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ORIGINAL RESEARCH

Dynamic investment in online advertising of multi-oligopoly competitive enterprises with spillover effect

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Abstract

This paper aims to provide solutions to the dynamic investment strategies of online advertising for multi-oligopoly enterprises. Specifically, by considering the spillover effect of online advertising, the investment cost function incorporating the characteristics of online advertising is constructed and then combined with external interference factors, and the dynamic investment decision-making model of online advertising of three oligarchic competitive enterprises is constructed. Subsequently, using the Hamilton–Jacobi–Bellman function, the Nash equilibrium solutions of the online advertising amount and profits are attained in symmetrical, semisymmetric, and asymmetric cases. We then calculated and empirically analysed the share of the market. Finally, the model is extended to *n*-dimensional space. Our study suggests that (1) investment in fixed-location online advertising is inversely proportional to the spillover effect, while the amount of pay-per-click online advertising investment is directly proportional to the spillover effect. (2) In the semisymmetric case, enterprises with a low initial share are easily affected by the spillover effect, while in the semisymmetric and asymmetric cases, dominant enterprises are more easily affected by the spillover effect. (3) The amount of investment in online advertising is inversely proportional to external interference factors. (4) When there are more than three enterprises in the market, the profit is negative, indicating that new enterprise should be cautious about entering the industry.

Keywords Multi-oligopoly competition · Online advertising · Spillover effect · Dynamic investment decision model · External interference factor

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1 Introduction

Over the past few years, the Internet has played an indispensable part in people's way of life, attracting an increasing number of enterprises to publish advertisements and carry out marketing activities on the Internet (Ivanov, 2020; Xue & Li, 2023). According to eMarketer,[1] the global online advertising market has rapidly grown from US $87.2 billion in 2011 to 563.00 billion yuan in 2022, with a month-on-month growth rate of 109%. The percentage of online advertising in the overall advertising market also increased from 21.84% in 2022 to 59.19% in 2019. Online advertising is developing rapidly. Shockingly, some traditional advertising spending has shown negative growth. For example, all-media advertising spending fell by 4.1 year-on-year in 2022, and TV advertising revenue decreased by 13.8% compared with last year.[2] Moreover, the bankruptcy of Touch Media, the suspension of publications of *Oriental Morning Post*, and the successive application of bankruptcy of Hong Kong ATV have further proven that traditional offline advertising is shifting to online advertising. Compared with traditional advertisements such as TV advertisements, newspaper advertisements, and radio advertisements, online advertisements have the advantage of recording multisource and full-process user behaviour data while breaking through geographical and time constraints through Internet dissemination (Chen & Stallaert, 2011; Zhang et al., 2023; Li et al., 2023). It has gradually replaced the traditional network to become an important means for enterprises to promote products and high-quality services, and it has also become the main way for some emerging industries to compete (Mishra et al., 2021). For example, the four giants of the game industry, AppLovin, ironSource, Unity Ads, Nativex (formerly Mobviata), have invested a total of US $77.2 billion in game advertisements, which has launched a fierce competition in various countries and online advertising platforms.[3]

Here, online advertising refers to a way of advertising operation that publishes or publishes advertisements on the Internet (including mobile networks) and delivers them to Internet users (including mobile networks) through the network using banners, text links and multimedia on the website or mobile end (Chen & Stallaert, 2011). With the continuous improvement of online advertising, its forms are also increasing, and many enterprises often apply different types of online advertising to promote products (Xu et al., 2012). Let us consider the following scenario: a traveller plans to go to Sanya for a holiday but is not familiar with the existing hotel options. He began to plan and entered "hotel" on the travel app to select and found that the "Hilton" hotel met his requirements, so he visited its website through the display advertisement of the "Hilton" hotel but took no further action. The next day, he continued to plan to search for "Hilton Sanya" directly on the site, and users entered through search engine advertisements, resulting in scheduled behaviour. This assumption indirectly implies that there is usually a specific internal effect between different types of online advertising, that is, the spillover effect (Rutz & Bucklin, 2011). It refers to the influence of consumers' browsing behaviour and clicking behaviour on other types of advertising browsing and even purchasing behaviour. If the spillover effect is ignored, the investment cost of online advertising continues to increase, resulting in huge losses. For example, Guazi launched various types of online advertising in 2017, with an advertising investment of 4.3 billion yuan but a gross profit of less than 1.4 billion yuan for the whole year. This kind of blind investment makes it difficult for enterprises to make ends meet.[4]

[1] Available at https://www.sohu.com/a/403008360_498848.

[2] Available at https://www.qianzhan.com/analyst/detail/220/210325-256ff5ef.html.

[3] Available at http://finance.sina.com.cn/stock/relnews/us/2021-02-03/doc-ikftpnny3649320.shtml.

[4] Available at https://www.sohu.com/a/368748083_120526963.

In addition, economists believe that the impact of any marketing means on the enterprise is not a single cycle but may last more cycles, commonly known as "carryover effects" (Cnops & Lyer, 2022; Kjersti et al., 2023). Most enterprises often ignore the "carryover effects" of online advertising, which makes the investment strategy of online advertising tend to be short-sighted. A typical case is the dispute over bike-sharing in the Chinese market. At the end of 2017, Xiaolan Bicycle and Mobike were acquired by Didi and Meituan, respectively. In the shared bicycle market, there is a tripartite situation among the Xiaolan Bicycle, Mobike and ofo. To achieve short-term profits, ofo enterprises took the lead on the app side to issue huge online advertising costs to win the consumer market, followed by Mobike, due to the burden of costs. At the end of 2018, Mobike lost 4 billion yuan, and ofo lost 30 billion yuan.[5] Based on the theory of carryover effect in marketing, the advertising expenditure will continue to affect sales in a few months, not only the current market share (Prasad & Sethi, 2004). Meanwhile, the time value of money should be considered since it has substantial influence on decision making. Therefore, it is more practical to study from a long-term and dynamic perspective. In game theory, differential game theory refers to a way that decision makers use differential equations or equations to dynamically describe certain phenomena or laws. Specifically, state variables or control variables evolve with time according to differential equations (Stone, 2008). On this basis, the present study uses the differential game model to discuss the dynamic online advertising strategy.

Existing studies have confirmed the existence of spillover benefits (Rutz & Bucklin, 2011; Xu et al., 2012). However, it is unclear how to reflect the spillover effect in the construction model and whether it impacts the profits and market share of the enterprise. Additionally, because of the complexity of the dynamic differential game model, the static model is often used to study the single-cycle online advertising strategy of enterprises (Gwang & IIkyeong, 2020; Robert et al., 2020). Considering the deficiency in management practice and theoretical research, this paper tries to solve the following problems:

(1) How can the influence between different types of online advertisements be described by introducing spillover effect? Further, will the spillover effect affect the amount, market share and profit of different types of online advertising?
(2) Under the spillover effect, how can a reasonable online advertising strategy be formulated for the three oligarchic competitive enterprises?
(3) When the three oligarchs are extended to multioligopoly competition, that is, when there are many network marketing enterprises in the industry, is the impact of the spillover effect be consistent with that of the three-oligopoly market, and what is the market access mechanism at this time? What are the conditions for the establishment of market share?

Accordingly, first, by introducing the spillover effect level of online advertising, this paper constructs an investment cost function that reflects online advertising. Subsequently, under the disturbance of the spillover effect, combined with noncompetitive factors, the dynamic investment strategy model of online advertising of three oligarchic competitive enterprises based on the spillover effect is constructed. Second, using the Hamilton–Jacobi–Bellman function, which comprehensively considers the influence of itself and competitive enterprises on profits, the Nash equilibrium solutions of the online advertising amount and profits in symmetrical, semisymmetric and asymmetric cases are obtained. In addition, the market share when the market is stable is calculated. Then, how the amount of investment, market share and profit of online advertising change with the spillover effect and time in symmetrical,

[5] Available at https://www.sohu.com/a/273155466_175648.

semisymmetric and asymmetric cases are analysed in detail, and the optimal dynamic investment strategy of online advertising in different situations is formulated. Finally, the model is extended to n-dimensional space: the competition in completely symmetrical, semisymmetric and asymmetric cases is extended to$n$-dimensional space, the number of industry-saturated network marketing enterprises and the sufficient conditions of market share $x \in [0, 1]$ are discussed, and the relevant management enlightenment is obtained.

This study brings four innovations to the literature. First, with the diversification of online advertising types, the charging mode of online advertising is no longer single, and new billing methods such as CPC have emerged (Sisodia & Sisodia, 2022). In this paper, the spillover effect is introduced to describe the influence of different types of online advertising. Then the cost function of online advertising is improved to solve the problem that different types of online advertising cannot be integrated because of the differences in billing modes. This point has not been mentioned in previous literature. Second, in terms of the dynamic delivery of traditional advertisements, most studies use the N-A model, which is a relatively simple form. However, goodwill can only be evaluated when the enterprise is undergoing mergers, bankruptcy liquidation, or other major events in actual management practice, making it difficult to conduct real-time monitoring. This study adopts an online advertising limited model that is more complex than the basic model to conduct our research and obtain our results, making them more suitable for management. Third, combined with the characteristics of online advertising and the advantages of the traditional differential game model, this study creatively proposes a dynamic strategy model of online advertising in the competitive market to address the problem of the multicycle online advertising strategy of enterprises and avoid the losses caused by short-sighted investment. Fourth, due to the simplicity of constructing, solving, and analyzing oligopoly market theory models, as well as the availability of single enterprise data, most current research on online advertising targets oligopoly markets. However, few e-commerce enterprises can monopolize the market. Therefore, this study applies the online advertising dynamic strategy model to comprehensively analyze the optimal online advertising investment strategy in three oligopoly/multi-oligopoly competitive markets, breaking through the theoretical research gap that only analyzes oligopoly and does not match the actual market environment. This approach effectively solves the problem of determining online advertising amounts under different market situations, improves the market research framework, and makes the theoretical research framework of this study more in line with the market environment of online advertising investment.

Our analysis yields several important findings. (1) Interestingly, this study concludes that the spillover effect disturbs different types of online advertising but has no impact on the total investment and profits of online advertising, a result that does not seem to be consistent with our understanding. (2) Overall, when enterprises are in a symmetric state, their market stability, unit profit, decay coefficient, and online advertising factors are not necessarily related and are all 1/3. In the semi-symmetric and asymmetric states, if a dominant enterprise blindly invests in online advertising, as long as external interference factors are strengthened, then the strategy will have a greater impact on the advantage enterprise. (3) In the symmetric case, enterprises with lower initial market share are more likely to be disturbed by the spillover effect. However, in the case of asymmetry and semi-symmetry, superior enterprises are more likely to be disturbed by the spillover effect while inferior enterprises are more likely to ignore the disturbance. (4) We prove a classic finding that new entrants will be squeezed out of the market when there are more than four enterprises in the industry because of losses (Naik et al., 2008).

The rest of this paper is organized as follows. Section 2 reviews the related literature. The third part designs the basic model of dynamic investment in online advertising under the competition of three oligarchs, and the fourth parts solve the dynamic investment model of online advertising in different situations. The fifth part carries out the empirical example to formulate the optimal online advertising strategy under different situations. In Sect. 6, we extend the three-oligarch model to multi-oligopoly competition. Finally, conclusions and managerial insights are given in Sect. 7. Proofs are provided in the Appendix.

2 Literature review

This study mainly involves online advertising, advertising dynamic investment and advertising spillover effect. Relevant literature can be divided into several streams.

2.1 Online advertising

The first related stream of literature is on online advertising. At present, the research on online advertising is mostly focused on advertising evaluation (Singal et al., 2022; Skiera et al., 2013; Ye et al., 2022; Dahooie et al., 2022), while the research on the investment strategy of online advertising in the competitive market is divided into two categories, namely, public and targeted online advertising. The former refers to the indiscriminate delivery of online advertising to the public, focusing on determining the amount of investment in online advertising, which is similar to the object and purpose of the text study. For example, Zhao & Nugurnry (2018) applied variational inequalities to explore how competitive enterprises in the same industry invest limited advertising budgets into different websites. Viscolani (2012) assumed that the demand function is a piecewise linear function of product reputation, constructs the online advertising competition model of complementary production enterprises, and concludes that advertising investment positively impacts its sales and a negative impact on the sales of competitors.

On the other hand, targeted online advertising uses technical means to select appropriate opportunities to send targeted product promotion information for different types of advertising audiences (Iyer et al., 2011). In the study of targeted online advertising and determining the investment of targeted online advertising, scholars focus on the advantages and disadvantages of targeted online advertising. For example: (1) Some scholars believe that targeted advertising can not only increase profits but also reduce waste. Among them, Shin & Shin (2022) were the first to point out that enterprises should put targeted online advertising into their advantageous markets to increase their balanced profits. After modelling and analysing the targeted online advertising of enterprise competitors, Recently, Zhang et al. (2022) discussed the competition of duopoly enterprises under cost asymmetry. This article reports that targeted advertising means less waste, more flexible strategies and less fierce competition and points out that targeted advertising indirectly narrows the cost gap between the two enterprises; Shin & Shin (2022) concluded that enterprises should avoid using offensive strategies if they want to make higher profits. In general, targeted advertising reduces the degree of competition among enterprises. (2) However, some scholars believe that targeted online advertising intensifies competition among enterprises and eventually lead to a decline in profits. Through the comparison of online and offline advertising, Bergemann et al. (2011) proved that the positive benefit of targeted advertising is at the cost of price discrimination against consumers and reducing consumer surplus. Moorthy & Tehrani (2023) and Cheng & Dogan (2023) held that when targeted advertisements are placed for consumers with different preferences and discriminated pricing is implemented, it leads to increased competition and a decline in profits.

2.2 Advertising dynamic investment

The second stream of relevant literature investigates dynamic advertising investment under competitive conditions. In management practice, the demand for advertising usually changes over time. How do enterprises dynamically allocate limited advertising budgets over time to achieve the purpose of increasing sales or maximizing profits? To solve this problem, the dynamic differential equation considering the time factor is usually used to describe this advertising dynamic investment problem. From the literature, it is rare to describe the dynamic decision-making process of online advertising, so the research related to the dynamic investment of traditional advertising is summarized.

Common advertising dynamic investment strategy models include the Nerlove-Arrow model (Nerlove & Arrow, 1962; Chintagunta, 1993; Frank & Bass, 2007; Amrouche et al., 2008; Tapiero, 1979; Gozzi et al., 2009; Gao & Souza, 2022; Ghosh & Shah, 2022; Du et al., 2015; Wang et al., 2020; Wu & Kung, 2020; Wu & Kung, 2020; Xu & Wang, 2022), Lanchester model (Martin-Herrán et al., 2012; Sorger, 1989), Vidale–Wolfe model (Erickson, 2009; Ozga, 1960; Ringbeck, 1985; Sethi, 1983; Sethi et al., 2008; Vidale & Wolfe, 1957), Bass model (Bass, 1969; Horsky & Simon, 1983; Cosguner & Seetharaman, 2022; Han et al., 2022; Zhang et al., 2022; Kang et al., 2022; Lin & Li, 2015) and other advertising dynamic models. Among them, the Nerlove-Arrow model and Lanchester model are often used in competitive markets.

