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ORIGINAL RESEARCH

The minimum covering Euclidean ball of a set of Euclidean balls in R^n

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Abstract

Primal and dual algorithms are developed for solving the n-dimensional convex optimization problem of finding the Euclidean ball of minimum radius that covers m given Euclidean balls, each with given center and radius. Each algorithm is based on a directional search method in which a search path may be a ray or a two-dimensional conic section in $\mathbb{R}^n$. At each iteration, a search path is constructed by the intersection of bisectors of pairs of points, where the bisectors are either hyperplanes or n-dimensional hyperboloids. The optimal stopping point along each search path is determined explicitly.

Keywords Location · Convex programming · Minimum covering ball · One-center location · Min–max location

1 Introduction

Let $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$ be a given set of m distinct points in $\mathbb{R}^n$, and for each point $\mathbf{p}_i \in P$, let r_i be a non-negative radius. Let $[\mathbf{p}_i, r_i] = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}_i\| \leq r_i\}$ denote the closed Euclidean ball where $\mathbf{p}_i$ is the center, r_i is the radius, and $\|\mathbf{x} - \mathbf{p}_i\|$ is the Euclidean distance between $\mathbf{p}_i$ and $\mathbf{x}$. The problem of determining the minimum covering Euclidean ball of a set of Euclidean balls requires the ball $[\mathbf{x}^*, z^*]$, with center $\mathbf{x}^*$ and minimum radius z^*, that covers, or contains, the balls $[\mathbf{p}_i, r_i]$ for all $\mathbf{p}_i \in P$. The problem is denoted by $M(P)$, and is written as:

$$
\begin{aligned}
M(P): \ & \min\ z \\
& \text{s.t.}\quad z \geq \|\mathbf{x} - \mathbf{p}_i\| + r_i, \quad \mathbf{p}_i \in P.
\end{aligned}
$$

Adding a real constant to all the radii yields an equivalent problem, so the assumption of non-negative radii is not necessary. However, non-negative radii are assumed to simplify the presentatiion. If all the radii are equal, then all the radii may be assumed to be zero, and the problem is to find the minimum covering ball of the points $\mathbf{p}_i \in P$, called the minimum

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covering ball problem. Problem $M(P)$ has the following equivalent representation:

$$M(P): \min_{\mathbf{x}\in I\!R^n} \max_{\mathbf{p}_i\in P}\{\|\mathbf{p}_i - \mathbf{x}\| + r_i\}.$$

This version of the problem is known as the min–max location problem with fixed distance and as the one-center delivery problem (Plastria, 2004). The center $\mathbf{x}$ is the location of a facility that minimizes the maximum service to points $\mathbf{p}_i \in P$, where service is measured as the travel distance from $\mathbf{x}$ to $\mathbf{p}_i$ plus a fixed travel distance r_i.

This paper presents primal and dual algorithms for solving problem $M(P)$. At each iteration of either algorithm, bisectors of pairs of points are intersected to construct a search path, that is either a ray or a two-dimensional conic section in $I\!R^n$. For the primal (dual) algorithm, points on the search path maintain primal (dual) feasibility and complementary slackness. The optimal stopping point along the search path is determined explicitly.

2 Literature

The problem of finding the minimum covering circle of a set of points in $I\!R^2$ was first posed by Sylvester (1857). Various geometric solutions were reported by Sylvester (1860), Chrystal (1885), Blumenthal and Wahlin (1941), Rademacher and Toeplitz (1990), and Elzinga and Hearn (1972a). Voronoi diagrams (Aurenhammer and Klein (2000)) have also been used to solve the problem in $I\!R^2$. Megiddo (1983a) reported an algorithm for $M(P)$, with $n = 3$, that is linear in m.

For the minimum covering ball problem of a set of points in $I\!R^n$, Elzinga and Hearn (1972a) solved the dual problem as a convex, quadratic programming problem. Hopp and Reeve (1996) extended the Sylvester (1860) and Chrystal (1885) algorithm to $I\!R^n$ relying on a heuristic without proof of convergence. Fischer et al. (2003) expanded the Hopp and Reeve algorithm, gave a proof of finite convergence, and were able to solve problems for m up to 10,000 and n up to 2000. Megiddo (1983b) reported an algorithm that is linear in m but exponential in n. Dyer (1992) improved the time-complexity of Meggido's algorithm. Dearing and Zeck (2009) reported a dual algorithm based on search paths constructed from bisectors of pairs of points. Cavaleiro and Alizadeh (2018) presented computational improvements to the Dearing and Zeck approach.

The literature for the minimum covering ball of a set of balls in $I\!R^n$ is more limited. Elzinga and Hearn (1972b) presented a geometrical algorithm for the problem in $I\!R^2$. Xu et al. (2003) reported computational results for four general approaches to problem $M(P)$ in $I\!R^2$: a second-order cone reformulation, a sub-gradient approach, a quadratic programming scheme, and a randomized incremental algorithm. Two approaches for problem $M(P)$ in $I\!R^n$ were reported by Zhou et al. (2005): an unconstrained convex program whose objective function approximates the maximum objective function, and a reformulation of $M(P)$ as a second-order cone programming problem. They solve problems with n up to 10,000, and m up to 5000. Applications of problem $M(P)$ are referenced in Fischer et al. (2003) and Megiddo (1983b). Plastria (2004) presents a survey of the min–max location problems. Cavaleiro and Alizadeh (2021) present a dual simplex-type algorithm for the smallest enclosing ball of balls using a search path parameterized by the objective function value.

3 Properties of problem $M(P)$

The following properties of problem $M(P)$ are used to develop the algorithms.

Property 1 There exists a unique minimum solution $\mathbf{x}^* \in I\!R^n$ to $M(P)$.

Proof The function $f(\mathbf{x}) = \max_{\mathbf{p}_i \in P}\{f_i(\mathbf{x})\}$, where $f_i(\mathbf{x}) = \|\mathbf{x}-\mathbf{p}_i\| + r_i$, is strictly convex, and therefore continuous, on $I\!R^n$ since each $f_i(\mathbf{x})$ is strictly convex. For any $\mathbf{x}_0 \in I\!R^n$, with $z_0 = \max\{f_i(\mathbf{x}_0)\}$, the sub-level set $L_{z_0}(f) = \{\mathbf{x} \in I\!R^n : f(\mathbf{x}) \leq z_0\}$ is non-empty and compact since it is the intersection of non-empty compact sub-level sets: $L_{z_0}(f) = \cap_{\mathbf{p}_i \in P}\{\mathbf{x} \in I\!R^n : f_i(\mathbf{x}) \leq z_0\}$. Thus, there exists a unique $\mathbf{x}^* \in L_{z_0}(f)$ that minimizes $f(\mathbf{x})$ over $L_{z_0}(f)$, and $\mathbf{x}^*$ minimizes $f(\mathbf{x})$ over $I\!R^n$ since for $\mathbf{x} \in I\!R^n \setminus L_{z_0}(f)$, $f(\mathbf{x}) > f(\mathbf{x}^*)$. □

