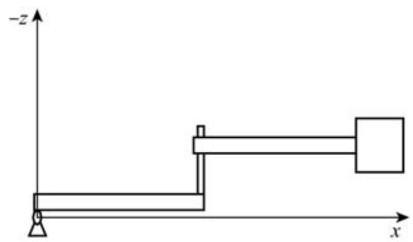

Fig.2 Top view of the flexible robot

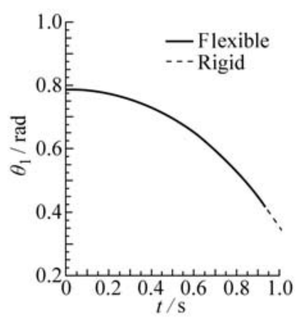

Fig.3 Time history of θ_1

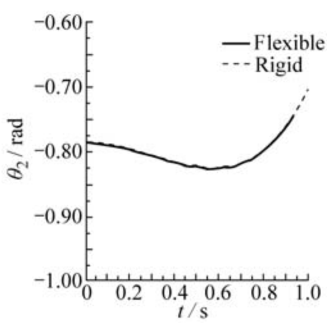

Fig.4 Time history of θ_2

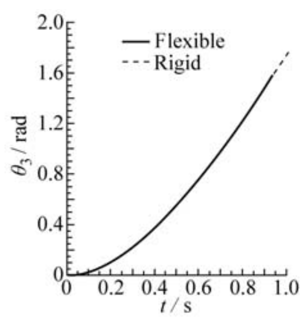

Fig.5 Time history of θ_3

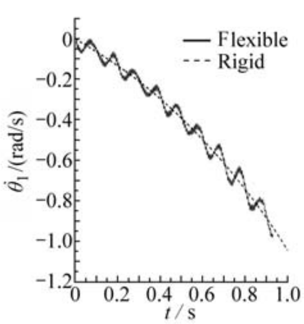

Fig.6 Angular velocity $\dot{\theta}_1$

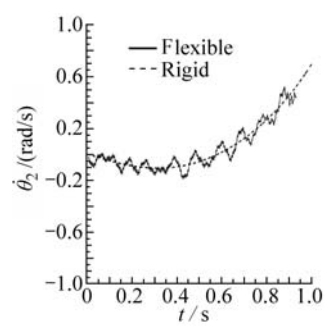

Fig.7 Angular velocity $\dot{\theta}_2$

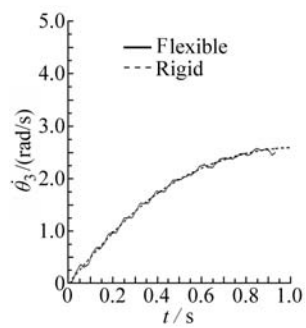

Fig.8 Angular velocity $\dot{\theta}_3$

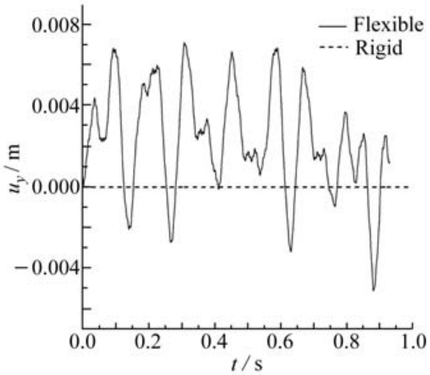

Fig.9 Tip deflection of link 2

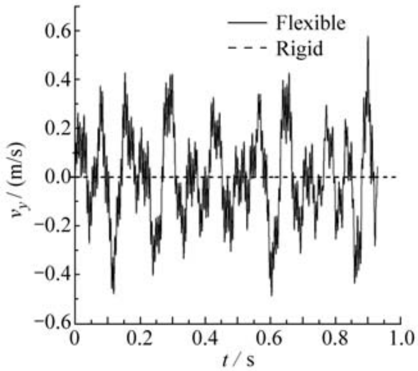

Fig.10 Tip deflection velocity of link 2

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