Nerlove-Arrow model: Nerlove & Arrow (1962) reported that investment in corporate advertising is an investment that helps to increase corporate goodwill. Through advertising to change consumer preferences so that the shape of the demand function is changed and moved upward, at the same time, due to the existence of competitor advertising, consumers turn to new consumer brands under these temptations, resulting in the decline of corporate goodwill. According to this idea, Nerlove and Arrow proposed the advertising goodwill model (Nmura model). Chintagunta (1993) and Bass (2007) further extended the N-A model to the multi-oligopoly market; however, the benchmark model does not consider the negative impact of competitors' advertising investment on their corporate goodwill. Given such a practical problem, Amrouche et al. (2008) took this factor into account in its research. Through the research, it is concluded that enterprises must invest much advertising to maintain high goodwill in the process of competition. Additionally, with the vigorous development of stochastic differential equation theory, the stochastic Nmura model has sprung up. Tapiero (1979) established a stochastic goodwill model of competitive enterprises by using stochastic dynamic optimization theory for the first time. The research suggests that the advertising investment of enterprises is a kind of venture capital, and enterprises with risk preference invest more advertising; otherwise, they invest less. Gozzi et al. (2009) explained the causes of random goodwill and used this model to study the influence of the advertising delay effect on the value function and optimal strategy. In recent years, N-A model has been widely used in the field of low-carbon emission reduction. On the one hand, carbon emission reduction is a dynamic process that changes over time, similar to the advertising process. Therefore, based on the principle of N-A model, a carbon emission reduction model is constructed (Gao & Souza, 2022; Ghosh & Shah, 2022; Du et al., 2015); On the other hand, the N-A model is directly applied to the process of joint emission reduction. The so-called joint emission reduction means that manufacturers are responsible for low-carbon production, while retailers are responsible for the advertising of low-carbon products (Wang et al., 2020; Wu, 2011; Wu & Kung, 2020; Xu & Wang, 2022).

Lanchester model In the competitive market, the role of advertising is to attract competitors' customers to expand the market share of their products. Based on this premise, Sorger (1989) proposed the Lanchester model and proved that in the duopoly market, the optimal advertising input is the decreasing function of enterprise's market share and the increasing function of the shadow price. Martin et al. (2012) discussed the interaction between offensive and defensive marketing activities, and the results imply that the amount of advertising investment depends to a large extent on the position of the enterprise in the market.

Other dynamic models Grosset et al. (2011) built a nonlinear quadratic differential game model by assuming that the advertisements of two competing enterprises have predatory effects (predatory phenomenon) and proposed the optimal advertising strategy according to Markov Nash equilibrium. Jiang et al. (2017) constructed a dynamic advertising competition model with the degree of promotion as a variable, and then the existence and stability of periodic T in the model is proven according to discrete mapping. Finally, central popularity theory is adopted to analyse the doubling bifurcation in the model to solve the optimal advertising investment of competitive enterprises.

2.3 Spillover effect

The third stream of relevant literature focuses on spillover effect. In recent years, there have been many studies on the spillover effect of traditional advertising in competitive enterprises (Li et al., 2023a, 2023b; Song & Li, 2011; Wu et al., 2022; Zheng & Huang, 2022). However, the traditional advertising spillover effect studied is limited to the indirect spillover effect of advertising on other complementary enterprises, rather than the internal spillover effect between traditional advertisements, because it is impossible to record and track the data of traditional advertising placement, browsing and purchase in real-time.

With the development of network technology, consumers' behaviour in online advertising can be recorded and tracked in time, which makes it possible to study the spillover effect between online advertisements. However, few scholars have studied the direct spillover effect between online advertisements within enterprises. Rutz & Bucklin (2011) discovered that overall search behaviour positively affects consumers' future brand search behaviour using a Bayesian model. Using the clickstream data of online sales of consumer electronics manufacturers' websites, Xu et al. (2012) established a Bayesian hierarchical model based on the dynamic interaction among many kinds of online advertisements and used independent random effects to represent consumer heterogeneity to study the spillover effect between different online advertisements and their effects on purchase conversion. Kireyev et al. (2015) studied the interaction between display advertising and search advertising through bank data and confirmed that display advertising significantly impacts search advertising. Li & Kannan (2014) proposed a measurement model that analyses consumers' acceptance of online advertising, the changes in consumers' access to these advertisements over time, and consumers' purchasing behaviour after visiting the site and then studied the spillover effect of advertising access on product purchases.

Our work follows the extant literature but differs in the following respects. First, the cost function of online advertising is improved to describe the impact between different types of online advertising by introducing spillover effect, which is different from the problem that most of the investment cost descriptions of online advertising only consider fixed costs (a sunk cost) in the previous literature. Second, taking the traditional advertising dynamic game model as the methodology, a dynamic investment model in line with the characteristics of online advertising is constructed. Third, the disturbing effects of spillover effect on different factors in different situations have been explored, and some interesting conclusions have been obtained, which can help enterprises develop optimal strategies in different situations. Fourth, in the expansion model, the competition in the three-oligopoly market is extended to multi-oligarchs. It is worth noting that few studies have examined the situation of multi-oligopoly competition because of the complexity of the differential game model.

3 Model formulation

According to the market research company eMarketer in 2021, Alibaba, JD.com and Pinduoduo have market shares of 58.2%, 16.3% and 5.2%,[6] respectively, in China's retail e-commerce field.[7] Its total market share has reached 79.7%, which can be approximately abstracted as a three-oligarch competitive enterprise. Among them, in the "Singles Day" shopping festival, the three giants compete for online consumers through a series of forms, such as display ads, search engine ads and e-mail ads, and the investment in online advertising has reached 40% of their income. Based on the above analysis, this paper studies the dynamic investment strategy of online advertising in three oligarchic competitive enterprises. Table 1 summarizes the key notations used throughout the paper.

Suppose that the online advertising investment of the online marketing enterprise i at time t is $A_i(t)$; then, according to the Lanchester model (Sorger, 1989), the dynamic relationship model between advertising investment and market share of the online marketing enterprise i is shown as follows:

$$\begin{cases} \dfrac{dx_i}{dt} = \rho_i A_i \sqrt{1-x_i} - \sum\limits_{j\in 3, j\neq i} \delta_j A_j \sqrt{1-x_j} - \xi\left(x_i - \dfrac{1}{n}\right) \\ x_i(0) = x_{i0} \\ s.t. \sum\limits_{i=3} x_i(t) = 1 \end{cases} \tag{1}$$

where $x_i(t)$ represents the market share of enterprise i at time $t.x_{i0}(t)$ represents the initial market share of the three e-commerce enterprises and satisfies the constraints $\sum x_i(t) = 1$. ρ_i represents the parameters of the online advertising effect,δ_i represents the attenuation coefficient of competitive online advertising, and ξ represents the interference coefficient.

What needs to be noted is that online advertising investment A mainly has the following two forms: one is the amount of advertising investment (A_0) per click, such as search engine advertising and mail advertising; the other is the amount of advertising investment (A_T) charged according to the playing time of fixed advertising spaces, such as portal banner ads, all kinds of display advertisements (Kim et al., 2021). In general, online marketing enterprises put advertisements on many types of online advertising channels at the same time, so at time t, the total investment in online advertising is expressed as $A(t) = A_0(t) + A_T(t)$. The advertising expenditure charged according to the broadcast time of the fixed advertising space is often similar to the one-time expenditure of traditional advertising, while the pay-per-click advertising expenditure is affected by the number of ad clicks (n_c) and the cost per thousand clicks (p_{cpc}), namely, $A_0(t) = n_c \times p_{cpc}$.

[6] Available at stock.10jqka.com.cn.

[7] Available at https://www.sohu.com/a/400072770_322372.

Table 1 List of notations

Symbol	Description
t	Time
n	The number of competing enterprises
$A(t)$	Online advertising investments at time t
$A_0(t)$	Click on charge advertising einvestments at time t
$A_\mathrm{T}(t)$	Fixed-positioned online advertising expenditures at time t
$u(t)$	Cost of online advertising at time t
$x(t)$	The market share at time t
$x_i(t)$	The market share of enterprise i at time t
$\overline{x}(t)$	Steady-state market shares
V_i	The net profit
$V_i(x_j)$	The marginal increment of firm i 's total discounted profit relative to enterprise j's market share gain
m_i	The unit profits of network marketing enterprise i
n_c	Advertising clicks
p_{cpc}	Cost-per-thousand impressions
ρ	The online advertising influence coefficient
δ	The market attenuation coefficient
ς	Interference coefficient
μ	The level of the online advertising's spillover effect
λ	The discount rates

Since display advertisements usually have strong spillover effect (Li & Kannan, 2014), we might also assume that the number of clicks on online advertisements is also partly affected by the investment in fixed advertising space, that is, $n_c = [n_0 + \mu A_T(t)]$. Among them, n_0 represents the number of clicks without fixed advertising for investment ads,μ represents the level of spillover effect of advertisements charged by fixed advertising space on pay-per-click advertising, $\mu \in [0, 1]$, the larger the μ, the stronger the level of the spillover effect of advertising. In the actual measurement, the influencing factor of the spillover effect depends on the location of the fixed-location paid advertisement of the enterprise. When the enterprise does not put the fixed-location advertisement, there is no spillover effect, that is,$\mu = 0$. When the enterprise puts the fixed-location advertisement in the core position of the web page, the spillover effect reaches the maximum, that is $\mu = 1$. If there is a partial overflow when placing fixed-location ads in other locations on the web page, that is, $\mu \in (0, 1)$. Accordingly, the online advertising investment function can be expressed as follows:

$$A = A_0 + A_T = [n_0 + \mu A_T(t)]p + A_T \tag{2}$$

where n_0 is a constant, and the general assumption is $n_0 = 0$. It is worth pointing out that even if $n_0 \neq 0$, it does not change the conclusion of this paper but only increases the complexity of mathematical processing.

At this time, substituting Eq. (2) into Eq. (1) to obtain the extended Lanchester model, as shown in Eq. (3):

$$\begin{cases} \frac{dx_i}{dt} = \rho_i(1+\mu_i p)A_{Ti}\sqrt{1-x_i} - \sum_{j\in 3, j\neq i} \delta_j\left(1+\mu_j p\right)A_{Tj}\sqrt{1-x_j} - \xi\left(x_i - \frac{1}{n}\right) \\ x_i(0) = x_{i0} \\ s.t. \sum_{i=3} x_i(t) = 1 \end{cases} \tag{3}$$

On the right side of Eq. (3): the first item is a positive number, indicating the market share gained due to online advertising. The second is negative, which indicates the loss of market share caused by online advertising competition, and the last indicates the change in market share due to noncompetitive factors (Dolgui et al., 2018). When the market share of an enterprise exceeds the industry average ($1/n$), the third item is negative, while when the market share is lower than the industry average, the third item is positive. The management meaning of its representative is as follows: when an enterprise has a higher market share, it is more likely to be disturbed by external factors, which makes the market share decrease gradually. For example, due to the global COVID-19 pandemic in 2020, Amazon, the largest online retailer in North America, lost its market share from 43.8% in 2019 to 43.8% in 2020 (Ivanov & Dolgui, 2020, 2021).

Proposition 1 The relationship between the online advertising effect parameters and the online advertising attenuation coefficient is $\rho_i = 2\delta_i$. This means that compared with competitors' online advertising, market share is more sensitive to their online advertising.

Proof see appendix A1

Substituting δ_i into Eq. (3), then Eq. (4) is:

$$\frac{dx_i}{dt} = \rho_i(1+\mu_i p)A_{Ti}\sqrt{1-x_i} - \sum_{j\in 3, j\neq i} \frac{\rho_j}{2}\left(1+\mu_j p\right)A_j\sqrt{1-x_j} - \xi\left(x_i - \frac{1}{n}\right). \tag{4}$$

Assuming that there is a definite planning time domain t, $t \in [0, T]$, and each enterprise seeks to maximize the discounted cash flow within time t, then the net profit of enterprise i is:

$$\begin{cases} V_1(x_1, x_2, x_3) = \max_{A_i(t)\geq 0} \int_0^T e^{-\lambda t}\left[m_1 x_1(t) - \frac{c}{2}(1+\mu_1 p)^2 A_{T1}^2\right]dt \\ V_2(x_1, x_2, x_3) = \max_{A_i(t)\geq 0} \int_0^T e^{-\lambda t}\left[m_2 x_2(t) - \frac{c}{2}(1+\mu_2 p)^2 A_{T2}^2\right]dt \\ V_3(x_1, x_2, x_3) = \max_{A_i(t)\geq 0} \int_0^T e^{-\lambda t}\left[m_3 x_3(t) - \frac{c}{2}(1+\mu_2 p)^2 A_{T3}^2\right]dt \end{cases} \tag{5}$$

where $m_i > 0$ represents the unit profit of network marketing enterprise i and λ represents the discount rate of the network marketing company. To simplify the comparison and analysis, this article assumes that the discount rates of the enterprises are equal (Soberman, 2004).

Online advertising investment is the level of an enterprise when placing online advertisements. The essence of the online advertising cost function $u_i(t) = {c[(1+\mu p)A_{Ti}]^2}/{2}$ is a production function for producing x (Wu & Kung, 2020). c measures advertising promotional efficiency (or cost coefficient of online advertising). In previous studies, the assumption of a quadratic cost function was confirmed (Kelly et al., 2023). Parameter c is dependent on the governmental policies on advertising; for instance, it includes the effects of a tax on advertising.

4 Equilibrium analysis

In the part of the model analysis, the two angles of complete symmetry and semisymmetry are focused on the analysis of the optimal online advertising investment and the optimal profit of each network marketing enterprise. In addition, the dynamic change process of online advertising investment and optimal profit are analysed in detail in the numerical simulation due to the complexity of the asymmetric state solution.

Lemma 1 $\eta_1, \eta_2, \eta_3, B_1, B_2, B_3, \gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{23}, \gamma_{31}, \gamma_{32}$ are constants and satisfy the system of equations:

$$\begin{cases} \lambda\eta_1 = \dfrac{1}{8}\left[\dfrac{8\xi B_1}{n} + \dfrac{8\xi\gamma_{12}}{n} + \dfrac{8\xi\gamma_{13}}{n} - \Gamma_1 - \Gamma_2 - \Gamma_3\right] \\ \lambda B_1 = \dfrac{1}{8}[8m_1 - 8\xi B_1 + \Gamma_1] \\ \lambda\gamma_{12} = \dfrac{1}{8}[\Gamma_2 - 8\xi V\gamma_{12}] \\ \lambda\gamma_{13} = \dfrac{1}{8}[\Gamma_3 - 8\xi\gamma_{13}] \end{cases}, \tag{6}$$

$$\begin{cases} \lambda\eta_2 = \dfrac{1}{8}\left[\dfrac{8\xi\gamma_{21}}{n} + \dfrac{8\xi B_2}{n} + \dfrac{8\xi\gamma_{23}}{n} - \Gamma'_1 - \Gamma'_2 - \Gamma'_3\right] \\ \lambda\gamma_{21} = \dfrac{1}{8}[\Gamma'_1 - 8\xi\gamma_{21}] \\ \lambda B_2 = \dfrac{1}{8}[8m_2 - 8\xi B_2 + \Gamma'_2] \\ \lambda\gamma_{23} = \dfrac{1}{8}[\Gamma'_3 - 8\xi\gamma_{23}] \end{cases}, \tag{7}$$

$$\begin{cases} \lambda\eta_3 = \dfrac{1}{8}\left[\dfrac{8\xi\gamma_{31}}{n} + \dfrac{8\xi\gamma_{32}}{n} + \dfrac{8\xi B_3}{n} - \Gamma''_1 - \Gamma''_2 - \Gamma''_3\right] \\ \lambda\gamma_{31} = \dfrac{1}{8}[\Gamma''_1 - 8\xi\gamma_{31}] \\ \lambda\gamma_{32} = \dfrac{1}{8}[\Gamma''_2 - 8\xi\gamma_{32}] \\ \lambda B_3 = \dfrac{1}{8}[8m_3 - 8\xi B_3 + \Gamma''_3] \end{cases}. \tag{8}$$

We can now prove the following result characterizing the optimal advertising investments and payoffs.