For points $\mathbf{p}_j, \mathbf{p}_k \in P$, the ball $[\mathbf{p}_k, r_k]$ is contained in the ball $[\mathbf{p}_j, r_j]$ if and only if $r_j - r_k \geq \|\mathbf{p}_k - \mathbf{p}_j\|$. In this case, the point $\mathbf{p}_k$ is redundant to $\mathbf{p}_j$, since any ball $[\mathbf{x}, z]$ that covers $[\mathbf{p}_j, r_j]$ also covers $[\mathbf{p}_k, r_k]$. The point $\mathbf{p}_k$ can be eliminated from the set P with no effect on the optimal solution. If each point in P is redundant to $\mathbf{p}_j$, then the ball $[\mathbf{x}, z] = [\mathbf{p}_j, r_j]$ is the optimal solution for $M(P)$ and is called the trivial solution.

The point $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_j$ if and only if $r_j - r_k < \|\mathbf{p}_k - \mathbf{p}_j\|$, and two points $\mathbf{p}_j, \mathbf{p}_k$ are (mutually) non-redundant if neither is redundant to the other, that is, $|r_j - r_k| < \|\mathbf{p}_j - \mathbf{p}_k\|$. Observe that if $\mathbf{p}_k$ is redundant to $\mathbf{p}_j$, then $\mathbf{p}_j$ is non-redundant to $\mathbf{p}_k$, and if $r_k \geq r_j$, $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_j$.

The bisector of two non-redundant points $\mathbf{p}_j, \mathbf{p}_k \in P$, with radii r_j and r_k respectively, is the set

$$B_{j,k} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}_j\| + r_j = \|\mathbf{x} - \mathbf{p}_k\| + r_k\}. \tag{1}$$

If $r_j = r_k$, then $\mathbf{p}_j$ and $\mathbf{p}_k$ are two non-redundant points (since P consists of distinct points), and $B_{j,k} = \{\mathbf{x} : (\mathbf{p}_j - \mathbf{p}_k)\mathbf{x} = (\mathbf{p}_j - \mathbf{p}_k)(\mathbf{p}_j + \mathbf{p}_k)/2\}$, is the hyperplane that is orthogonal to the line through $\mathbf{p}_j$ and $\mathbf{p}_k$, and contains the midpoint between $\mathbf{p}_j$ and $\mathbf{p}_k$.

If $r_j > r_k$, and if $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_j$, then the bisector $B_{j,k} = \{\mathbf{x} : \|\mathbf{p}_k - \mathbf{x}\| - \|\mathbf{p}_j - \mathbf{x}\| = r_j - r_k\}$ is one sheet of an n-dimensional hyperboloid of two sheets, symmetric about the line through the focal points $\mathbf{p}_j$ and $\mathbf{p}_k$. This follows by setting $2a_{j,k} = r_j - r_k > 0$, and noting that $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_j$ if and only if $2a_{j,k} < \|\mathbf{p}_j - \mathbf{p}_j\|$. Then the set $HB_{j,k} = \{\mathbf{x} : |\|\mathbf{p}_k - \mathbf{x}\| - \|\mathbf{p}_j - \mathbf{x}\|| = 2a_{j,k}\}$ satisfies the definition, given by equation (32) in the “Appendix”, of an n-dimensional hyperboloid of two sheets with focal points $\mathbf{p}_j$ and $\mathbf{p}_k$, and the bisector $B_{j,k}$ satisfies the definition, given by equation (33), of the sheet of $HB_{j,k}$ closest to $\mathbf{p}_j$.

The bisector $B_{j,k}$ is characterized by the axis vector $\mathbf{v}_{j,k} = (\mathbf{p}_j - \mathbf{p}_k)/\|\mathbf{p}_j - \mathbf{p}_k\|$, center $\mathbf{c}_{j,k} = (\mathbf{p}_j + \mathbf{p}_k)/2$, vertex $\mathbf{a}_{j,k} = \mathbf{c}_{j,k} + a_{j,k}\mathbf{v}_{j,k}$, which is the intersection of $B_{j,k}$ and the line through the focal points, focal distance $c_{j,k} = \|\mathbf{p}_j - \mathbf{p}_k\|/2$, and eccentricity $\epsilon_{j,k} = c_{j,k}/a_{j,k}$.

Each point $\mathbf{x} \in B_{j,k}$ is the center of a ball $[\mathbf{x}, z_{\mathbf{x}}]$, with radius $z_{\mathbf{x}} = \|\mathbf{p}_j - \mathbf{x}\| + r_j = \|\mathbf{p}_k - \mathbf{x}\| + r_k$, that contains, and is internally tangent to, the two balls $[\mathbf{p}_j, r_j]$ and $[\mathbf{p}_k, r_k]$. If $B_{j,k}$ is the sheet of the hyperboloid closest to $\mathbf{p}_j$, its vertex $\mathbf{a}_{j,k}$ is the center of the smallest ball containing, and internally tangent to, the two balls $[\mathbf{p}_j, r_j]$ and $[\mathbf{p}_k, r_k]$. If $B_{j,k}$ is the hyperplane, then $(\mathbf{p}_j + \mathbf{p}_k)/2$ is the center of the smallest ball containing, and internally tangent to, the two balls $[\mathbf{p}_j, r_j]$ and $[\mathbf{p}_k, r_k]$.

Property 2 For non-redundant points $\mathbf{p}_j, \mathbf{p}_k \in P$, and for each $\mathbf{x} \in B_{j,k}$, $\mathbf{x} \neq \mathbf{p}_j$ and $\mathbf{x} \neq \mathbf{p}_k$.

Proof The Property is true for any $\mathbf{x}$ not on the line through $\mathbf{p}_j$ and $\mathbf{p}_k$. The only point on the line through $\mathbf{p}_j$ and $\mathbf{p}_k$ and on $B_{j,k}$ is the vertex $\mathbf{a}_{j,k}$. Substitution of the vector and parameter definitions implies $\|\mathbf{a}_{j,k} - \mathbf{p}_j\| = c_{j,k} - a_{j,k} > 0$ and $\|\mathbf{a}_{j,k} - \mathbf{p}_k\| = c_{j,k} + a_{j,k} > 0$. □

The primal and dual algorithms developed here are examples of the active set method of mathematical programming, Gill et al. (1991). A set $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\} \subset P$, with its points ordered by non-decreasing radii, $r_{i_1} \geq \ldots \geq r_{i_s}$, is an active set of a ball $[\mathbf{x}_S, z_S]$ if S satisfies the following three conditions: (1) the points in S are affinely independent, (2) the constraint of $M(P)$ corresponding to each point in S holds at equality, that is, $\|\mathbf{p}_{i_j} - \mathbf{x}_S\| + r_{i_j} = z_S$, for each $\mathbf{p}_{i_j} \in S$, and (3) if $s > 1$, $\mathbf{p}_{i_j}$ is non-redundant to $\mathbf{p}_{i_1}$, for $j = 2, \ldots, s$, that is, B_{i_1, i_j} is a bisector for $j = 2, \ldots, s$. Observe that if a set $S \subset P$ satisfies conditions (1) and (2), but not (3), S may be transformed into an active set by deleting the points $\mathbf{p}_{i_j}$ that are redundant to $\mathbf{p}_{i_1}$. Also, there may be more than one active set corresponding to a ball, and any subset of an active set of a ball is also an active set of that ball.