(1) The optimal fixed-position advertising investment and total online advertising investment of online marketing enterprises are:

$$\begin{cases} A^*_{T1} = \dfrac{\rho_1[2B_1 - \gamma_{12} - \gamma_{13}]\sqrt{1-x_1}}{2c(1+\mu_1 p)} \\ A^*_{T2} = \dfrac{\rho_2[2B_2 - \gamma_{21} - \gamma_{23}]\sqrt{1-x_2}}{2c(1+\mu_2 p)} \\ A^*_{T3} = \dfrac{\rho_2[2B_3 - \gamma_{31} - \gamma_{32}]\sqrt{1-x_3}}{2c(1+\mu_3 p)} \end{cases}, \tag{9}$$

$$\begin{cases} A_1^* = \dfrac{\rho_1\left[2B_1 - \gamma_{12} - \gamma_{13}\right]\sqrt{1-x_1}}{2c} \\ A_2^* = \dfrac{\rho_2\left[2B_2 - \gamma_{21} - \gamma_{23}\right]\sqrt{1-x_2}}{2c} \\ A_3^* = \dfrac{\rho_2\left[2B_3 - \gamma_{31} - \gamma_{32}\right]\sqrt{1-x_3}}{2c} \end{cases}. \tag{10}$$

(2) The optimal profits of network marketing enterprises are:

$$\begin{cases} V_1(x) = \eta_1 + B_1x_1 + \gamma_{12}x_2 + \gamma_{13}x_2 \\ V_2(x) = \eta_2 + B_2x_2 + \gamma_{21}x_1 + \gamma_{23}x_3 \\ V_m(x) = \eta_3 + B_3x_3 + \gamma_{31}x_1 + \gamma_{32}x_2 \end{cases}, \tag{11}$$

Wherein: $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_1\prime, \Gamma_2\prime, \Gamma_3\prime, \Gamma_1^{''}, \Gamma_2^{''}, \Gamma_3^{''}$ are constants.

Proof see appendix A2:

When $\rho_1 \neq \rho_2 \neq \rho_3, \mu_1 \neq \mu_2 \neq \mu_3$,$m_1 \neq m_2 \neq m_3$, the enterprise is in an asymmetric state. The exact solution cannot be given because 12 equations need to be solved because of the complexity of mathematical operations. As a result, the case of the numerical solution of the asymmetric retailer and the related properties are mainly introduced in the simulation and derivation section.

4.1 Solution of symmetric network marketing enterprise model

When the enterprise is in a symmetric situation, there are $m_1 = m_2 = m_3 = m$, $\rho_1 = \rho_2 = \rho_3 = \rho$ and $\mu_1 = \mu_2 = \mu_3 = \mu$. According to symmetry, the profit of enterprise i at this time satisfies the equation:

$$V_{\mathrm{i}}(x_1, x_2, x_3) = \max_{A_i(t)\geq 0} \int_0^T e^{-\lambda t}\left[mx_i(t) - \frac{c}{2}(1+\mu p)^2 A_{Ti}^2\right]dt. \tag{12}$$

When competing enterprises are in a completely symmetric state, their optimal online advertising investment and profits are shown in Proposition 3:

Lemma 2 In the case of symmetry:

(1) The optimal fixed-position advertising investment and total online advertising investment of online marketing enterprises are:

$$\begin{cases} A_{Ti}^* = \dfrac{\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)} \\ A_i^* = \dfrac{\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2} \end{cases}. \tag{13}$$

(2) The optimal profits of network marketing enterprises are:

$$V_i = \frac{-B^2n\rho^2 + 2Bc\xi + 2Bn\rho^2\gamma + 4c\delta\gamma - n\rho^2\gamma^2}{2cn\lambda} + \frac{2m\rho^2(\xi+\lambda) - c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2}x_i$$
$$+ \frac{2m\rho^2(\xi+\lambda) + c(\xi+\lambda)^3 - \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2}\sum_{j=1, j\neq i}^{n} x_j, \tag{14}$$

$$\textbf{Among:}\begin{cases} B=\dfrac{2m\rho^2(\xi+\lambda)-c(\xi+\lambda)^3+\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2} \\ \gamma=\dfrac{2m\rho^2(\xi+\lambda)+c(\xi+\lambda)^3-\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2} \end{cases}.$$

Proof see appendix A3

Corollary 1 In the symmetric state, the spillover effect is inversely proportional to the fixed location online advertising investment, directly proportional to the pay-per-click online advertising investment and has nothing to do with the total amount of online advertising. The investment in online advertising is inversely proportional to the interference coefficient ξ and directly proportional to the ρ parameter of the online advertising effect.

Proof. See Appendix A3

4.2 Solution to semisymmetric network marketing enterprise model

When the enterprise is in the case of semisymmetry, it is generally assumed that enterprise 2 and enterprise 3 are in a state of symmetry, that is, $m_2 = m_3 = m$, $\rho_2 = \rho_3=\rho$ and $\mu_2 = \mu_3 = \mu$. Then the profit function of the enterprise is:

$$\begin{cases} V_1(x_1,x_2,x_3)=\max\limits_{A_i(t)\geq 0}\displaystyle\int_0^T e^{-\lambda t}\left[m_1x_1(t)-\frac{c}{2}(1+\mu_1 p)^2A_{T1}^2\right]dt \\ V_2(x_1,x_2,x_3)=\max\limits_{A_i(t)\geq 0}\displaystyle\int_0^T e^{-\lambda t}\left[mx_2(t)-\frac{c}{2}(1+\mu p)^2A_{T2}^2\right]dt \\ V_3(x_1,x_2,x_3)=\max\limits_{A_i(t)\geq 0}\displaystyle\int_0^T e^{-\lambda t}\left[mx_3(t)-\frac{c}{2}(1+\mu p)^2A_{T3}^2\right]dt \end{cases}. \tag{15}$$

When the competitive enterprise is in a semisymmetric state, its optimal online advertising investment and profit are shown in Proposition 4:

Lemma 3 In the case of semi-symmetry:

(1) The optimal fixed advertising investment and total online advertising investment of online marketing enterprises are:

$$\begin{cases} A_{T1}^*=\dfrac{(\beta_1-\psi_1)\rho_1\sqrt{1-x_1}}{c(1+\mu_1 p)} \\ A_{T2}^*=\dfrac{(2\beta-\psi-\varphi)\rho\sqrt{1-x_2}}{2c(1+\mu_2 p)} \\ A_{T3}^*=\dfrac{(2\beta-\psi-\varphi)\rho\sqrt{1-x_3}}{2c(1+\mu_3 p)} \end{cases},$$

$$\begin{cases} A_1^*=\dfrac{(\beta_1-\psi_1)\rho_1\sqrt{1-x_1}}{c} \\ A_2^*=\dfrac{(2\beta-\psi-\varphi)\rho\sqrt{1-x_2}}{2c} \\ A_3^*=\dfrac{(2\beta-\psi-\varphi)\rho\sqrt{1-x_3}}{2c} \end{cases}. \tag{16}$$

(2) The optimal profits of network marketing enterprises are:

$$\begin{cases} V_1 = \alpha_1 + \beta_1 x_1 + \psi_1 (x_2 + x_3) \\ V_2 = \alpha + \beta x_2 + \psi x_1 + \varphi x_3 \\ V_3 = \alpha + \beta x_3 + \psi x_1 + \varphi x_2 \end{cases}. \tag{17}$$

where all $\alpha_1, \alpha, \beta_1, \beta, \psi_1, \psi, \varphi_1, \varphi$ are constant and satisfy Eq. (40).

Proof. See Appendix A4

Corollary 2: In the semisymmetric state, the spillover effect is inversely proportional to the fixed-location online advertising investment, directly proportional to the pay-per-click online advertising investment and has nothing to do with the total amount of online advertising.

This proof is similar to corollary 1 and is not repeated.

4.3 Steady-state market shares

The meaning of "stability" in this paper means that the market share of online marketing enterprises in long-term competition no longer changes with time, and the mathematical meaning is $\dot{x}_i(t)=0$. Substituting the equilibrium online advertising investment amount of Eq. (10) into Eq. (4), the system state equation of market share is:

$$\begin{cases} \dot{x}_1(t) = \dfrac{6\left[2\left(c\xi + B_1\rho_1^2\right) - \rho_1^2(\gamma_{12}+\gamma_{13})\right](1-x_1) - 3\rho_2^2(2B_2-\gamma_{21}-\gamma_{23})(1-x_2) - 3\rho_3^2(2B_3-\gamma_{31}-\gamma_{32})(1-x_3)}{12c} - \dfrac{2}{3}\xi \\ \dot{x}_2(t) = \dfrac{6\left[2\left(c\xi + B_2\rho_2^2\right) - \rho_2^2(\gamma_{21}+\gamma_{23})\right](1-x_2) - 3\rho_1^2(2B_1-\gamma_{12}-\gamma_{13})(1-x_1) - 3\rho_3^2(2B_3-\gamma_{31}-\gamma_{32})(1-x_3)}{12c} - \dfrac{2}{3}\xi \\ \dot{x}_3(t) = \dfrac{6\left[2\left(c\xi + B_3\rho_3^2\right) - \rho_3^2(\gamma_{31}+\gamma_{32})\right](1-x_3) - 3\rho_1^2(2B_1-\gamma_{12}-\gamma_{13})(1-x_1) - 3\rho_2^2(2B_2-\gamma_{21}-\gamma_{23})(1-x_2)}{12c} - \dfrac{2}{3}\xi \end{cases}. \tag{18}$$

Then, the solution of the equation is obtained, that is, the expression of the stable market share of the enterprise when the system is stable.

According to Eq. (18), when the system is in a stable state, the vector of the market share and the augmented matrix are in the form of:

$$\overline{x}(t) = \left[\begin{array}{ccc|c} \frac{2\left(c\xi+B_1\rho_1^2\right)-\rho_1^2(\gamma_{12}+\gamma_{13})}{2c} & -\frac{\rho_2^2(2B_2-\gamma_{21}-\gamma_{23})}{4c} & -\frac{\rho_3^2(2B_3-\gamma_{31}-\gamma_{32})}{4c} & \frac{2}{3}\xi \\ -\frac{\rho_1^2(2B_1-\gamma_{12}-\gamma_{13})}{4c} & \frac{2\left(c\xi+B_2\rho_2^2\right)-\rho_2^2(\gamma_{21}+\gamma_{23})}{2c} & -\frac{\rho_3^2(2B_3-\gamma_{31}-\gamma_{32})}{4c} & \frac{2}{3}\xi \\ -\frac{\rho_1^2(2B_1-\gamma_{12}-\gamma_{13})}{4c} & -\frac{\rho_2^2(2B_2-\gamma_{21}-\gamma_{23})}{4c} & \frac{2\left(c\xi+B_3\rho_3^2\right)-\rho_3^2(\gamma_{31}+\gamma_{32})}{2c} & \frac{2}{3}\xi \end{array}\right] \begin{bmatrix} 1-x_1 \\ 1-x_2 \\ 1-x_3 \end{bmatrix} = 0 \tag{19}$$

Substituting the values of $V_i\left(x_j\right)$ in Propositions 1, 2 and 3 into Eqs. (18) and (19), the stable market share in symmetric, semisymmetric and asymmetric states is shown in Table 5 (Proof. See Appendix B1.):

According to Table 5, in the symmetric case, when the network marketing enterprise is in a stable state, its market share is $^1/_3$. That is, when the enterprise is in a symmetric state, its market stability is not necessarily related to the unit profit, attenuation coefficient or online advertising factor. Additionally, when the enterprise is in an asymmetric state because the result of the solution is too complex, its properties are analysed in detail in the simulation. In addition, for the case of semisymmetry, we can draw a more interesting inference, namely, corollary 3.

Corollary 3 When the network marketing enterprise is in a semisymmetric situation, if the online advertising investment of network marketing enterprise 1 is higher than that of enterprises 2 and 3, then the market share of enterprise 1 is inversely proportional to the interference factor, and the market share of enterprises 2 and 3 is proportional to the interference factor. If the online advertising investment of enterprise 1 is lower than that of enterprises 2 and 3, the market share of enterprise 1 is directly proportional to the interference factor, and the market share of enterprises 2 and 3 is inversely proportional to the interference factor. Specifically:

$$D[\overline{x}_1(\xi)] = \begin{cases} \nearrow, A_{2,3} > A_1 \\ \searrow, A_{2,3} \leq A_1 \end{cases}, \quad D\left[\overline{x}_{2,3}(\xi)\right] = \begin{cases} \nearrow, A_{2,3} < A_1 \\ \searrow, A_{2,3} \geq A_1 \end{cases}. \tag{20}$$

To some extent, enterprises put in a large number of advertisements, resulting in fierce competition. At this time, when the external interference factors are enhanced, they counteract the negative effects of competition on enterprise 1, and the market share of enterprise 1 increases, while enterprise 2 or 3 is blindly put into practice because of its online advertising. This, coupled with the enhancement of external interference factors, makes its market share decline, bringing adverse effects. For example, the lawsuit caused by the vicious advertising competition between Jiaduobao and Wong Lo Kat has achieved Heqizheng. Proof. See Appendix A5**.**

To Sum up, what is surprising is that the total amount of online advertising is only related to external disturbances. If external uncertainties increase, companies need to reduce online advertising no matter what situation they are in. Once the total amount of online advertising is determined. If the spillover effect of online advertising in the industry is strong, then it is necessary to increase the pay-per-click online advertising investment and decrease the fixed-location online advertising investment. The presentation of such a conclusion has greater significance for companies with limited cost budgets. It can maximize the effectiveness of online advertising in the case of a certain budget. In addition, the more aggressive a company is, the more likely it is to be hit by the competition. But for rising enterprises, however, can seize this opportunity. That is, they should properly reduce the amount of online advertising to reduce the operating costs of enterprises, and strive to achieve the status of the leading enterprises.

5 Empirical example

According to the equilibrium solution, the spillover effect disturbs different types of online advertising but does not affect the total amount of online advertising. Therefore, the empirical example explores the changes in market share, total investment and profit of online advertising with time and interference factors in symmetrical, semisymmetric and asymmetric cases. It should be noted that the change trend of different types of online advertising investment is consistent with the total online advertising investment, so we take the total online advertising investment as an example. Second, it analyses in detail the disturbance degree of spillover effect on different types of online advertising in different situations and combines these two points to formulate the optimal dynamic investment strategy of online advertising in different situations.

In this paper, the parameters p, c are based on Baidu alliance in nearly a year to get the average fee. ρ, ξ are obtained through behavioral data such as visits to consumer sites, searches, page views, favorites, and purchases. The parameter μ is based on the purchasing behavior of consumers (Han et al., 2023). The parameter m is obtained according to the 2021 financial annual report of the enterprise. Among them, when the competitive enterprises are in the symmetric state, the monitoring is the 2021 online advertising data of China Mobile, China Unicom and China Telecom; When the enterprise is in a semisymmetric state, it monitors the 2021 online advertising data of Didi, Huaxiaozhu Dache and T3 Chuxing; When the company is in the asymmetric state, the monitoring is Gree, Midea and Aux' online AD delivery data in 2021. Because we want to explore the disturbance of spillover effect on online advertising, the value of parameter μ will be changed in the process of analysis. Pay attention to the following sections for specific assignments.

5.1 Symmetric network marketing enterprise

When the network marketing enterprise is symmetrical, its assignment is shown in Table 2:

Enterprise 1 represents China Mobile, enterprise 2 represents China Unicom and enterprise 3 represents China Telecom. According to the values of Table 2 and Eqs. (4), (16) and (17), the changes in market share, online advertising investment and profits over time are shown in Figs. 4, 5 and 6:

According to Fig. 1, when the three enterprises are in a symmetric situation, their market shares are all $1/3$ in equilibrium, which is consistent with Table 5. As seen from Fig. 2, the amount of investment in online advertising is inversely proportional to the interference factor, which is consistent with corollary 1; from Fig. 3, it can be seen that profit is proportional to the interference factor. Specifically:

(1) Whether it is the amount of investment or profit in online advertising, they are all equal in the end. This is due to the indifference between enterprises, and the result is consistent with the "cartel collusion" in the Bertrand Paradox. That is, when the market is in equilibrium, the equilibrium price should be higher than the marginal cost, which is consistent with the reality of management. China Mobile, China Unicom and China Telecom have formed a monopoly situation among the three giants, but after a period of competition, the market share of the three giants tends to stabilize and gradually divide the market equally.

(2) The investment in online advertising is inversely proportional to the disturbance factor, and the profit is directly proportional to the disturbance factor. From the aspect of management, when the enterprise is in a symmetric state, the smaller the disturbance factor is, the

Table 2 Enterprise-related parameter values in symmetric mode

Parameters	m	ρ	ξ	p	λ	c
Value	0.25	0.3	0.4	4	0.05	2

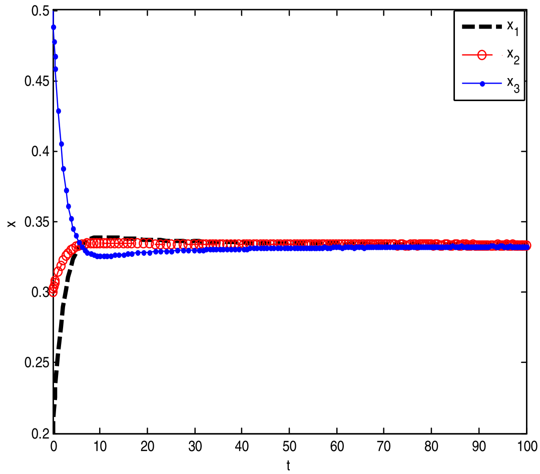

Fig. 1 Trend chart of market share changing with time when symmetrical

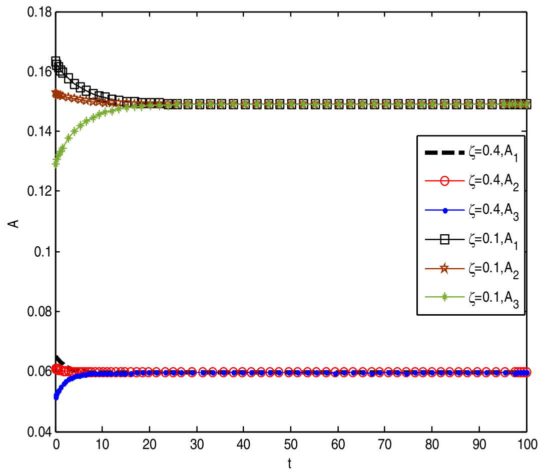

Fig. 2 Trend chart of total investment in online advertising over time when symmetrical

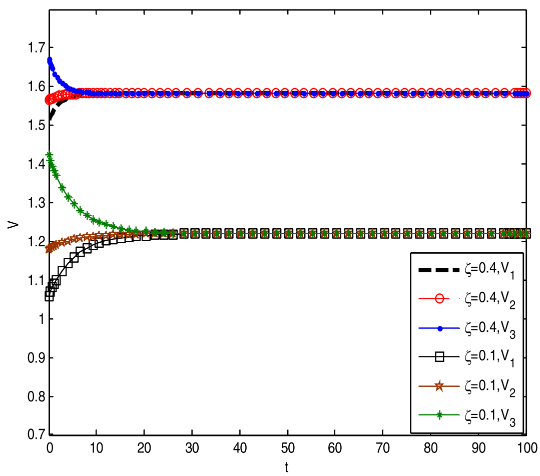

Fig. 3 trend chart of profit with time when symmetrical

smaller the uncertain factors in the outer market are, and the more advertisements are put in. When the disturbance factor is larger, and the uncertainty of the external market is larger, the assumption that an enterprise is a rational person reduces the delivery of online advertising. Enterprises become less aggressive when the external disturbance factor increases, which reduces the investment in online advertising, but profits increase because the market share remains the same. This indirectly warns enterprises that when the market is relatively stable, they should not blindly put online advertising, should comprehensively consider the cost and other factors, and control it within a certain range.

5.2 Semisymmetric network marketing enterprise

When the network marketing enterprise is symmetrical, its assignment is shown in Table 3:

Enterprise 1 represents Didi, enterprise 2 represents Huaxiaozhu Dache, and enterprise 3 represents T3 Chuxing. According to the values of Table 3 and Eqs. (4), (16) and (17), the changes in market share, online advertising investment and profits over time are shown in Figs. 4, 5 and 6:

(1) First, based on the parameters of Table 3 and according to Table 5, when the interference factor is δ=0.4, the stable market share of Didi, Huaxiaozhu Dache and T3 chuxing is x_1=0.38132, x_2=0.30934, x_3=0.30934, respectively. When the interference factor is $\delta = 0.1$, the stable market share of Didi, Huaxiaozhu Dache and T3 chuxing is x_1=0.515092, x_2=0.242454, x_3=0.242454, which is consistent with Fig. 4. Given its absolute dominance, it is understandable that Didi's market share is higher than that of Huaxiaozhu Dache and T3 Chuxing in stable times.

Table 3 Enterprise-related parameter values in semisymmetric mode

Parameters	m_1	m	ρ_1	ρ	ξ	p	λ	c
Value	0.35	0.25	0.4	0.3	0.4	0.4	0.05	2

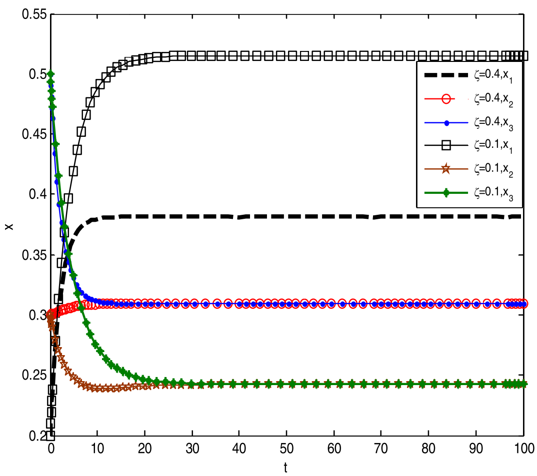

Fig. 4 Trend chart of market share changing with time under semisymmetry

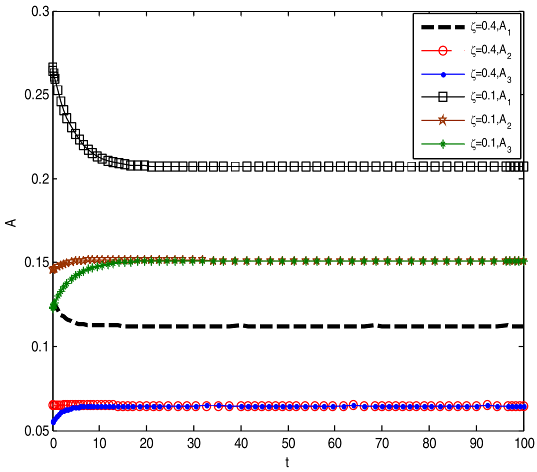

Fig. 5 Trend chart of total investment in online advertising over time under semisymmetry

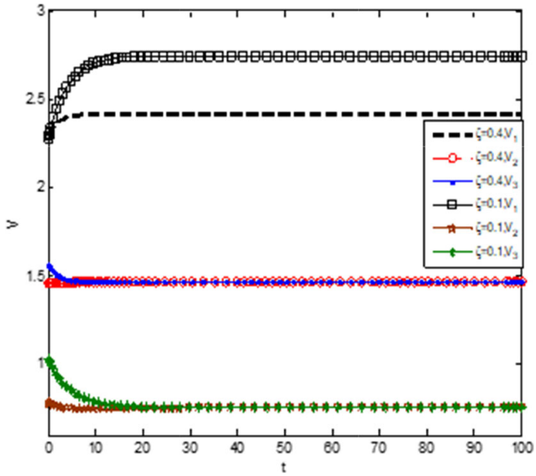

Fig. 6 Trend chart of profit with time under semisymmetry

(2) Second, Fig. 4 implies that the market share of Didi is inversely proportional to the external interference factor, while the market share of Huaxiaozhu Dache and T3 Chuxing is directly proportional to the external interference factor, which is also consistent with corollary 3. In the past 2 years, Didi has formed an absolute dominant position in the market. Huaxiaozhu Dache and T3 Chuxing have jointly cracked down on Didi, which has seen its market share rapidly decline. This brings enlightenment to the competitive enterprises under the condition of semisymmetry. It is not difficult to see from Figs. 4, 5 that when attacked by disadvantaged enterprises, enterprises in a dominant position should correspondingly reduce the investment in online advertising to ensure that their profits will not fall significantly. At this time, inferior enterprise capture the changes in the market and then reduce the amount of advertising, but it is worth noting that the extent of their reduction is less than that of superior enterprise. This way enables them to gain access to part of the market and increase profits.

Table 4 Enterprise-related parameter values in asymmetric mode

Parameters	m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	ξ	p	λ	c
Value	0.45	0.35	0.25	0.4	0.3	0.2	0.4	0.4	0.05	2

5.3 Asymmetric network marketing enterprise

When the network marketing enterprise is asymmetrical, its assignment is shown in Table 4:

Enterprise 1 represents Gree, enterprise 2 represents Midea, and enterprise 3 represents Aux. According to the values of Table 4 and Eqs. (4), (9) and (11), the changes in market share, online advertising investment and profits over time are shown in (7), (8) and (9):

(1) Based on the parameters of Table 4, according to Table 5, when the interference factor is δ=0.4, the stable market share of enterprises Gree, Media and Aux is x_1=0.404688, x_2=0.325004, x_3=0.270309, respectively. When the interference factor is ξ=0.1, the stable market share of Gree, Media and Aux is x_1=0.602375, x_2=0.38723, x_3=0.010347, which is consistent with Fig. 6.

(2) Gree air conditioners and Midea air conditioners quickly established a leading position in the industry because of their excellent technology and strong marketing means, while AUX shared the rest of the market, which lasted for nearly a decade. However, with the rise of the e-commerce industry at this time, Aux opened up an online sales platform to attack Gree and Midea, carving up the original market share of Midea and Aux, but its market share increased rapidly, and to compete with Gree and Midea. This point is also confirmed in Figs. 7, 8 and 9. Therefore, for disadvantaged enterprises, when external interference factors increase, they should first look for new sales means, and then appropriately reduce the amount of online advertising. As for the competitive enterprises, they are more susceptible to external influences, so they should strictly control the amount of online advertising and sales cost in the following investment.

In general, regardless of the state in which the competitive enterprises are in when the external interference factors are enhanced, the competitive enterprises reduce their investment

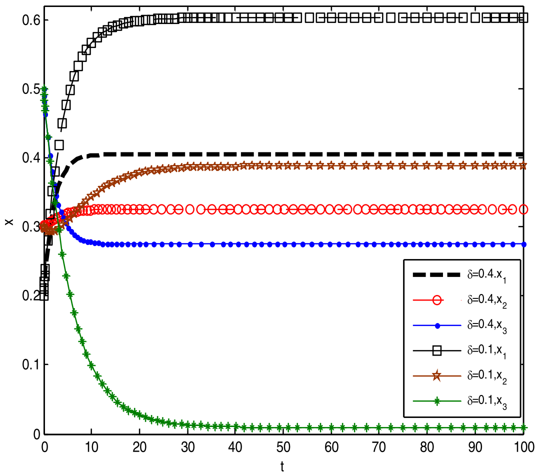

Fig. 7 Trend chart of market share changing with time under asymmetry

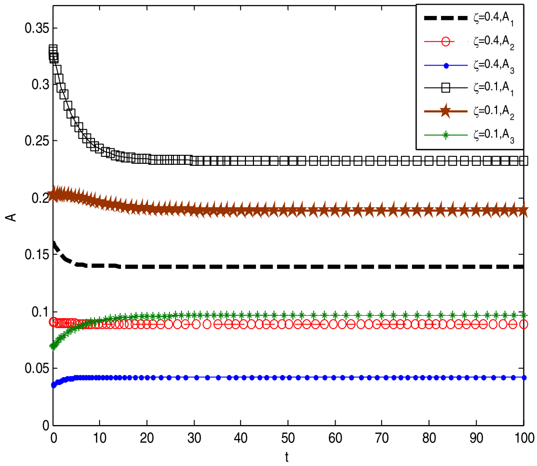

Fig. 8 Trend chart of total investment in online advertising over time under asymmetry

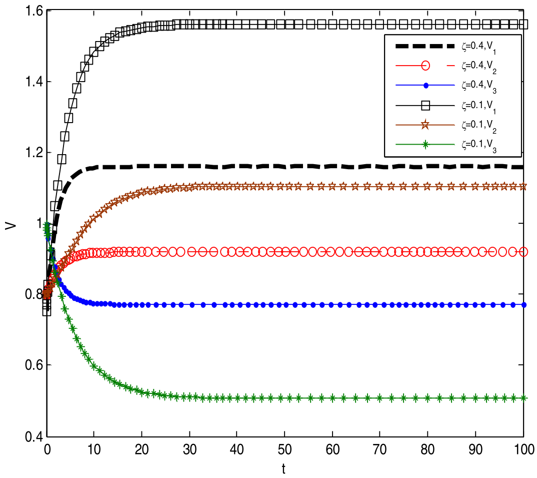

Fig. 9 Trend chart of profit over time under asymmetry

in online advertising; at the same time, if the enterprises are in a state of incomplete symmetry when the external interference factors increase, the market share and profits of the network marketing enterprises in the dominant position decline.

5.4 Analysis of the disturbance of spillover effect on different types of online advertising investment

This section focuses on exploring the sensitivity of different types of online advertising to spillover effect in different situations to provide more comprehensive suggestions for enterprises' investment strategies.

(1) Disturbance analysis of the spillover effect on fixed location online advertising investment.