Each iteration of the primal and dual algorithms corresponds to a ball $[\mathbf{x}_S, z_S]$ and an active set S. The points in S are used to check for optimality and to determine the search path. The size of S may increase, decrease, or remain unchanged at each iteration.

Observe that a ball $[\mathbf{x}_S, z_S]$ and an active set S with $s = 1$ are optimal if and only if $[\mathbf{x}_S, z_S]$ is the trivial solution, that is, $[\mathbf{x}_S, z_S] = [\mathbf{p}_{i_1}, r_{i_1}]$ and $S = \{\mathbf{p}_{i_1}\}$.

Property 3 At any iteration of the primal or the dual algorithm, let $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, with points ordered by non-increasing radii, and with $s > 1$, be an active set corresponding to the ball $[\mathbf{x}_S, z_S]$. Then $\mathbf{x}_S \neq \mathbf{p}_{i_j}$, for $j = 1, \ldots, s$.

Proof For each $\mathbf{p}_{i_j} \in S$, B_{i_1, i_j} is a bisector and by Property 2, $\mathbf{x}_S \neq \mathbf{p}_{i_1}$ and $\mathbf{x}_S \neq \mathbf{p}_{i_j}$. □

Property 3 implies that at each iteration of the primal or the dual algorithm with active set S, and for each $\mathbf{p}_{i_j} \in S$, the Euclidean distance $\|\mathbf{x} - \mathbf{p}_{i_j}\|$ is differentiable with respect to $\mathbf{x}$ over the set $I\!R^n \setminus S$. Then the Karush Kuhn Tucker (KKT) conditions for optimality, Gill et al. (1991), may be written as follows.

Property 4 A ball $[\mathbf{x}_S, z_S]$, with active set S and $|S| > 1$, is the optimal solution to $M(P)$ if and only if there exist variables $\lambda_j \geq 0$ for $\mathbf{p}_{i_j} \in S$ and $\lambda_j = 0$ for $\mathbf{p}_{i_j} \notin S$ such that

$$z_S \geq \|\mathbf{x}_S - \mathbf{p}_j\| + r_j \qquad \mathbf{p}_j \in P \tag{2}$$

$$\sum_{\mathbf{p}_{i_j} \in S} \lambda_j = 1 \tag{3}$$

$$\sum_{\mathbf{p}_{i_j} \in S} \frac{(\mathbf{x}_S - \mathbf{p}_{i_j})}{\|\mathbf{x}_S - \mathbf{p}_{i_j}\|} \lambda_j = \mathbf{0} \tag{4}$$

$$\lambda_j \geq 0 \qquad \mathbf{p}_{i_j} \in S \tag{5}$$

$$(z_S - \|\mathbf{x}_S - \mathbf{p}_j\| - r_j)\lambda_j = 0 \qquad \mathbf{p}_j \in P. \tag{6}$$

Property 5 At optimality, statements (3), (4), and (5) of the KKT conditions are equivalent to:

$$\sum_{\mathbf{p}_{i_j} \in S} \pi_j = 1 \tag{7}$$

$$\sum_{\mathbf{p}_{i_j} \in S} (\mathbf{x}_S - \mathbf{p}_{i_j}) \pi_j = \mathbf{0} \tag{8}$$

$$\pi_j \geq 0 \qquad \mathbf{p}_{i_j} \in S. \tag{9}$$

Proof Since $\|\mathbf{x}_S - \mathbf{p}_{i_j}\| > 0$, for all $\mathbf{p}_{i_j} \in S$, the change of variables $\pi_j = \frac{\lambda_j/\|\mathbf{x}_S - \mathbf{p}_{i_j}\|}{\sum_{\mathbf{p}_{i_j} \in S} \lambda_j/\|\mathbf{x}_S - \mathbf{p}_{i_j}\|}$ and $\lambda_j = \frac{\|\mathbf{x}_S - \mathbf{p}_{i_j}\|\pi_j}{\sum_{\mathbf{p}_{i_j} \in S} \|\mathbf{x}_S - \mathbf{p}_{i_j}\|\pi_j}$ for each $\mathbf{p}_{i_j} \in S$, shows the equivalence of conditions (3), (4), and (5) to conditions (7), (8), and (9). □

Properties 4 and 5 lead to additional properties of an optimal ball $[\mathbf{x}_S, z_S]$.

Property 6 The non-trivial minimum covering ball $[\mathbf{x}_S, z_S]$ for problem $M(P)$ is determined by an active set S of at most $n + 1$ affinely independent points in P, and the optimal center $\mathbf{x}_S \in \text{conv}(S)$, the convex hull of S.

Proof Conditions (7) and (8) determine a linear system with $n+1$ linear equations. A solution is determined by at most $n+1$ linearly independent columns of the system which correspond to at most $n+1$ affinely independent points from P. Conditions (7), (8), and (9) are equivalent to $\mathbf{x}_S \in \text{conv}(S)$. □

The complementary slackness conditions (6) imply the following necessary condition for optimality.

Property 7 If $[\mathbf{x}_S, z_S]$ is optimal to problem $M(P)$ with active set S, and $|S| > 1$, then $\mathbf{x}_S$ is on at least one bisector of a pair of points in S.

If S is an active set of size s corresponding to an optimal covering ball $[\mathbf{x}_S, z_S]$, there may be more than s constraints that are active. At each iteration of either the primal or dual algorithm, the points in the active set S are sufficient to determine the ball $[\mathbf{x}_S, z_S]$.

If S is an active set corresponding of the ball $[\mathbf{x}_S, z_S]$, then $\mathbf{x}_S \in B_{i_j,i_k}$ for each pair of points $\mathbf{p}_{i_j}, \mathbf{p}_{i_k} \in S$, so that $\mathbf{x}_S$ is on the intersection of bisectors B_{i_j,i_k} over all pairs of points $\mathbf{p}_{i_j}, \mathbf{p}_{i_k} \in S$, denoted by B_S. That is, $\mathbf{x}_S \in B_S = \cap_{\{\mathbf{p}_{i_j},\mathbf{p}_{i_k}\} \subseteq S} B_{i_j,i_k}$.