①According to Fig. 10, the amount of investment in (a–c), fixed-location online advertising is inversely proportional to the spillover effect, which is consistent with inferences 1, 2, and

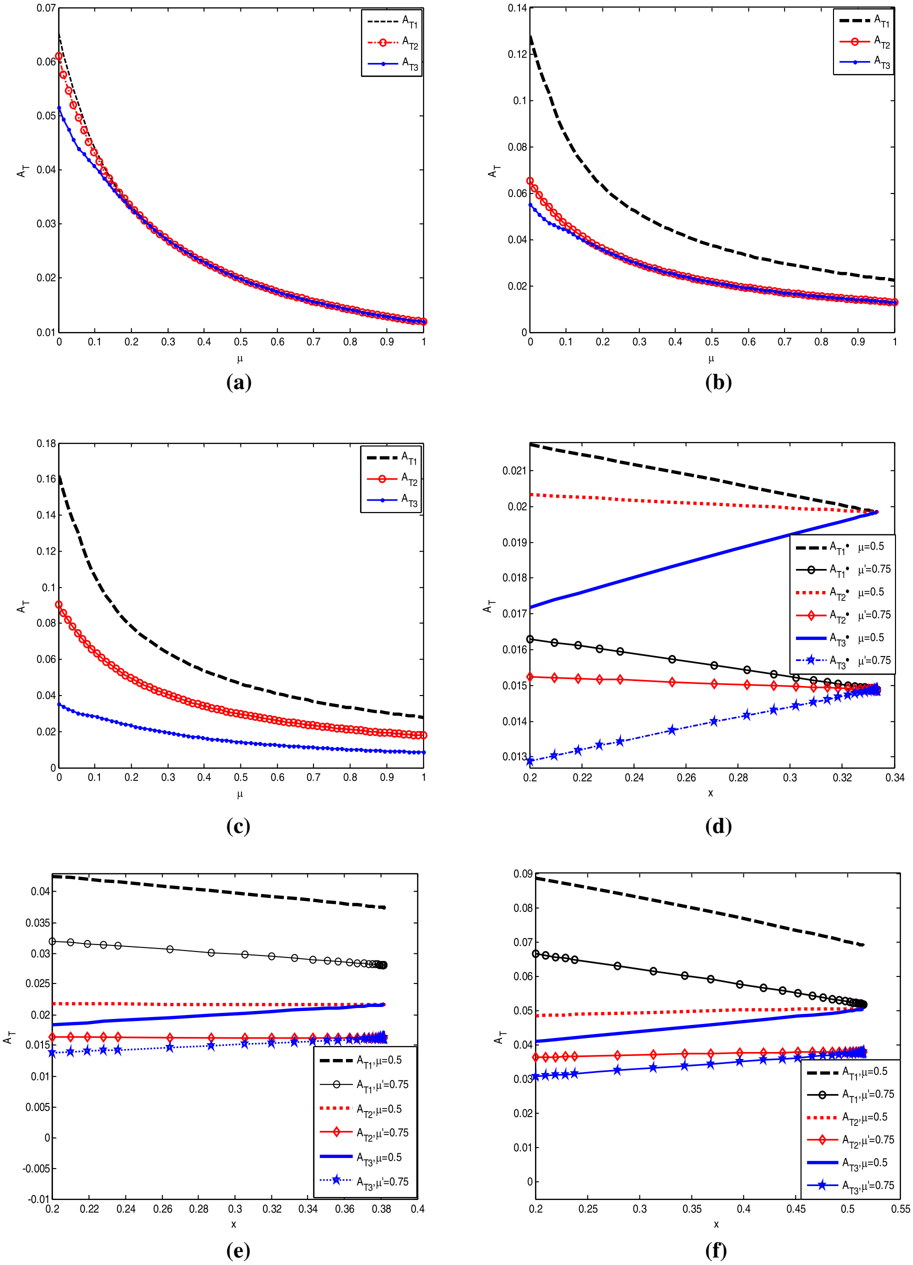

Fig. 10 Trend chart of fixed position online advertising investment changing with spillover effect

6. Specifically, when the network marketing enterprise is in a symmetric state, the enterprise with a lower initial market share is more sensitive to the change in the spillover effect. When the network marketing enterprises are in a state of semisymmetry and asymmetry, the enterprises in the dominant position are the most sensitive to the change in spillover effect. In contrast, the enterprises in the inferior position are weaker to the spillover effect. ②.Figs. 10d–f show that the amount of fixed-location online advertising can be changed according to the market share in the closed-loop state. Additionally, the network marketing enterprises with lower initial shares have the highest fixed advertising investment, which is similar to the conclusion drawn by duopoly competition. In addition, it is found that the amount of fixed-location online advertising is inversely proportional to the spillover effect.

(2) Disturbance analysis of spillover effect on the amount of investment in pay-per-click online advertising

① According to Figs. 11a–c, the amount of pay-per-click online advertising investment is proportional to the spillover effect, consistent with corollaries 1 and 2. In the case of semisymmetric and asymmetric advertising, the relationship between the amount of investment and the spillover effect of pay-per-click online advertising is similar to the analysis of fixed-location online advertising. That is, the network marketing enterprises in the dominant position pay more attention to the spillover effect in the case of semisymmetry and asymmetry, while the enterprises in the inferior position are more likely to ignore the spillover effect. Unlike fixed-position online advertising, in the symmetric case, the pay-per-click online advertising of each enterprise has a different disturbance on the spillover effect, but it is not apparent.

②Similarly, Figs. 11e–f show that in a closed-loop, the amount of pay-per-click online advertising can be changed according to market share. The online marketing enterprises with a lower initial share have the highest pay-per-click advertising investment. In addition, it is found that the amount of pay-per-click online advertising is proportional to the spillover effect.

With regard to the disturbance of the comprehensive spillover effect on different types of online advertising investment, in the symmetric case, enterprises with lower initial market share are more likely to be disturbed by the spillover effect. However, in the case of asymmetry and semisymmetry, superior enterprises are more likely to be disturbed by spillover effect, while inferior enterprises are more likely to ignore the disturbance of spillover effect.

(i) Overall, when enterprises are in a symmetric state, their market stability, unit profit, decay coefficient, and online advertising factors are not necessarily related and are all 1/3. In the semi-symmetric and asymmetric states, if a dominant enterprise blindly invests in online advertising, as long as external interference factors are strengthened, then the strategy will have a greater impact on the advantage enterprise.
(ii) In the symmetric case, enterprises with lower initial market share are more likely to be disturbed by the spillover effect. However, in the case of asymmetry and semi-symmetry, superior enterprises are more likely to be disturbed by the spillover effect while inferior enterprises are more likely to ignore the disturbance.

6 Model extension

6.1 Extension of the model in the case of symmetry

The solution is relatively easy in a symmetric environment, and it is easy to see its related properties. Therefore, in a symmetric environment, the situation of three online marketing

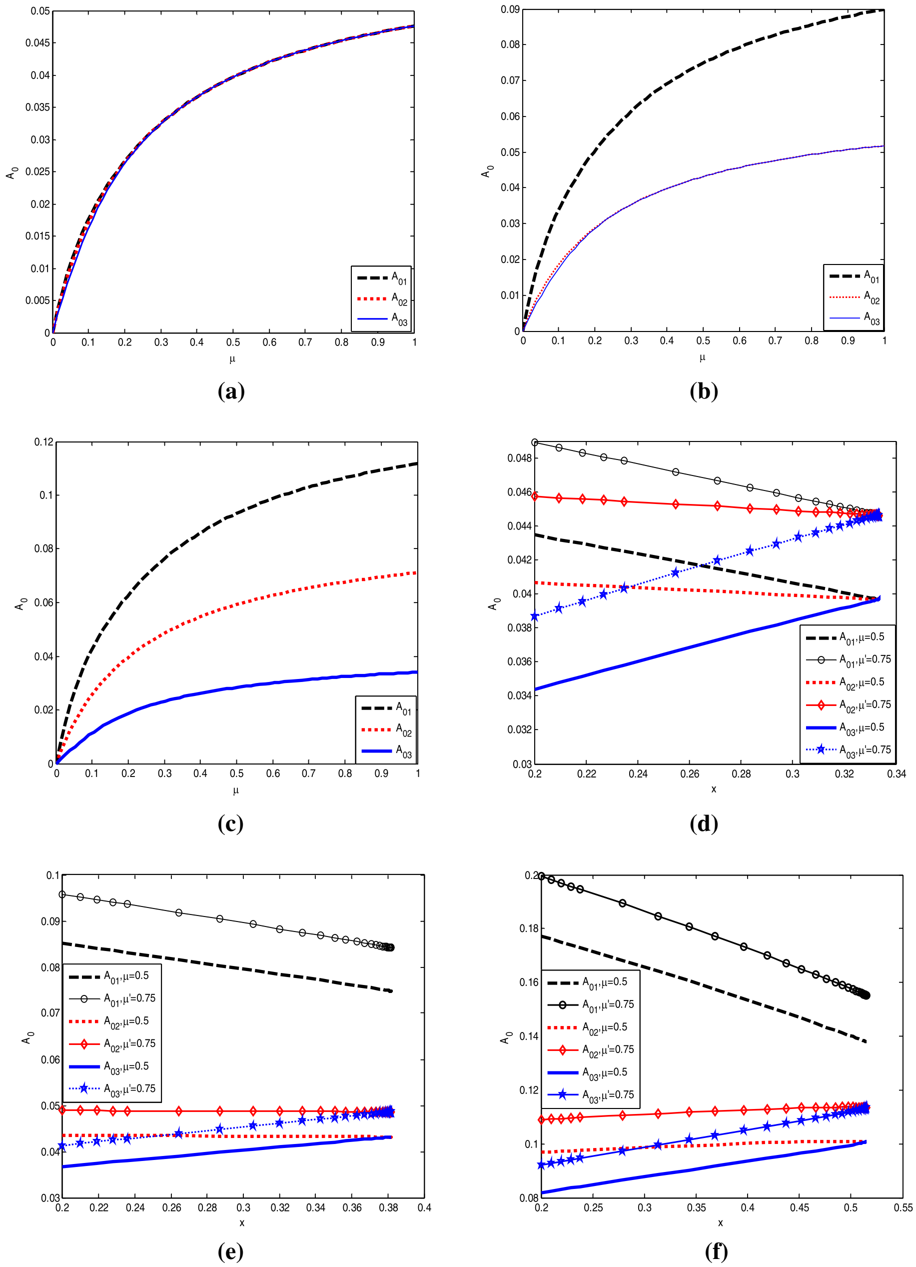

Fig. 11 Trend chart of pay-per-click online advertising investment changing with spillover effect

enterprises is extended to $n(n \geq 2)$-dimensional space. In the symmetric case, $m_1 = m_2 = m_3 = \cdots = m$, $\rho_1 = \rho_2 = \rho_3 = \cdots = \rho$ and $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu$, and based on Eq. (4). The market share of enterprise i is:

$$\begin{cases} \frac{dx_i}{dt} = \rho_i(1 + \mu_i p)A_{Ti}\sqrt{1 - x_i} - \sum_{j \in I, j \neq i} \frac{\rho_j}{2}\left(1 + \mu_j p\right)A_{Tj}\sqrt{1 - x_j} - \xi\left(x_i - \frac{1}{n}\right) \\ x_i(0) = x_{i0} \\ s.t. \sum_{i=1}^{n} x_i(t) = 1 \end{cases}, \tag{21}$$

wherein: $n \geq 2$, $i, j \in I \equiv \{1, 2, 3 \cdots n\}$.

The net profit of the enterprise i is:

$$V_i(x_1, x_2, x_3, \cdots, x_i) = \max_{A_i(t) \geq 0} \int_0^{\infty} e^{-\lambda t}\left[m x_i(t) - \frac{c}{2}(1 + \mu p)^2 A_{Ti}^2\right]dt. \tag{22}$$

Lemma 4 Under n-dimensional space, in the case of symmetry:

(1) The optimal fixed advertising investment and total online advertising investment of online marketing enterprises are:

$$\begin{cases} A_{Ti}^* = \rho\left[G - \frac{1}{2}(n - 1)\right]\sqrt{1 - x_i} \Big/ c(1 + \mu p) \\ A_i^* = \rho\left[G - \frac{1}{2}(n - 1)\right]\sqrt{1 - x_i} \Big/ c \end{cases}. \tag{23}$$

(2) The optimal profits of network marketing enterprises are:

$$V_i(x_1, x_2, x_3, \cdots, x_i) = \Omega + G x_i + Q \sum_j x_j. \tag{24}$$

where Ω, G, Q are constants and satisfy the equation:

$$\begin{cases} \lambda\Omega = \frac{[-2G + (n - 1)Q][2G(n - 2) + (n - 1)(2n - 7)Q]\rho^2}{8c} + \frac{[G + (n - 1)Q]\xi}{n} \\ \lambda G = m - \rho^2\left[G - \frac{(n-1)}{2}Q\right]^2 \Big/ 2c - \xi G \\ \lambda Q = \rho^2[G - (4 - n)Q]\left[G - \frac{(n-1)}{2}Q\right] \Big/ 2c - \xi Q \end{cases}. \tag{25}$$

Proof. See Appendix A6

Corollary 4 If n^* is the saturated number of online marketing enterprises, then the relationship satisfied by n^* is as follows:

$$F\left(n^*\right) = \frac{-3(n - 3)Q\rho + \sqrt{Q}\sqrt{9(n - 3)^2 Q\rho^2 + 32c(\xi + \lambda)}}{4\rho}. \tag{26}$$

For Eq. (26), when the network marketing enterprise is a duopoly competition or a three-oligarch competition, it is always established, and its advertising investment is always greater than 0. However, when there are more than three competitive online marketing enterprises in the market, it cannot be guaranteed that the equation is true, which may lead to vicious competition and negative profits. Proof. See Appendix A6.

According to Table 2 and Eq. (25), the trend analysis of n=4, 5, 6 is carried out, where when n=5, $\begin{cases} G = 1.5197 + 0.i \\ Q = 0.2697 + 0.i \end{cases}$, there is no solution in the real part, so n=4, 6 is analysed. The result is shown in Figs. 12 and 13:

Combined with Figs. 12 and 13, when there are 4 or 6 network marketing enterprises in the market, the profit is negative, and the enterprise gradually withdraws from the market, thus verifying the correctness of inference 4. This is also consistent with management practice; in general, the same type of network marketing enterprise does not exceed 3.

Corollary 4 In the case of symmetry, when the network marketing enterprises are in a stable state, their market share is $\frac{1}{n}$.

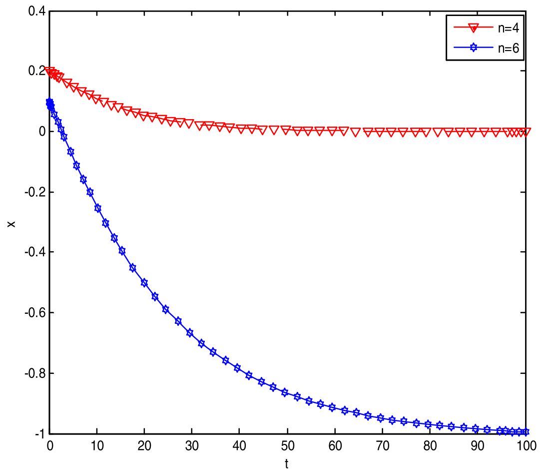

Fig. 12 Trend chart of market share changes over time when n = 4,6

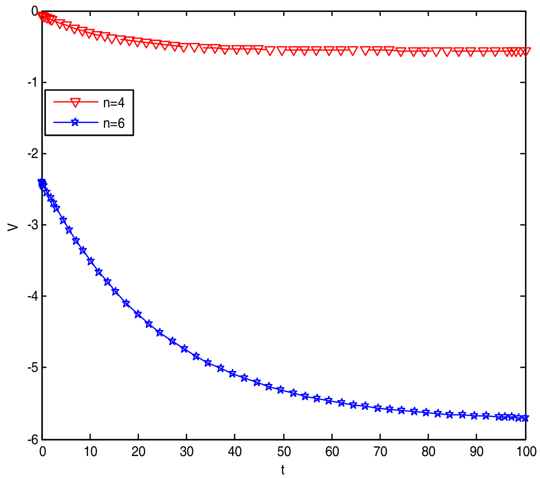

Fig. 13 Trend chart of profit over time when n = 4,6

Corollary 5 shows that when the enterprise is in a symmetric state, its market share is the average market share no matter how the external environment changes. Proof. See Appendix A6.

6.2 Extension of the model in the case of semi-symmetry

In the semisymmetric environment, because of the complexity of the model, it is difficult to obtain the specific online advertising volume and profit equivalent as in the symmetric case, so when the $n(n > 2)$ enterprises are in the semisymmetric case, we should try to explore the relevant properties from the market point of view. For the convenience of expression, without losing the general assumption, there are $m_1 \neq m_2 = m_3 = \cdots = m_n = m$, $\rho_1 \neq \rho_2 = \rho_3 = \cdots = \rho_n = \rho$ and $\mu_1 \neq \mu_2 = \mu_3 = \cdots = \mu_n = \mu$, that is, the e-commerce enterprise $(2, 3, \cdots n)$ is in a symmetric position. According to Eq. (4), the market share of enterprise i is:

$$\begin{cases} \dfrac{dx_1}{dt} = \rho_1(1+\mu_1 p)A_{T1}\sqrt{1-x_1} - \displaystyle\sum_{j\in I, j\neq i} \frac{\rho}{2}(1+\mu p)A_{Tj}\sqrt{1-x_j} - \xi\left(x_1 - \frac{1}{n}\right) \\ \dfrac{dx_j}{dt} = \rho(1+\mu p)A_{Tj}\sqrt{1-x_j} - \dfrac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1} \\ - \displaystyle\sum_{k\in I, k\neq i,j} \frac{\rho}{2}(1+\mu p)A_{Tk}\sqrt{1-x_k} - \xi\left(x_j - \frac{1}{n}\right) \\ x_i(0) = x_{i0} \\ s.t. \displaystyle\sum_{i=1}^{n} x_i(t)=1 \end{cases}, \tag{27}$$

wherein: $n \geq 3$, $j, k \in I \equiv \{2, 3 \cdots n\}$.