At each iteration of the primal and dual algorithms a search path $X_S = \{\mathbf{x}(\alpha), \alpha_S \le \alpha \le \alpha_m\}$ is constructed so that $X_S \subset B_S$. Property 23 shows that $\mathbf{x}(\alpha) \in B_{i_j,i_k}$ for all $\{\mathbf{p}_{i_j}, \mathbf{p}_{i_k}\} \subseteq S$, and $\alpha_S \le \alpha \le \alpha_m$, and that the complementary slackness conditions (6) are maintained at each point on the search path X_S.

If all the points in S have equal radii, then all the bisectors are hyperplanes, and B_S is a linear manifold of dimension $n - |S| + 1$. In this case, the search path is a ray.

If some points in S have unequal radii, then at least one bisector is a hyperboloid. Property 22 shows that the vectors and parameters of B_S may be computed by intersecting the hyperboloid bisector with $|S|-2$ hyperplanes, and Property 19 shows that the resulting intersection is a conic section of dimension $n - |S| + 2$.

4 Primal algorithm

At each iteration of the primal algorithm there is a ball $[\mathbf{x}_S, z_S]$ and an active set S that satisfy expressions (2) and (6) of the KKT conditions, but do not satisfy expressions (3), (4), and (5) of the KKT conditions. That is, S and $[\mathbf{x}_S, z_S]$ are primal feasible but not dual feasible.

Since $[\mathbf{x}_S, z_S]$ is primal feasible at each iteration, z_S is an upper bound on the optimal objective function value of $M(P)$. Assuming non-degeneracy, the radius z_S is shown to decrease at each iteration of the primal algorithm.

The primal algorithm may be initialized by choosing $\mathbf{x}_S$ to be any point in $I\!R^n$, computing $z_S = \max_{\mathbf{p}_i \in P} \|\mathbf{x}_S - \mathbf{p}_i\| + r_i = \|\mathbf{x}_S - \mathbf{p}_j\| + r_j$, for some $\mathbf{p}_j \in P$, and choosing $S = \{\mathbf{p}_j\}$.

Observe that $[\mathbf{x}_S, z_S]$ is primal feasible, but $\mathbf{x}_S \notin \text{aff}(S)$, which implies $\mathbf{x}_S \notin \text{conv}(S)$ so that $\mathbf{x}_S$ is not dual feasible.

Primal search phase

Given a primal feasible ball $[\mathbf{x}_S, z_S]$ and an active set S, a search path $X_S = \{\mathbf{x}(\alpha) : \alpha_S \leq \alpha \leq \alpha_m\}$ is constructed so that $X_S \subset B_S$, $\mathbf{x}(\alpha_S) = \mathbf{x}_S$, and $\mathbf{x}(\alpha_m) \in \text{aff}(S)$. If all the points in S have equal radii, the search path will be a ray, but if some points in S have unequal radii, the search path will be a two-dimensional conic section in $I\!R^n$.

For a search path X_S that is either a ray or a conic section, Property 23 shows that S is an active set for the ball $[\mathbf{x}(\alpha), z(\alpha)]$ for $\alpha \geq \alpha_S$, and that S and $[\mathbf{x}(\alpha), z(\alpha)]$ satisfy the complementary slackness conditions (6). Property 24 shows that $z(\alpha)$ is decreasing for $\alpha \geq \alpha_S$.

For each $\mathbf{p}_k \in P \setminus S$, the parameter $\alpha_k \geq \alpha_S$ is determined, if it exists, so that X_S intersects the bisector $B_{i_1,k}$ at $\mathbf{x}(\alpha_k)$. If $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_1,k}$, then $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_j,k}$ for each $\mathbf{p}_{i_j} \in S$. That is, X_S simultaneously intersects the bisectors $B_{i_j,k}$ at $\mathbf{x}(\alpha_k)$. Thus, it suffices to consider the intersection of X_S with only $B_{i_1,k}$. At the point $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_1,k}$, the constraint corresponding to the point $\mathbf{p}_k$ is active. The Update Phase chooses $\alpha^* = \min\{\alpha_m, \min_{\mathbf{p}_k \in P \setminus S} \alpha_k\}$, and the ball $[\mathbf{x}(\alpha^*), z(\alpha^*)]$ is checked for optimality.

Case 1: All points in S have equal radii

The points in the active set are denoted by $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, where $r_{i_j} = r_{i_1}$ for $\mathbf{p}_{i_j} \in S$. In this case, the search path is the ray

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha \mathbf{d}_S, \alpha_S \leq \alpha \leq \alpha_m\}, \tag{10}$$

where $\alpha_S = 0$, $\alpha_m = [(\mathbf{p}_{i_1} - \mathbf{x}_S)\mathbf{d}_S]/\|\mathbf{d}_S\|^2 > 0$, and $\mathbf{d}_S \leftarrow \text{PROJECTION}((\mathbf{p}_{i_1} - \mathbf{x}_S), R)$, with $R = \{(\mathbf{p}_{i_1} - \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})\}$. If $|S| = 1$, $\mathbf{d}_S = (\mathbf{p}_{i_1} - \mathbf{x}_S)/\|\mathbf{p}_{i_1} - \mathbf{x}_S\|$.

Algorithm PROJECTION($\mathbf{v}$, R)

Input: Vector $\mathbf{v}$ and set of vectors R

Output: Vector $\mathbf{u}$, the normalized component of $\mathbf{v}$ orthogonal to the projection of $\mathbf{v}$ onto span(R)

1: $\mathbf{u}' \leftarrow \mathbf{v} - \text{Proj}_{\text{span}(R)}\mathbf{v}$

2: $\mathbf{u} \leftarrow \mathbf{u}'/\|\mathbf{u}'\|$

For each $\mathbf{p}_k \in P \setminus S$, the parameter $\alpha_k \geq \alpha_S$ is determined, if it exists, so that X_S intersects the bisector $B_{i_1,k}$ at $\mathbf{x}(\alpha_k)$. There are two sub-cases to consider for computing α_k depending on whether the radius r_{i_1} equals the radius r_k.

Case 1a: $r_{i_1} = r_k$ Then $B_{i_1,k}$ is a hyperplane. The point $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_1,k}$ is determined by solving for α in the equation $(\mathbf{p}_{i_1} - \mathbf{p}_k)\mathbf{x}(\alpha) = (\mathbf{p}_{i_1} - \mathbf{p}_k)(\mathbf{p}_{i_1} + \mathbf{p}_k)/2$. If $(\mathbf{p}_{i_1} - \mathbf{p}_k)\mathbf{d}_S = 0$, $\alpha_k \leftarrow \infty$, else

$$\alpha_k = \frac{(\mathbf{p}_{i_1} - \mathbf{p}_k)(\mathbf{p}_{i_1} + \mathbf{p}_k)/2 - (\mathbf{p}_{i_1} - \mathbf{p}_k)\mathbf{x}_S}{(\mathbf{p}_{i_1} - \mathbf{p}_k)\mathbf{d}_S}. \tag{11}$$

If $\alpha_k < 0$, $\alpha_k \leftarrow \infty$.

Case 1b(i) $r_{i_1} > r_k$. Then $\mathbf{p}_{i_1}$ is non-redundant to $\mathbf{p}_k$. If $\mathbf{p}_k$ is redundant to $\mathbf{p}_{i_1}$, eliminate $\mathbf{p}_k$ from P, and continue. Otherwise, $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_{i_1}$ so that $B_{i_1,k}$ is a hyperboloid and the parameter α_k is determined so that $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_1,k}$.

Case 1b(ii) $r_{i_1} < r_k$. Then $\mathbf{p}_k$ is non-redundant to $\mathbf{p}_{i_1}$. The point $\mathbf{p}_{i_1}$ is redundant to $\mathbf{p}_k$ if and only if $r_k - r_{i_1} = \|\mathbf{p}_{i_1} - \mathbf{p}_k\|$, in which case, $\alpha_k = 0$. Otherwise, $\mathbf{p}_{i_1}$ is non-redundant to $\mathbf{p}_k$ so that $B_{i_1,k}$ is a hyperboloid and the parameter α_k is determined so that $\mathbf{x}(\alpha_k) \in X_S \cap B_{k,i_1}$.

The intersection of X_S and $B_{i_1,k}$ is determined by substituting $\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha\mathbf{d}_S$ for $\mathbf{x}$, and $c_{i_1,k}$, $\mathbf{c}_{i_1,k}$, $a_{i_1,k}$, $\mathbf{v}_{i_1,k}$, and $\epsilon_{i_1,k}$, for c, $\mathbf{c}$, a, $\mathbf{v}$, and ϵ, respectively, in expression (36) for $B_{i_1,k}$ as a quadratic form. This gives the quadratic equation

$$A\alpha^2 + B\alpha + C = 0, \tag{12}$$

where $A = (\mathbf{d}_S)^2 - \epsilon^2_{i_1,k}(\mathbf{d}_S\mathbf{v}_{i_1,k})^2$, $B = 2(\mathbf{x}_S - \mathbf{c}_{i_1,k})\mathbf{d}_S - 2\epsilon^2_{i_1,k}[(\mathbf{x}_S - \mathbf{c}_{i_1,k})\mathbf{v}_{i_1,k}][\mathbf{d}_S\mathbf{v}_{i_1,k}]$, and $C = (\mathbf{x}_S - \mathbf{c}_{i_1,k})^2 - \epsilon^2_{i_1,k}[(\mathbf{x}_S - \mathbf{c}_{i_1,k})\mathbf{v}_{i_1,k}]^2 - a^2_{i_1,k} + c^2_{i_1,k}$. The parameter α_k is chosen as the smallest positive real solution to (12).

Case 2: At least two points in S have unequal radii

The points in the active set are denoted by $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, where $r_{i_1} \geq \cdots \geq r_{i_s}$. By assumption $r_{i_1} > r_{i_s}$, and by construction B_{i_1,i_s} is a bisector. Property 22 shows that $B_S = B_{i_1,i_s} \cap_{k=2}^{s-1} H_k$, where each $H_k = \{\mathbf{x} : \mathbf{h}_k\mathbf{x} = \mathbf{h}_k\mathbf{d}_k\}$ is a hyperplane such that $B_{i_1,i_k} \cap B_{i_1,i_s} = B_{i_1,i_k} \cap H_k = B_{i_1,i_s} \cap H_k$, for $k = 2 \ldots, s-1$. Thus, B_S is a conic section of dimension $n - s + 2$.

Algorithm Intersections gives the closed-form expressions, based on Property 19, for computing the parameters and vectors of B_S: $(\mathbf{v}_S, \mathbf{c}_S, \epsilon_S, a_S, b_S, c_S) \leftarrow$ Intersections $(S, \{r_{i_j} : \mathbf{p}_{i_j} \in S\})$. The algorithm starts with the vectors and parameters of B_{i_1,i_s}, and each iteration computes the vectors and parameters of $B_{i_1,i_s} \cap_{j=2}^{k-1} H_j \cap H_k$ given the vectors and parameters of $B_{i_1,i_s} \cap_{j=2}^{k-1} H_j$, for $k = 2, \ldots, s-1$.

If $\epsilon_S > 1$, then B_S is one branch of a hyperboloid; if $\epsilon_S < 1$, B_S is an ellipsoid; and if $\epsilon_S = 1$, B_S is a paraboloid.

Algorithm BisectorParameters$(\mathbf{p}_j, \mathbf{p}_k, r_j, r_k)$

Input: Points $\mathbf{p}_j$, $\mathbf{p}_k$ with radii $r_j \geq r_k$
Output: Hyperboloid parameters $\mathbf{c}$, $\mathbf{v}$, $\mathbf{d}$, ϵ, a, b, and c

1: $a \leftarrow |r_j - r_k|/2$
2: $c \leftarrow \|\mathbf{p}_j - \mathbf{p}_k\|/2$
3: $b \leftarrow \sqrt{c^2 - a^2}$
4: $\epsilon \leftarrow c/a$
5: $\mathbf{c} \leftarrow (\mathbf{p}_j + \mathbf{p}_k)/2$
6: $\mathbf{v} \leftarrow (\mathbf{p}_j - \mathbf{p}_k)/\|\mathbf{p}_j - \mathbf{p}_k\|$
7: $\mathbf{d} \leftarrow \mathbf{c} + (a^2/c)\mathbf{v}$

Case 2a: $\epsilon_S > 1$ Then B_S is a hyperboloid. Property 14 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S\sec(\alpha)\mathbf{v}_S + b_S\tan(\alpha)\mathbf{u} : -\pi < \alpha < \pi\}$ is one branch of a two-dimensional hyperbola with $\hat{X}_S \subset B_S \cap \mathrm{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$. Also, $\hat{X}_S$ has the same vectors and parameters as B_S, with vertex $\mathbf{x}(0) = \mathbf{a}_S$.