According to the above analysis, the expressions of the net profit of enterprises $1, j$ are:

$$\begin{cases} V_1 = \Theta_1 + \Upsilon_1 x_1 + \mathrm{O}_1 \displaystyle\sum_j x_j \\ V_j = \Theta_2 + \Upsilon x_j + \mathrm{Z} x_j + \mathrm{O} \displaystyle\sum_k x_k \end{cases}, \tag{28}$$

wherein: $\Theta_1, \Theta_2, \Upsilon, \Upsilon_1, \mathrm{O}, \mathrm{O}_1$ are constants.

According to the previous analysis, the investment amount of corporate online advertising is $A_1 = \frac{\rho_1[2\Upsilon_1-(n-1)\mathrm{O}_1]}{2c}\sqrt{1-x_1}$ and $A_j=\frac{\rho[2\Upsilon-\mathrm{Z}-(n-2)\mathrm{O}]}{2c}\sqrt{1-x_j}$, respectively. Since e-commerce enterprise $(2, 3, \cdots n)$ is in a symmetric position, $x_j = \frac{1}{n-1}(1-x_1)$ is obtained.Substituting the amount of online advertising and market share into Eq. (27), then:

$$\begin{cases} \dfrac{dx_1}{dt} = \left\{ \dfrac{\rho_1^2[2\Upsilon_1 - (n-1)\mathrm{O}_1]}{2c} + \dfrac{\rho^2[2\Upsilon_1 - \mathrm{O} - (n-2)\mathrm{Z}]}{4c} + \xi \right\}(1 - x_1) \\ - \dfrac{\rho^2}{4c}[2\Upsilon_1 - \mathrm{O} - (n-2)\mathrm{Z}](n-3) \\ - \xi\left(1 - \dfrac{1}{n}\right) \\ \dfrac{dx_j}{dt} = \left\{ \dfrac{\rho^2(4-n)[2\Upsilon_1 - \mathrm{O} - (n-2)\mathrm{Z}] - \rho_1^2[2\Upsilon_1 - (n-1)\mathrm{O}_1]}{4c} + \dfrac{\xi}{n-1} \right\}(1 - x_1) \\ - \dfrac{\rho^2(4-n)}{4c(n-1)}[2\Upsilon_1 - \mathrm{O} - (n-2)\mathrm{Z}](n-3) + \dfrac{\xi}{n} \end{cases} . \tag{29}$$

According to the second formula of (29), when n=3, $\frac{dx_1}{dt} > 0$ is established. The more network marketing enterprises there are, the lower the probability of survival of the enterprise. From the second formula of (29), it can be seen that when $3 \le n \le 4$, it is always established. Similarly, the more network marketing enterprises there are, the more difficult it is for enterprises to survive. This is consistent with a symmetric environment. From this, Coroally 6 is derived.

Corollary 6 In the semisymmetric situation, the more network marketing enterprises there are, the more difficult it is to enter the market.

6.3 Extension of the model in the asymmetric case

Similar to the semisymmetric case, in the asymmetric case, due to the complexity of the model, it is also difficult to obtain specific online advertising investment and profit equivalents. Therefore, when $n(n \ge 2)$ enterprises are in an asymmetric situation, this article also attempts to explore their relevant properties from the perspective of market share. In the asymmetric case, $m_1 \ne m_2 \ne m_3 \ne \cdots \ne m_n$, $\rho_1 \ne \rho_2 \ne \rho_3 \ne \cdots \ne \rho_n$ and $\mu_1 \ne \mu_2 \ne \mu_3 \ne \cdots \ne \mu_n$.

Proposition 2 In the asymmetric situation, when $\Xi_1 < \Xi_2 < \Xi_3 < \cdots \Xi_n$, the conditions for $x_0 \in [0, 1]$ to be established are:

$$\Xi_1 \ge \sum_{j>3} \frac{\Xi_j}{2} - \frac{\xi}{n}, \tag{30}$$

wherein: $\Xi_i = \frac{\rho_i^2}{2c}\left(2\frac{\partial V_i}{\partial x_i} - \sum_j \frac{\partial V_i}{\partial x_j}\right)$, $\Xi_j = \frac{\rho_j^2}{4c}\left(2\frac{\partial V_j}{\partial x_j} - \frac{\partial V_j}{\partial x_i} - \sum_k \frac{\partial V_j}{\partial x_k}\right)$.

According to Eq. (30), when $n \to +\infty$, that is, when there are infinite network marketing enterprises in the market, there is $\lim_{n\to+\infty} \frac{\xi}{n} = 0$, then $\Xi_1 \ge \sum_{j>3} \frac{\Xi_j}{2}$. This formula indicates that enterprise 1 has formed a huge advantage at this time, and the weaker network marketing enterprise only occupies a very small market share or even no market share so that it eventually withdraws from the entire market. This is consistent with the conclusion obtained in the symmetric case. Proof. See Appendix A7.

Corollary 7 Regardless of symmetry, semisymmetry or asymmetry, in the n-dimensional space, the spillover effect is inversely proportional to the fixed-location online advertising investment, directly proportional to the pay-per-click online advertising investment, and has nothing to do with the total amount of online advertising.

This proof is similar to corollary 1 and cannot be repeated.

7 Conclusion

In this paper, differential game theory is used to study the dynamic investment of online advertising in the competition of three oligopoly network marketing enterprises under the disturbance of the spillover effect and considering the interference of external factors, and an extended Lanchester model is established. The equilibrium online advertising investment and profit in the symmetric state, semisymmetric state and asymmetric state are obtained using the Hamilton–Jacobi–Bellman function, and the three games are compared. Furthermore, the competition in the case of complete symmetry, semisymmetry and asymmetry is extended to the n-dimensional space, and the number of saturated network marketing enterprises in the industry and the sufficient conditions of market share $x \in [0, 1]$ are discussed.

Our results are summarized as follows:

(1) With regard to the disturbance of the spillover effect on the amount of investment, market share and profit of online advertising, regardless of the state of symmetry, semisymmetry or asymmetry, the number of online marketing enterprises is inversely proportional to the amount of fixed online advertising investment and the spillover effect μ, and the amount of pay-per-click online advertising investment is directly proportional to the spillover effect μ. In the case of symmetry, enterprises with lower initial market share are more sensitive to the disturbance of spillover effect. In the case of asymmetry and semisymmetry, the dominant enterprises are more sensitive to the disturbance of spillover effect. In addition, the spillover effect is not directly related to the total investment, market share and profit of online advertising, which is similar to the conclusion of the two oligarchs

(2) The influence of the interference coefficient on the investment, market share and profit of online advertising: when there are a limited number of online marketing enterprises in the market, there are three. a. Regarding the amount of investment in online advertising, the amount of investment in online advertising is inversely proportional to the interference coefficient ξ; when the external interference coefficient ξ is weak, enterprises increase their investment in online advertising. b. Regarding market share, if the enterprise is in a state of incomplete symmetry, when the external interference coefficient increases, the market share of the dominant position of network marketing enterprises decrease. c. Regarding profit, in the case of complete symmetry, the profit is proportional to the external interference coefficient ξ; when the enterprise is in the case of incomplete symmetry, when the external interference coefficient ξ increases, the profit of the dominant network marketing enterprise decreases

(3) The relevant conclusions in the n-dimensional space: regardless of symmetry, semisymmetry or asymmetry, when the number of network marketing enterprises in the market increases, up to four network marketing enterprises, the competition becomes fiercer.

Finally, this leads to the gradual withdrawal of enterprises with small market shares and weak competitiveness from the market.

The managerial insights of this paper are as follows:

(1) When enterprises put fixed-location online advertising in the core position of the web page, they can appropriately reduce the investment in fixed-location online advertising and increase the investment in pay-per-click online advertising such as search engines
(2) Enterprises with low initial market share should pay more attention to the disturbance of the spillover effect when placing online advertisements under symmetric conditions, and dominant enterprises should pay more attention to the disturbance of spillover effect when placing online advertisements under asymmetric and semi-symmetric conditions. That is, there should be a more reasonable and rigorous allocation of fixed-location online advertising and pay-per-click online advertising ratio
(3) When external interference factors are enhanced, enterprises should reduce the total amount of online advertising and save costs. Additionally, enterprises in a weak position should seize the opportunity and not excessively reduce the amount of investment in online advertising
(4) When enterprises want to enter a new field, they should fully examine the number of existing competitive enterprises in the industry, and if there are more than three strong enterprises in the industry, they should carefully enter the industry.

There are several potential limitations in our study. On the one hand, the forms of online advertising are diversified, and the trajectory of consumers buying products according to online advertising is more complex, which makes the spillover effect not only exist between display advertising and pay-per-click advertising. For example, snack brands such as "Three Squirrels" and "Baicaowei" not only place display advertisements on Baidu affiliates but also advertise on Douyin and Taobao homepages. Then, among the various types of advertising, will the corporate online advertising investment strategy change? On the other hand, there are often external spillover effect among competitive enterprises, and it is easy to form a "hitchhiking phenomenon" in the process of online advertising. Therefore, in the case of careful consideration of internal and external spillover effect, how can online advertising be put into place? Future studies can discuss and solve these limitations.

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Declarations

Conflict of interest The authors declare no conflict of interests.

Human or animal rights This article does not contain any studies with human participants performed by any of the authors.

Appendix. Proofs of propositions and Table 5

Appendix. Proofs of propositions

A1. Proof of proposition 1

Proof since $\sum_{i=3} x_i(t) = 1$, then $\sum_{i=3} \dot{x}_i(t) = 0$. According to Eq. (1), it is:

$$(\rho_1 - 2\delta_1)(1+\mu_1 p)A_{T1}\sqrt{1-x_1} + (\rho_2 - 2\delta_2)(1+\mu_2 p)A_{T2}\sqrt{1-x_2} + (\rho_3 - 2\delta_3)(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \delta(x_1 + x_2 + x_3 - 1) = 0. \tag{31}$$

According to Eq. (31): $\delta_1 = \frac{\rho_1}{2}$, $\delta_2 = \frac{\rho_2}{2}$, $\delta_3 = \frac{\rho_3}{2}$. The proof is complete.

A2. Proof of Lemma 1

Proof Using the reverse induction method, according to the optimal control theory, the optimal profit function of the network marketing company is $V_{Ri}(x)$, which satisfies the Hamilton–Jacobi-Bellman (HJB) equation, and the first and second orders of $V_i(x_j)$ can be guided, then:

$$\lambda V_1(x) = \max_{A_i}\left\{\begin{array}{l} m_1 x_1(t) - \frac{c}{2}(1+\mu_1 p)^2 A_{T1}^2 \\ +V_1\prime(x_1)\left[\begin{array}{l}\rho_1(1+\mu_1 p)A_{T1}\sqrt{1-x_1} - \frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2} \\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \zeta\left(x_1 - \frac{1}{n}\right)\end{array}\right] \\ +V_1\prime(x_2)\left[\begin{array}{l}\rho_2(1+\mu_2 p)A_{T2}\sqrt{1-x_2} - \frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1} \\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \xi\left(x_2 - \frac{1}{n}\right)\end{array}\right] \\ +V_1\prime(x_3)\left[\begin{array}{l}\rho_3(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1} \\ -\frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2} - \xi\left(x_3 - \frac{1}{n}\right)\end{array}\right]\end{array}\right\}, \tag{32}$$

$$\lambda V_2(x) = \max_{A_i}\left\{\begin{array}{l} m_2 x_2(t) - \frac{c}{2}(1+\mu_2 p)^2 A_{T2}^2 \\ +V_2\prime(x_1)\left[\begin{array}{l}\rho_1(1+\mu_1 p)A_{T1}\sqrt{1-x_1} - \frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2} \\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \xi\left(x_1 - \frac{1}{n}\right)\end{array}\right] \\ +V_2\prime(x_2)\left[\begin{array}{l}\rho_2(1+\mu_2 p)A_{T2}\sqrt{1-x_2} - \frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1} \\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \xi\left(x_2 - \frac{1}{n}\right)\end{array}\right] \\ +V_2\prime(x_3)\left[\begin{array}{l}\rho_3(1+\mu_3 p)A_{T3}\sqrt{1-x_3} - \frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1} \\ -\frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2} - \xi\left(x_3 - \frac{1}{n}\right)\end{array}\right]\end{array}\right\}, \tag{33}$$

$$\lambda V_3(x)=\max_{A_i}\left\{\begin{array}{l} m_3x_3(t)-\frac{c}{2}(1+\mu_3 p)^2A_{T3}^2\\ +V_3\prime(x_1)\left[\begin{array}{l}\rho_1(1+\mu_1 p)A_{T1}\sqrt{1-x_1}-\frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2}\\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3}-\xi\left(x_1-\frac{1}{n}\right)\end{array}\right]\\ +V_3\prime(x_2)\left[\begin{array}{l}\rho_2(1+\mu_2 p)A_{T2}\sqrt{1-x_2}-\frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1}\\ -\frac{\rho_3}{2}(1+\mu_3 p)A_{T3}\sqrt{1-x_3}-\xi\left(x_2-\frac{1}{n}\right)\end{array}\right]\\ +V_3\prime(x_3)\left[\begin{array}{l}\rho_3(1+\mu_3 p)A_{T3}\sqrt{1-x_3}-\frac{\rho_1}{2}(1+\mu_1 p)A_{T1}\sqrt{1-x_1}\\ -\frac{\rho_2}{2}(1+\mu_2 p)A_{T2}\sqrt{1-x_2}-\xi\left(x_3-\frac{1}{n}\right)\end{array}\right]\end{array}\right\}. \tag{34}$$

where the value function V_{ix_j} can be understood as the marginal increase of the total discounted profit of enterprise i relative to the increase in the market share of the enterprise i; the same is true below.