The search path X_S is constructed by computing the vector $\mathbf{u}_S \leftarrow$ Projection$((\mathbf{c}_S - \mathbf{x}_S), \{\mathbf{v}_S\})$, so that $\mathbf{u}_S$ is orthogonal to $\mathbf{v}_S$, and by restricting the domain to $\alpha_S \leq \alpha \leq \alpha_m$, where $\alpha_S = \arctan\{(\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/b_S\} < 0$, and $\alpha_m = 0$. Then $\mathbf{x}_S = \mathbf{x}(\alpha_S)$, and $\mathbf{x}(0) = \mathbf{a}_S \in \mathrm{aff}(S)$. The search path is defined by

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S\sec(\alpha)\mathbf{v}_S + b_S\tan(\alpha)\mathbf{u}_S : \alpha_S \leq \alpha \leq \alpha_m\}. \tag{13}$$

For each $\mathbf{p}_k \in P \setminus S$, compute α_k, if it exists, such that $\mathbf{x}(\alpha_k) \in X_S \cap B_{i_1,k}$. If $\mathbf{p}_k$ is redundant to $\mathbf{p}_{i_1}$, eliminate $\mathbf{p}_k$ from P and continue to the next point in $P \setminus S$. If $\mathbf{p}_{i_1}$ is redundant to $\mathbf{p}_k$, $\alpha_k = 0$, and the algorithm continues to the next point in $P \setminus S$. Else, $\mathbf{p}_k$ is not redundant to $\mathbf{p}_{i_1}$ so that $B_{i_1,k}$ is a bisector and the parameter α_k is determined so that $\mathbf{x}(\alpha) \in X_S \in B_{i_1,k}$.

Algorithm INTERSECTIONS$(S, \{r_{i_j} : \mathbf{p}_{i_j} \in S\})$

Input: Active set S and radii $\{r_{i_j} : \mathbf{p}_{i_j} \in S\}$
Output: Vectors and Parameters $\mathbf{c}_S, \mathbf{v}_S, \epsilon_S, a_S, b_S, c_S$

$(\mathbf{c}_1, \mathbf{v}_1, \mathbf{d}_1, \epsilon_1, a_1, b_1, c_1) \leftarrow$ BISECTORPARAMETERS$\big(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, r_{i_1}, r_{i_s}\big)$
$\mathbf{hp}_1 \leftarrow \mathbf{0}$
for $k = 2, \ldots, (s-1)$
  $\big(\mathbf{c}_{1,k}, \mathbf{v}_{1,k}, \mathbf{d}_{1,k}, \epsilon_{1,k}, a_{1,k}, b_{1,k}, c_{1,k}\big) \leftarrow$ BISECTORPARAMETERS$\big(\mathbf{p}_{i_1}, \mathbf{p}_{i_k}, r_{i_1}, r_{i_k}\big)$
  if $r_{i_1} > r_{i_k}$
    $\mathbf{hr}_k \leftarrow (\mathbf{p}_{i_1} - \mathbf{p}_{i_k})/(r_{i_1} - r_{i_k}) - (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})/(r_{i_1} - r_{i_s})$
    $\mathbf{h}_k \leftarrow \mathbf{hr}_k / \|\mathbf{hr}_k\|$
  else $\mathbf{h}_k \leftarrow (\mathbf{p}_{i_1} - \mathbf{p}_{i_k})/\|\mathbf{p}_{i_1} - \mathbf{p}_{i_k}\|$
  $\mathbf{hp}_k \leftarrow$ PROJECTION$\big(\mathbf{h}_k, \{\mathbf{hp}_2, \ldots, \mathbf{hp}_{k-1}\}\big)$
  $\mathbf{w}_k \leftarrow$ PROJECTION$\big(\mathbf{hp}_k, \{\mathbf{v}_{k-1}\}\big)$
  $\mathbf{v}_k \leftarrow$ PROJECTION$\big(\mathbf{v}_{k-1}, \{\mathbf{hp}_k\}\big)$
  $\mathbf{d}_k \leftarrow \mathbf{d}_{k-1} + \left(\frac{\mathbf{v}_{1,k}(\mathbf{d}_{1,k} - \mathbf{d}_{k-1})}{\mathbf{v}_{1,k}\mathbf{w}_k}\right)\mathbf{w}_k$
  $\hat{h}_k \leftarrow \mathbf{hp}_k(\mathbf{d}_k - \mathbf{c}_{k-1})$
  $\rho_k \leftarrow \mathbf{v}_{k-1}\mathbf{v}_k$
  $\sigma_k \leftarrow \mathbf{v}_{k-1}\mathbf{hp}_k$
  $\epsilon_k \leftarrow \epsilon_{k-1}\rho_k$
  if $\epsilon_k = 1$
    $\tilde{c}_k \leftarrow \epsilon_{k-1}\sigma_k\hat{h}_k/2$
    $\hat{c}_k \leftarrow \left((1 - \epsilon_{k-1}^2\sigma_k^2)\hat{h}_k^2 + b_{k-1}^2\right)/(4\tilde{c}_k)$
    $\mathbf{c}_k \leftarrow \mathbf{c}_{k-1} + \hat{h}_k\mathbf{hp}_k + \hat{c}_k\mathbf{v}_k$
  else
    $\tilde{c}_k \leftarrow \epsilon_{k-1}^2\rho_k\sigma_k\hat{h}_k/(1 - \epsilon_k^2)$
    $a_k^2 \leftarrow \left((1 - \epsilon_{k-1}^2)(a_{k-1}^2(1 - \epsilon_k^2) - \hat{h}_k^2)\right)/(1 - \epsilon_k^2)^2$
    $\mathbf{c}_k \leftarrow \mathbf{c}_{k-1} + \hat{h}_k\mathbf{hp}_k + \tilde{c}_k\mathbf{v}_k$
    if $\epsilon_k > 1$
      $b_k^2 \leftarrow -a_k^2(1 - \epsilon_k^2)$
      $c_k^2 \leftarrow a_k^2 + b_k^2$
    else
      $b_k^2 \leftarrow a_k^2(1 - \epsilon_k^2)$
      $c_k^2 \leftarrow a_k^2 - b_k^2$
end for
$(\mathbf{c}_S, \mathbf{v}_S, \epsilon_S, a_S, b_S, c_S) \leftarrow (\mathbf{c}_{s-1}, \mathbf{v}_{s-1}, \epsilon_{s-1}, a_{s-1}, b_{s-1}, c_{s-1})$

Property 20 shows that $X_S \cap B_{i_1,k} = X_S \cap H_k$ for the hyperplane $H_k = \{\mathbf{x} : \mathbf{h}_k\mathbf{x} = \mathbf{h}_k\mathbf{d}_k\}$, where $(\mathbf{h}_k, \mathbf{d}_k) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k, r_{i_1}, r_{i_s}, r_k)$. The point $\mathbf{x}(\alpha_k) \in X_S \cap H_k$ is determined by solving for α using the equation $\mathbf{h}_k\mathbf{x}(\alpha) = \mathbf{h}_k\mathbf{d}_k$, which is equivalent to the equation $a_S\mathbf{h}_k\mathbf{v}_S \sec(\alpha) + b_S\mathbf{h}_k\mathbf{u}_S \tan(\alpha) = \mathbf{h}_k(\mathbf{d}_k - \mathbf{c}_S)$. Multiplying through by $\cos(\alpha)$, which is positive for $-\pi/2 < \alpha < 0$, and rearranging gives

$$A\cos(\alpha) + B\sin(\alpha) = C, \tag{14}$$

where $A = \mathbf{h}_k(\mathbf{d}_k - \mathbf{c}_S)$, $B = -b_S\mathbf{h}_k\mathbf{u}_S$, and $C = a_S\mathbf{h}_k\mathbf{v}_S$. Define $\phi = \sin(\alpha)$ for $-\pi/2 < \alpha < 0$. Note that $\phi_S = \sin(\alpha_S)$. The solution α_k is determined by $\phi_k \leftarrow$ COMPUTE $\phi(\phi_S, A, B, C)$, and $\alpha_k = \arcsin(\phi_k)$.

Algorithm HYPERPLANE$\left(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k, r_{i_1}, r_{i_s}, r_{i_k}\right)$

Input: Points $\mathbf{p}_{i_1}, \mathbf{p}_{i_s}$, and $\mathbf{p}_k$, and radii r_{i_1}, r_{i_s}, and r_k
Output: Normal vector $\mathbf{h}_k$ and point $\mathbf{d}_k$ of hyperplane $H_k = \{\mathbf{x} : \mathbf{h}_k\mathbf{x} = \mathbf{h}_k\mathbf{d}_k\}$, such that $X_S \cap B_{i_1,k} = X_S \cap H_k$.

$\left(\mathbf{p}_{j_1}, \mathbf{p}_{j_2}, \mathbf{p}_{j_3}\right) \leftarrow$ SORT$\left(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k\right)$, such that $r_{j_1} \geq r_{j_2} \geq r_{j_3}$
if $\left(r_{j_1} = r_{j_2} > r_{j_3}\right)$
    $\mathbf{h}_k \leftarrow \left(\mathbf{p}_{j_1} - \mathbf{p}_{j_2}\right)/\|\mathbf{p}_{j_1} - \mathbf{p}_{j_2}\|$
    $\mathbf{d}_k \leftarrow \left(\mathbf{p}_{j_1} + \mathbf{p}_{j_2}\right)/2$
else if $\left(r_{j_1} > r_{j_2} = r_{j_3}\right)$
    $\mathbf{h}_k \leftarrow \left(\mathbf{p}_{j_2} - \mathbf{p}_{j_3}\right)/\|\mathbf{p}_{j_2} - \mathbf{p}_{j_3}\|$
    $\mathbf{d}_k \leftarrow \left(\mathbf{p}_{j_2} + \mathbf{p}_{j_3}\right)/2$
else
    $\left(\mathbf{v}_{j_1,j_2}, \epsilon_{j_1,j_2}\right) \leftarrow$ BISECTORPARAMETERS$\left(\mathbf{p}_{j_1}, \mathbf{p}_{j_2}, r_{j_1}, r_{j_2}\right)$
    $\left(\mathbf{v}_{j_1,j_3}, \epsilon_{j_1,j_3}\right) \leftarrow$ BISECTORPARAMETERS$\left(\mathbf{p}_{j_1}, \mathbf{p}_{j_3}, r_{j_1}, r_{j_3}\right)$
    $\mathbf{h}_k \leftarrow \left(\epsilon_{j_1,j_2}\mathbf{v}_{j_1,j_2} - \epsilon_{j_1,j_3}\mathbf{v}_{j_1,j_3}\right)/\|\epsilon_{j_1,j_2}\mathbf{v}_{j_1,j_2} - \epsilon_{j_1,j_3}\mathbf{v}_{j_1,j_3}\|$
    $\mathbf{w}_k \leftarrow$ PROJECTION$\left(\mathbf{h}_k, \{\mathbf{v}_{j_1,j_3}\}\right)$
    $\mathbf{d}_k \leftarrow \mathbf{d}_{j_1,j_3} + \left(\frac{\mathbf{v}_{j_1,j_2}(\mathbf{d}_{j_1,j_2} - \mathbf{d}_{j_1,j_3})}{\mathbf{v}_{j_1,j_2}\mathbf{w}_k}\right)\mathbf{w}_k$
end if

Algorithm COMPUTE$\phi(\phi_S, A, B, C)$

Input: ϕ_S, A, B, C
Output: ϕ

if $A^2 + B^2 - C^2 > 0$
    $D \leftarrow \sqrt{A^2 + B^2 - C^2}$
    $\theta' \leftarrow (AC + BD)/(A^2 + B^2)$
    $\theta'' \leftarrow (AC - BD)/(A^2 + B^2)$
    $\phi' \leftarrow (BC - AD)/(A^2 + B^2)$
    $\phi'' \leftarrow (BC + AD)/(A^2 + B^2)$
    if $\left(\theta' < 0 \text{ **OR** } \phi' < \phi_S\right)$ **then** $\phi' \leftarrow \infty$
    if $\left(\theta'' < 0 \text{ **OR** } \phi'' < \phi_S\right)$ **then** $\phi'' \leftarrow \infty$
    $\phi \leftarrow \min\left\{\phi', \phi''\right\}$
else if $A^2 + B^2 - C^2 = 0$
    $\phi \leftarrow B/C$
    $\theta \leftarrow A/C$
    if $(\theta < 0$ **OR** $\phi < \phi_S)$ **then** $\phi \leftarrow \infty$
else $\phi \leftarrow \infty$
end if

Case 2b: $\epsilon_S < 1$ Then B_S is an ellipsoid. Corollary 1 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S \cos(\alpha)\mathbf{v}_S + b_S \sin(\alpha)\mathbf{u} : 0 \leq \alpha \leq 2\pi\}$ is a two-dimensional ellipse with $\hat{X}_S \subset B_S \cap \text{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$.

The search path X_S is constructed by computing $\mathbf{u}_S \leftarrow$ PROJECTION$((\mathbf{c}_S - \mathbf{x}_S), \{\mathbf{v}_S\})$, so that $\mathbf{u}_S$ is orthogonal to $\mathbf{v}_S$, and by restricting the domain to $\alpha_S \leq \alpha \leq \alpha_m$, where $\alpha_S = \arcsin\{(\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/b_S\} < 0$, and $\alpha_m = 0$. Then $\mathbf{x}_S = \mathbf{x}(\alpha_S)$, and $\mathbf{x}(\alpha_m) = \mathbf{a}_S \in \text{aff}(S)$. The search path is defined by

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S \cos(\alpha)\mathbf{v}_S + b_S \sin(\alpha)\mathbf{u}_S : \alpha_S \leq \alpha \leq \alpha_m\}. \tag{15}$$

The analysis for the elliptic search path is analogous to the hyperbolic search path. For each $\mathbf{p}_k \in P \setminus S$, the parameter α_k is determined, if it exists, so that X_S intersects the bisector $B_{i_1,k}$ at $\mathbf{x}(\alpha_k)$, with $\alpha_S \leq \alpha \leq 0$. If $\mathbf{p}_k$ is redundant to $\mathbf{p}_{i_1}$, eliminate $\mathbf{p}_k$ from P and continue to the next point in $P \setminus S$. If $\mathbf{p}_{i_1}$ is redundant to $\mathbf{p}_k$, $\alpha_k = 0$, and the algorithm continues to the next point in $P \setminus S$. Else, $\mathbf{p}_k$ is not redundant to $\mathbf{p}_{i_1}$ so that $B_{i_1,k}$ is a bisector and the parameter α_k is determined so that $\mathbf{x}(\alpha) \in X_S \in B_{i_1,k}$.