Obtain the maximum value of A_{T1}, A_{T2}, A_{T3} on the right side of Eqs. (32), (33) and (34), that is, $(V_1, V_2, V_3)\left(A_{T1}^*, A_{T2}^*, A_{T3}^*\right) = \arg\max V_i\left[(V_1, V_2, V_3)\middle|\left(A_{T1}^*, A_{T2}^*, A_{T3}^*\right)\right]$. From the first-order condition:

$$\begin{cases} A_{T1}^*=\dfrac{\rho_1[2V_1\prime(x_1)-V_1\prime(x_2)-V_1\prime(x_3)]\sqrt{1-x_1}}{2c(1+\mu_1 p)}\\ A_{T2}^*=\dfrac{\rho_2[2V_2\prime(x_2)-V_2\prime(x_1)-V_2\prime(x_3)]\sqrt{1-x_2}}{2c(1+\mu_2 p)}\\ A_{T3}^*=\dfrac{\rho_2[2V_3\prime(x_3)-V_3\prime(x_1)-V_2\prime(x_2)]\sqrt{1-x_3}}{2c(1+\mu_3 p)}\end{cases}. \tag{35}$$

Substituting Eq. (35) into Eqs. (32)–(34), and merging similar terms for x_1, x_2, x_3, we can obtain:

$$\begin{aligned}\lambda V_1(x)=&\frac{1}{8}\left[8m_1-8\xi V_1'(x_1)+\Gamma_1\right]x_1+\frac{1}{8}[\Gamma_2-8\xi V_1\prime(x_2)]x_2+\frac{1}{8}\left[\Gamma_3-8\xi V_1'(x_3)\right]x_3\\ &+\frac{1}{8}\left[\frac{8\xi V_1'(x_1)}{n}+\frac{8\xi V_1'(x_2)}{n}+\frac{8\xi V_1'(x_3)}{n}-\Gamma_1-\Gamma_2-\Gamma_3\right],\end{aligned} \tag{36}$$

$$\begin{aligned}\lambda V_2(x)=&\frac{1}{8}\left[8m_2-8\xi V_2'(x_2)+\Gamma_2\prime\right]x_2+\frac{1}{8}\left[\Gamma_1'-8\xi V_2'(x_1)\right]x_1+\frac{1}{8}\left[\Gamma_3'-8\xi V_2'(x_3)\right]x_3\\ &+\frac{1}{8}\left[\frac{8\xi V_2'(x_1)}{n}+\frac{8\xi V_2'(x_2)}{n}+\frac{8\xi V_2'(x_3)}{n}-\Gamma_1'-\Gamma_2'-\Gamma_3'\right],\end{aligned} \tag{37}$$

$$\begin{aligned}\lambda V_3(x)=&\frac{1}{8}\left[8m_3-8\xi V_3\prime(x_3)+\Gamma_3''\right]x_3+\frac{1}{8}\left[\Gamma_1''-8\xi V_3\prime(x_1)\right]x_1+\frac{1}{8}\left[\Gamma_2''-8\xi V_3\prime(x_2)\right]x_2\\ &+\frac{1}{8}\left[\frac{8\xi V_3\prime(x_1)}{n}+\frac{8\xi V_3\prime(x_2)}{n}+\frac{8\xi V_3\prime(x_3)}{n}-\Gamma_1''-\Gamma_2''-\Gamma_3''\right],\end{aligned} \tag{38}$$

$$\text{wherein: }\begin{cases}\Gamma_1 = -\frac{8[V_1\prime(x_1)]^2\rho_1^2}{c} + \frac{2\rho_1^2 V_1\prime(x_2)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} \\ + \frac{2\rho_1^2 V_1\prime(x_3)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} \\ + \frac{4V_1\prime(x_1)\rho_1^2[V_1\prime(x_2) + V_1\prime(x_3)]}{c} + \frac{\rho_1^2[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]^2}{c} \\ \Gamma_2 = -\frac{8V_2\prime(x_2)\rho_2^2 V_1\prime(x_2)}{c} + \frac{2\rho_2^2 V_1\prime(x_1)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} \\ + \frac{2\rho_2^2 V_1\prime(x_3)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} + \frac{4\rho_2^2 V_1\prime(x_2)[V_2\prime(x_1) + V_2\prime(x_3)]}{c} \\ \Gamma_3 = -\frac{8V_3\prime(x_3)\rho_3^2 V_1\prime(x_3)}{c} + \frac{2\rho_3^2 V_1\prime(x_1)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} \\ + \frac{2\rho_3^2 V_1\prime(x_2)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} + \frac{4\rho_3^2 V_1\prime(x_3)[V_3\prime(x_1) + V_3\prime(x_2)]}{c}\end{cases},$$

$$\begin{cases}\Gamma_1\prime = -\frac{8V_1\prime(x_1)\rho_1^2 V_2\prime(x_1)}{c} + \frac{2\rho_1^2 V_2\prime(x_2)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} \\ + \frac{2\rho_1^2 V_2\prime(x_3)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} + \frac{4\rho_1^2 V_2\prime(x_1)[V_1\prime(x_2) + V_2\prime(x_3)]}{c} \\ \Gamma_2\prime = -\frac{8[V_2\prime(x_2)]^2\rho_2^2}{c} + \frac{2\rho_2^2 V_2\prime(x_1)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} \\ + \frac{2\rho_2^2 V_2\prime(x_3)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} + \frac{4V_2\prime(x_2)\rho_2^2[V_2\prime(x_1) + V_2\prime(x_3)]}{c} + \frac{\rho_2^2[V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]^2}{c} \\ \Gamma_3\prime = -\frac{8V_3\prime(x_3)\rho_3^2 V_2\prime(x_3)}{c} + \frac{2\rho_3^2 V_2\prime(x_2)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} \\ + \frac{2\rho_3^2 V_2\prime(x_1)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} + \frac{4\rho_3^2 V_2\prime(x_3)[V_3\prime(x_1) + V_2\prime(x_2)]}{c}\end{cases},$$

$$\begin{cases}\Gamma_1'' = -\frac{8V_1\prime(x_1)\rho_1^2 V_3\prime(x_1)}{c} + \frac{2\rho_1^2 V_3\prime(x_3)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} \\ + \frac{2\rho_1^2 V_3\prime(x_2)[2V_1\prime(x_1) - V_1\prime(x_2) - V_1\prime(x_3)]}{c} + \frac{4\rho_1^2 V_3\prime(x_1)[V_1\prime(x_2) + V_1\prime(x_3)]}{c} \\ \Gamma_2'' = -\frac{8V_2\prime(x_2)\rho_2^2 V_3\prime(x_2)}{c} + \frac{2\rho_2^2 V_3\prime(x_3)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} \\ + \frac{2\rho_2^2 V_3\prime(x_1)[2V_2\prime(x_2) - V_2\prime(x_1) - V_2\prime(x_3)]}{c} + \frac{4\rho_2^2 V_3\prime(x_2)[V_2\prime(x_1) + V_2\prime(x_3)]}{c} \\ \Gamma_3'' = -\frac{8[V_3\prime(x_3)]^2\rho_3^2}{c} + \frac{2\rho_3^2 V_3\prime(x_1)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} \\ + \frac{2\rho_3^2 V_3\prime(x_2)[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]}{c} \\ + \frac{4V_3\prime(x_3)\rho_3^2[V_3\prime(x_1) + V_3\prime(x_2)]}{c} + \frac{\rho_3^2[2V_3\prime(x_3) - V_3\prime(x_1) - V_3\prime(x_2)]^2}{c}\end{cases}.$$

From Eq. (32–34), it can be seen that the linear optimal profit function of $V_1(x)$, $V_2(x)$, $V_3(x)$ with respect to x_1, x_2, x_3 is Eq. (11), which is the solution of the enterprise HJB equation. Substituting $V_1(x)$, $V_2(x)$, $V_3(x)$ and its derivative to x_i into Eqs. (32)–(34), according to the method of undetermined coefficients, the equations can be obtained as Eqs. (6)–(8). The proof is complete.

A3. Proof of lemma 2 and Corollary 1

Proof Proposition 2

When the enterprise is in a symmetric situation, namely, $m_1 = m_2 = m_3 = m$, $\rho_1 = \rho_2 = \rho_3 = \rho$ and $\mu_1 = \mu_2 = \mu_3 = \mu$, the linear equation satisfied by the value function at this time is:

$$\begin{cases}V_1 = \eta + Bx_1 + \gamma(x_2 + x_3) \\ V_2 = \eta + Bx_2 + \gamma(x_1 + x_3) \\ V_3 = \eta + Bx_3 + \gamma(x_1 + x_2)\end{cases}. \tag{39}$$

According to Proposition 2, the advertising investment of enterprise i is $A_{Ti}^{*} = \frac{\rho(B-\gamma)\sqrt{1-x_i}}{2c(1+\mu p)}$. Additionally, according to Eqs. (6)–(8), the equation satisfied by η, B, γ is:

$$\begin{cases} \lambda\eta = \dfrac{-B^2n\rho^2 + 2Bc\xi + 2Bn\rho^2\gamma + 4c\xi\gamma - n\rho^2\gamma^2}{2cn} \\ \lambda B = \dfrac{2cmn - Bn^2\rho^2 - 2Bcn\xi + 2Bn\gamma\rho^2 - n\gamma^2\rho^2}{2cn} \\ \lambda\gamma = \dfrac{B^2n\rho^2 - 2Bn\rho^2\gamma - 2cn\xi\gamma + n\gamma^2\rho^2}{2cn} \end{cases}. \tag{40}$$

then: $$\begin{cases} \eta = \dfrac{-B^2n\rho^2 + 2Bc\xi + 2Bn\rho^2\gamma + 4c\xi\gamma - n\rho^2\gamma^2}{2cn\lambda} \\ B = \dfrac{2m\rho^2(\xi+\lambda) - c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2} \\ \gamma = \dfrac{2m\rho^2(\xi+\lambda) + c(\xi+\lambda)^3 - \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}}{4\rho^2(\xi+\lambda)^2} \end{cases}.$$

Substituting η, B, γ into $A_{Ti}^{*} = \frac{\rho(B-\gamma)\sqrt{1-x_i}}{2c(1+\mu p)}$, then:

$A_{Ti}^{*} = \frac{\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)}$, as shown in Eq. (13); at the same time, substituting η, B, γ into V_i, the expression of network marketing enterprise profit is obtained, as shown in Eq. (14). The proof is complete.

Corollary 1

Proof* (1) In the case of symmetry, according to Eq. (13), the expression of the known fixed position of the amount of online advertising investment and the total amount of online advertising investment:

$$\begin{cases} A_{Ti}^{*} = \dfrac{\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)} \\ A_i^{*} = \dfrac{\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2} \end{cases}.$$

According to Eq. (2), it is not difficult to find that the amount of pay-per-click online advertising investment is:

$A_{0i}^{*} = \frac{\mu p\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)}$

$$\frac{\partial A_{0i}^{*}}{\partial \mu} = \frac{\mu p\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)}$$
$$= \frac{p\left\{-c(\xi+\lambda)^3 + \sqrt{c(\xi+\lambda)^4\left[4m\rho^2 + c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i}}{2c\rho(\xi+\lambda)^2(1+\mu p)^2} > 0$$

.

Therefore, the pay-per-click online advertising is proportional to the spillover effect.

(2) $\frac{\partial A_i^*}{\partial \xi} = \frac{\left\{c(\xi+\lambda)^3-\sqrt{c}(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]\right\}\sqrt{1-x_i}}{2\rho(1+\mu p)\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}}$.Since $A_{Ti}^* > 0$, and $2c\rho(\xi+\lambda)^2$ $(1+\mu p) > 0$, it is obtained that $\left\{-c(\xi+\lambda)^3+\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i} > 0$. Then, $\left\{c(\xi+\lambda)^3-\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}\right\}\sqrt{1-x_i} < 0$. Wherein, $\frac{\partial A_{Ti}^*}{\partial \xi} < 0$.

(3) $\frac{\partial A_i^*}{\partial \rho} = -\frac{(\xi+\lambda)\left(c(\xi+\lambda)^3-\sqrt{c(\xi+\lambda)^4\left(4m\rho^2+c(\xi+\lambda)^2\right)}\right)\sqrt{1-x_i}}{2\rho^2\sqrt{c(\xi+\lambda)^4\left[4m\rho^2+c(\xi+\lambda)^2\right]}} > 0$. The proof is complete.

A4. Proof of Lemma 3

Proof When the enterprise is in the case of semisymmetry, the linear equation satisfied by the value function is $\begin{cases} V_1=\alpha_1+\beta_1 x_1+\psi_1(x_2+x_3) \\ V_2=\alpha+\beta x_2+\psi x_1+\varphi x_3 \\ V_3=\alpha+\beta x_3+\psi x_1+\varphi x_2 \end{cases}$.

According to Proposition 2, we know that the amount of online advertising investment of enterprise i is shown in Eq. (16).

Additionally, according to Eqs. (6) – (8), the seven equations satisfied by α, α_1, β_1, β, φ, ψ are

$$\begin{cases} \lambda\alpha_1=\dfrac{\begin{array}{c}-2\beta(\beta_1-\psi_1)n\rho^2-\psi\psi_1 n\rho^2+n\rho_1^2\beta_1^2-2\beta_1\psi_1 n\rho_1^2+n\psi_1^2\rho_1^2+2\beta_1 c\xi\\ +4c\psi_1\xi-\psi_1 n\rho^2\varphi+\psi_1 n\rho^2(\psi+\varphi)\end{array}}{2cn} \\ \lambda\alpha=\dfrac{\begin{array}{c}-6\beta\psi n\rho^2+3\psi^2 n\rho^2+8\beta_1\psi n\rho_1^2-4\beta(\beta_1-\psi_1)n\rho_1^2-8\psi\psi_1 n\rho_1^2+\\ 8\beta c\xi+8c\psi\xi+6\beta n\rho^2\varphi-4\beta_1 n\varphi\rho_1^2+4\psi_1 n\rho_1^2\varphi+8c\xi\varphi-3n\rho^2\varphi^2\end{array}}{8cn} \\ \lambda\beta_1=m_1-\dfrac{\rho_1^2(\beta_1-\psi_1)^2}{2c}-\beta_1 c\xi \\ \lambda\psi_1=\dfrac{2\beta(\beta_1-\psi_1)\rho^2-4c\delta\psi_1+(\psi_1-\beta_1)\rho^2(\psi+\varphi)}{4c} \\ \lambda\beta=\dfrac{\beta(\beta_1-\psi_1)\rho_1^2-2\psi\left[(\psi_1-\beta_1)\rho_1^2+c\xi\right]+(\psi_1-\beta_1)\rho_1^2\varphi}{2c} \\ \lambda\psi=m-\beta\xi-\dfrac{\rho^2(-2\rho+\psi+\varphi)^2}{8c} \\ \lambda\varphi=\dfrac{2\beta^2\rho^2+\beta\psi\rho^2-\psi^2\rho^2-5\beta\rho^2\varphi+\psi\rho^2\varphi+2\varphi\left(-2c\xi+\rho^2\varphi\right)}{4c} \end{cases}. \quad (41)$$

The proof is complete.

A5. Proof of Corollary 2

Proof

$$\frac{d\overline{x}_1}{d\xi} = \frac{16c\left[(2\beta-\psi-\varphi)\rho^2 - 2(\beta_1-\psi_1)\rho_1^2\right]}{3\left[4(-\beta_1+\psi_1)\rho_1^2 - 4c\xi + \rho^2(-2\beta+\psi+\varphi)\right]^2},$$

$\frac{d\overline{x}_{2,3}}{d\xi}=\frac{8c\left[2(\beta_1-\psi_1)\rho_1^2-(2\beta-\psi-\varphi)\rho^2\right]}{3\left[4(-\beta_1+\psi_1)\rho_1^2-4c\xi+\rho^2(-2\beta+\psi+\varphi)\right]^2}$.. According to the above two formulas, it is necessary to compare $(2\beta-\psi-\varphi)\rho^2$ and $2(\beta_1-\psi_1)\rho_1^2$ to judge the change in market share with the attenuation coefficient and divide the above two formulas by $2c$. Then, $\frac{(2\beta-\psi-\varphi)\rho^2}{2c}=A_2$, A_3 and $\frac{(\beta_1-\psi_1)\rho_1^2}{c}=A_1$, so Corollary 2 can be obtained. The proof is complete.

A6 Proof of Lemma 4, Corollary 4 and Corollary 5

Lemma 4

Proof According to Eqs. (21) and (22), the HJB equation of enterprise i is:

$$\begin{aligned}
&\lambda V_i(x_1,x_2,x_3,\cdots,x_i) = mx_i(t) - \frac{c}{2}(1+\mu p)^2 A_{Ti}^2\\
&-\rho(1+\mu p)A_{Ti}\left[\frac{\partial V_i}{\partial x_i} - \frac{1}{2}(n-1)\frac{\partial V_i}{\partial x_j}\right]\sqrt{1-x_i}\\
&+\rho(1+\mu p)\left[\frac{\partial V_i}{\partial x_j} - \frac{1}{2}\frac{\partial V_i}{\partial x_j} - \frac{1}{2}\sum_{k\in I,k\neq i,j}\frac{\partial V_i}{\partial x_k}\right]\sum_{j\in I,j\neq i}A_{Tj}\sqrt{1-x_j}\\
&-(n-1)\frac{\partial V_i}{\partial x_j}\xi\sum_{j\in I,j\neq i}\left(x_j-\frac{1}{n}\right) - \frac{\partial V_i}{\partial x_i}\xi\left(x_i-\frac{1}{n}\right)
\end{aligned}\tag{42}$$

Find the maximum value of A_{Ti} at the right end of Eq. (42), which can be obtained from the first-order condition: $A_{Ti}^* = \frac{\rho\left[\frac{\partial V_i}{\partial x_i} - \frac{n-1}{2}\frac{\partial V_i}{\partial x_j}\right]\sqrt{1-x_i}}{c(1+\mu p)}$.

Substituting A_i into Eq. (42), and merging similar terms on x_i, we can get Eq. (43):

$$\begin{aligned}
&\lambda V_i(x_1,x_2,x_3,\cdots,x_i) = mx_i(t) + \frac{\rho^2}{2c}\left[\frac{\partial V_i}{\partial x_i} - \frac{1}{2}(n-1)\frac{\partial V_i}{\partial x_j}\right]^2(1-x_i)\\
&+\frac{\rho^2}{2c}\left[(4-n)\frac{\partial V_i}{\partial x_j} - \frac{\partial V_i}{\partial x_i}\right]\\
&\left[\frac{\partial V_i}{\partial x_j} - \frac{1}{2}\frac{\partial V_i}{\partial x_j} - \frac{1}{2}\sum_{k\in I,k\neq i,j}\frac{\partial V_i}{\partial x_k}\right]\sum_{j\in I,j\neq i}(1-x_j)\\
&-\xi(n-1)\frac{\partial V_i}{\partial x_j}\sum_{j\in I,j\neq i}\left(x_j-\frac{1}{n}\right) - \xi\frac{\partial V_i}{\partial x_i}\left(x_i-\frac{1}{n}\right).
\end{aligned}\tag{43}$$

From Eq. (42), we can see that the linear optimal profit function of $V_i(x)$ with respect to x_1, x_2, x_3 is $V_i(x_1,x_2,x_3,\ldots,x_i) = \Omega + Gx_i + Q\sum_j x_j$, and the above equation is the solution of the enterprise HJB equation. Substituting V_i and its derivative with respect to x_i, x_j into Eq. (43), the system of Eqs. (25) can be obtained according to the method of undetermined coefficients. The proof is complete.

Corollary 3

Proof According to Eq. (25), the second and third formulas can be obtained:

$$\begin{cases} Q = \frac{1}{2}(n-1)G + \frac{-c(\lambda+\xi) + \sqrt{c}\sqrt{2m\rho^2 + (\lambda+\xi)\left[(1-n)G\rho^2 + c(\lambda+\xi)\right]}}{\rho^2} \\ G = \frac{(n-7)Q\rho + \sqrt{Q}\sqrt{9(n-3)^2 Q\rho^2 + 32c}}{4\rho} \end{cases}. \tag{44}$$

Since $A_i > 0$, then,$G > \frac{1}{2}(n-1)Q$. According to Eq. (44), it is: $\frac{-c(\lambda+\xi)+\sqrt{c}\sqrt{2m\rho^2+(\lambda+\xi)\left[(1-n)G\rho^2+c(\lambda+\xi)\right]}}{\rho^2} > 0$is always established. $\frac{(n-7)Q\rho+\sqrt{Q}\sqrt{9(n-3)^2Q\rho^2+32c}}{4\rho} - \frac{1}{2}(n-1)Q = \frac{-3(n-3)Q\rho+\sqrt{Q}\sqrt{9(n-3)^2Q\rho^2+32c}}{4\rho}$. To obtain $A_i > 0$, there must be $\frac{-3(n-3)Q\rho+\sqrt{Q}\sqrt{9(n-3)^2Q\rho^2+32c}}{4\rho} > 0$. The proof is complete.

Corollary 4

Proof Substituting Eq. (23) into Eq. (21), the market share of enterprise i can be expressed as:

$$\frac{dx_i}{dt} = \xi\left(\frac{1}{n} - x_i\right) - \frac{\rho^2(G-Q)}{2c}\left[2x_i - \sum_{j\in I, j\neq i} x_j\right]. \tag{45}$$

Solving Eq. (45), it is obtained that $x_i = \frac{1}{n}$. The proof is complete.

A7. Proof of proposition 7

Proof According to the correlation analysis of the symmetric case and the semisymmetric case, in the asymmetric case, $A_{Ti}^* = \frac{\rho_i\left[2\frac{\partial V_i}{\partial x_i} - \sum_j \frac{\partial V_i}{\partial x_j}\right]\sqrt{1-x_i}}{2c(1+\mu_i p)}$, $A_{Tj}^* = \frac{\rho_j\left[2\frac{\partial V_j}{\partial x_j} - \frac{\partial V_j}{\partial x_i} - \sum_k \frac{\partial V_j}{\partial x_k}\right]\sqrt{1-x_j}}{2c(1+\mu_j p)}$. The proof process will not be repeated. Substituting A_{Ti}^* and A_{Tj}^* into Eq. (21), we can get:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\rho_i^2}{2c}\left(2\frac{\partial V_i}{\partial x_i} - \sum_j \frac{\partial V_i}{\partial x_j}\right)(1-x_i) \\ &- \sum_{j\in I, j\neq i} \frac{\rho_j^2}{4c}\left(2\frac{\partial V_j}{\partial x_j} - \frac{\partial V_j}{\partial x_i} - \sum_k \frac{\partial V_j}{\partial x_k}\right)(1-x_j) - \xi\left(x_i - \frac{1}{n}\right). \end{aligned} \tag{46}$$

Combining Eq. (46) with the same type, then:

$$\frac{dx_i}{dt} = \left[\frac{\rho_i^2}{2c}\left(2\frac{\partial V_i}{\partial x_i} - \sum_j \frac{\partial V_i}{\partial x_j}\right) + \xi\right](1 - x_i) - \sum_{j \in I, j \neq i} \frac{\rho_j^2}{4c}\left(2\frac{\partial V_j}{\partial x_j} - \frac{\partial V_j}{\partial x_i} - \sum_k \frac{\partial V_j}{\partial x_k}\right)(1 - x_j) - \frac{n-1}{n}\xi \quad (47)$$

In Eq. (47), let $\Xi_i = \frac{\rho_i^2}{2c}\left(2\frac{\partial V_i}{\partial x_i} - \sum_j \frac{\partial V_i}{\partial x_j}\right), \Xi_j = \frac{\rho_j^2}{4c}\left(2\frac{\partial V_j}{\partial x_j} - \frac{\partial V_j}{\partial x_i} - \sum_k \frac{\partial V_j}{\partial x_k}\right)$, and suppose $\Xi_1 < \Xi_2 < \Xi_3 < \cdots \Xi_n$, then $\frac{dx_i}{dt} = [\Xi_i + \xi](1 - x_i) - \sum_{j \in I, j \neq i} \frac{\Xi_j}{2}(1 - x_j) - \frac{n-1}{n}\xi$. To make $x_i \in [0, 1]$, two conditions must be met: namely, $\dot{x}_i\big|_{x_i=1} \leq 0$ and $\dot{x}_i\big|_{x_i=0} \geq 0$. For $\forall n \geq 2$, it $x_i = 1$, then $\dot{x}_i = -\sum_{j \in I, j \neq i} \frac{\Xi_j}{2}(1 - x_j) - \frac{n-1}{n}\xi < 0$. $\dot{x}_i\big|_{x_i=1} \leq 0$ is always established. Next, verify the conditions for the establishment of $\dot{x}_i\big|_{x_i=0} \geq 0$. When $n = 2$, $\dot{x}_i = \Xi_i + \xi - \frac{n-1}{n}\xi = \Xi_i + \frac{1}{n}\xi > 0$ is always established. When $n > 2$, $\dot{x}_i = [\Xi_i + \xi] - \sum_{j \in I, j \neq i} \frac{\Xi_j}{2}(1 - x_j) - \frac{n-1}{n}\xi \geq 0 \Rightarrow \Xi_i \geq \sum_{j \in I, j \neq i} \frac{\Xi_j}{2}(1 - x_j) - \frac{\xi}{n}$. For the inequality to hold constant, the minimum value on the left side of the inequality must be greater than the maximum value after the inequality; thus, $\Xi_1 \geq \sum_{j>2} \frac{\Xi_j}{2} - \sum_{j>2} \frac{\Xi_j}{2} x_j - \frac{\xi}{n} \geq \sum_{j>2} \frac{\Xi_j}{2} - \frac{\Xi_2}{2} \sum_{j>2} x_j - \frac{\xi}{n} \geq \sum_{j>3} \frac{\Xi_j}{2} - \frac{\xi}{n}$. The proof is complete.

Appendix. Table 5

B1 Table 5

See Table 5.

Table 5 The stable market

	symmetric	semi-symmetric
Enterprise 1	1/3	$\frac{-6\beta\rho^2+3\psi\rho^2+12(\psi_1-\varphi_1)\rho_1^2+4c\xi+3\rho^2\varphi}{3\left[2\beta\rho^2+4(\psi_1-\varphi_1)\rho_1^2+4c\xi-\rho^2(\psi+\varphi)\right]}$
Enterprise 2	1/3	$-\frac{-6\beta\rho^2+3\psi\rho^2-4c\xi+3\rho^2\varphi}{\left[2\beta\rho^2+4(\psi_1-\varphi_1)\rho_1^2+4c\xi-\rho^2(\psi+\varphi)\right]}$
Enterprise 3	1/3	$-\frac{-6\beta\rho^2+3\psi\rho^2-4c\xi+3\rho^2\varphi}{\left[2\beta\rho^2+4(\psi_1-\varphi_1)\rho_1^2+4c\xi-\rho^2(\psi+\varphi)\right]}$
Asymmetric		
Enterprise 1	$\left\{\begin{array}{l}16c^2\xi^2+3(-8c\rho_1^2\xi\gamma_{12}-8c\rho_1^2\delta\gamma_{13}+3\rho_1^2\rho_2^2(\gamma_{12}\gamma_{21}+\gamma_{13}\gamma_{31}+\gamma_{12}\gamma_{23}+\gamma_{12}\gamma_{23})\\+6B_3\rho_3^2\left[-\rho_1^2(\gamma_{12}+\gamma_{13})+\rho_2^2(\gamma_{21}+v\gamma_{23})\right]+3\rho_1^2\rho_3^2(\gamma_{12}\gamma_{31}+\gamma_{13}\gamma_{31})-3\rho_2^2\rho_3^2(\gamma_{21}\gamma_{31}+\gamma_{23}\gamma_{31})\\+3\rho_3^2\gamma_{32}\left[\rho_1^2(\gamma_{12}+\gamma_{13})-\rho_2^2(\gamma_{21}+v\gamma_{23})\right]-6B_2\rho_2^2\left[2B_3\rho_3^2+\rho_1^2(\gamma_{12}+\gamma_{13})-\rho_3^2(\gamma_{31}+\gamma_{32})\right]\\+6B_1\rho_1^2\left\{6B_2\rho_2^2+6B_3\rho_3^2+8c\xi-3\left[\rho_2^2(\gamma_{21}+\gamma_{23})+\rho_3^2(\gamma_{31}+\gamma_{32})\right]\right\}\end{array}\right\}\Big/\Theta$	
Enterprise 2	$\left\{\begin{array}{l}16c^2\xi^2-24c\rho_2^2\xi\gamma_{21}+9\rho_1^2\rho_2^2(\gamma_{12}\gamma_{21}+\gamma_{13}\gamma_{21}+\gamma_{12}\gamma_{23}+\gamma_{13}\gamma_{23})+9\rho_2^2\rho_3^2\gamma_{31}(\gamma_{21}+\gamma_{23})\\+18B_3\rho_3^2\left[\rho_1^2(\gamma_{12}+\gamma_{13})-\rho_2^2\gamma_{21}-\rho_2^2\gamma_{23}\right]-24c\rho_2^2\xi\gamma_{23}-9\rho_1^2\rho_3^2\gamma_{31}(\gamma_{12}+\gamma_{13})\\+9\rho_3^2\gamma_{32}\left[-\rho_1^2(\gamma_{12}+\gamma_{13})+\rho_2^2(\gamma_{21}+\gamma_{23})\right]+18B_1\rho_1^2\left[2B_2\rho_2^2-\rho_2^2(\gamma_{21}+\gamma_{23})+\rho_3^2(-2B_3+\gamma_{31}+\gamma_{32})\right]\\+6B_2\rho_2^2\left\{6B_3\rho_3^2+8c\xi-3\left[\rho_1^2(\gamma_{12}+\gamma_{13})+\rho_3^2(\gamma_{31}+\gamma_{32})\right]\right\}\end{array}\right\}\Big/\Theta$	

Table 5 (continued)

	symmetric	semi-symmetric
Enterprise 3	$-\left\{\begin{array}{l} 48B_3c\rho_3^2\xi-16c^2\xi^2+18B_3\rho_1^2\rho_3^2(\gamma_{12}+\gamma_{13})+18B_3\rho_2^2\rho_3^2(\gamma_{21}+\gamma_{23}) \\ +9\rho_1^2\rho_2^2(\gamma_{12}\gamma_{21}+\gamma_{21}\gamma_{13}+\gamma_{12}\gamma_{23}+\gamma_{13}\gamma_{23}) \\ +24c\rho_3^2\delta\gamma_{31}-9\rho_1^2\rho_3^2(\gamma_{12}\gamma_{31}+\gamma_{13}\gamma_{31})-9\rho_2^2\rho_3^2(\gamma_{21}\gamma_{31}+\gamma_{23}\gamma_{31}) \\ +3\rho_3^2\gamma_{32}\left\{8c\xi-3\left[\rho_1^2(\gamma_{12}+\gamma_{13})+\rho_2^2(\gamma_{21}+\gamma_{23})\right]\right\} \\ -18B_2\rho_2^2\left[2B_3\rho_3^2+\rho_1^2(\gamma_{12}+\gamma_{13})-\rho_3^2(\gamma_{31}+\gamma_{32})\right] \\ +18B_1\rho_1^2\left[2B_2\rho_2^2-\rho_2^2(\gamma_{21}+\gamma_{23})+\rho_2^2(-2B_3+\gamma_{31}+\gamma_{32})\right] \end{array}\right\}\Big/\Theta$	
Wherein:	$\Theta=3\left\{\begin{array}{l} 16B_3c\rho_3^2\delta+16c^2\delta^2-6B_3\rho_1^2\rho_3^2(\gamma_{12}+\gamma_{13})-8c\rho_1^2\xi(\gamma_{12}+\gamma_{13}) \\ -8c\xi\left(\rho_2^2\gamma_{21}+\rho_3^2\gamma_{31}\right)+3\rho_1^2\rho_2^2(\gamma_{12}\gamma_{21}+\gamma_{13}\gamma_{21}+\gamma_{12}\gamma_{23}+\gamma_{13}\gamma_{23})-6B_3\rho_2^2\rho_3^2\gamma_{23}-8cr2^2\xi\gamma_{23} \\ +3\rho_1^2\rho_3^2(\gamma_{12}\gamma_{31}+\gamma_{13}\gamma_{31})+3\rho_2^2\rho_3^2(\gamma_{21}\gamma_{31}+\gamma_{23}\gamma_{31})-6B_3\rho_2^2\rho_3^2\gamma_{21}+\rho_3^2\gamma_{32}\left[\begin{array}{l}-8c\xi+3\rho_1^2(\gamma_{12}+\gamma_{13}) \\ +3\rho_2^2(\gamma_{21}+\gamma_{23})\end{array}\right]+ \\ 2B_2\rho_2^2\left\{6B_3\rho_3^2+8c\xi-3\left[\rho_1^2(\gamma_{12}+\gamma_{13})+\rho_3^2(\gamma_{31}+\gamma_{32})\right]\right\}+2B_1\rho_1^2\left\{\begin{array}{l}6B_2\rho_2^2+6B_3\rho_3^2+8c\xi \\ -3\left[\rho_2^2(\gamma_{21}+\gamma_{23})+\rho_3^2(\gamma_{31}+\gamma_{32})\right]\end{array}\right\} \end{array}\right\}$	

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