Property 20 shows that $X_S \cap B_{i_1,k} = X_S \cap H_k$ for the hyperplane $H_k = \{\mathbf{x} : \mathbf{h}_k\mathbf{x} = \mathbf{h}_k\mathbf{d}_k\}$, where $(\mathbf{h}_k, \mathbf{d}_k) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k, r_{i_1}, r_{i_s}, r_k)$. The point $\mathbf{x}(\alpha_k) \in X_S \cap H_k$ is determined by solving for α using the equation $\mathbf{h}_k\mathbf{x}(\alpha) = \mathbf{h}_k\mathbf{d}_k$, which is equivalent to the equation

$$A\cos(\alpha) + B\sin(\alpha) = C, \tag{16}$$

where $A = a_S\mathbf{h}_k\mathbf{v}_S$, $B = b_S\mathbf{h}_k\mathbf{u}_S$, and $C = \mathbf{h}_k(\mathbf{d}_e - \mathbf{c}_S)$. Define $\phi = \sin(\alpha)$ for $-\pi/2 < \alpha < \pi/2$. Note that $\phi_S = \sin(\alpha_S)$. The solution α_k is determined by $\phi_k \leftarrow$ COMPUTE$\phi(\phi_S, A, B, C)$, and $\alpha_k = \arcsin(\phi_k)$.

Case 2c: $\epsilon_S = 1$ Then B_S is a paraboloid. Property 17 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + \tilde{c}_S\alpha^2\mathbf{v}_S + 2\tilde{c}_S\alpha\mathbf{u} : -\infty < \alpha < \infty\}$ is a two-dimensional parabola with $\hat{X}_S \subset B_S \cap \text{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$, and $\hat{X}_S$ has the same vectors and parameters as B_S. The search path X_S is constructed by computing the vector $\mathbf{u}_S \leftarrow$ PROJECTION$((\mathbf{c}_S - \mathbf{x}_S), \{\mathbf{v}_S\})$. Also, the domain is restricted to $\alpha_S \leq \alpha \leq \alpha_m$, where $\alpha_S = -(\mathbf{u}_S(\mathbf{c}_S - \mathbf{x}_S))/(2\tilde{c}_S) \leq 0$, and $\alpha_m = 0$. Then $\mathbf{x}(\alpha_S) = \mathbf{x}_S$, $\mathbf{x}(\alpha_m) = \mathbf{c}_S \in \text{aff}(S)$, and the search path is defined by

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + \tilde{c}_S\alpha^2\mathbf{v}_S + 2\tilde{c}_S\alpha\mathbf{u}_S : \alpha_S \leq \alpha \leq \alpha_m\}. \tag{17}$$

For each $\mathbf{p}_k \in P \setminus S$, the parameter α_k is determined, if it exists, so that X_S intersects the bisector $B_{i_1,k}$ at $\mathbf{x}(\alpha_k)$ for $\alpha_k \leq \alpha \leq 0$.

Property 20 shows that $X_S \cap B_{i_1,k} = X_S \cap H_k$ for the hyperplane $H_k = \{\mathbf{x} : \mathbf{h}_k\mathbf{x} = \mathbf{h}_e\mathbf{d}_k\}$ where $(\mathbf{h}_k, \mathbf{d}_k) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k, r_{i_1}, r_{i_s}, r_k)$. The point $\mathbf{x}(\alpha_k) \in X_S \cap H_k$ is determined by solving the equation $\mathbf{h}_e\mathbf{x}(\alpha_k) = \mathbf{h}_k\mathbf{d}_k$, for α_k, which is equivalent to the quadratic equation

$$A\alpha^2 + B\alpha = C, \tag{18}$$

where $A = \tilde{c}_S\mathbf{h}_k\mathbf{v}_S$, $B = 2\tilde{c}_S\mathbf{h}_k\mathbf{u}_S$, and $C = \mathbf{h}_k(\mathbf{d}_k - \mathbf{c}_S)$. Then α_k is chosen as the smallest real solution such that $\alpha_S \leq \alpha_k$.

Primal update phase

Let $\alpha^* = \min\{\alpha_m, \min_{\mathbf{p}_k \in P\setminus S} \alpha_k\}$. If $\alpha^* = \alpha_m$, then $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_m), z(\alpha_m)]$ and $\mathbf{x}_S \in \text{aff}(S)$. If $\mathbf{x}_S \in \text{conv}(S)$, that is, there exists a solution to equations (3) and (4) of the KKT conditions over the set S, where $\lambda_{i_j} \geq 0$ for all $p_{i_j} \in S$, then $[\mathbf{x}_S, z_S]$ is the optimal solution. Else, $\mathbf{x}_S \notin \text{conv}(S)$, and $\lambda_{i_l} < 0$, for some $\mathbf{p}_{i_l} \in S$. Set $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$. The algorithm returns to the Search Phase with $[\mathbf{x}_S, z_S]$ and active set S.

If $\alpha^* = \min_{\mathbf{p}_k \in P\setminus S}\{\alpha_k\}$, set $\alpha_e \leftarrow \alpha^*$, $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$, and $S \leftarrow S \cup \{\mathbf{p}_e\}$. In this case, α_e is the smallest parameter such that the ball $[\mathbf{x}_S, z_S]$ remains primal feasible and contains the ball $[\mathbf{p}_e, r_e]$. If $\mathbf{x}_S \in \text{conv}(S)$, then $[\mathbf{x}_S, z_S]$ is the optimal solution. If $\mathbf{x}_S \notin \text{conv}(S)$, but $\mathbf{x}_S \in \text{aff}(S)$, then $\lambda_{i_l} < 0$ for some $\mathbf{p}_{i_l} \in S$ and $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$. The algorithm returns to the Search Phase with $[\mathbf{x}_S, z_S]$ and active set S. Else, $\mathbf{x}_S \notin \text{aff}(S)$, then the points in S are affinely independent, and the algorithm returns to the Search Phase with $[\mathbf{x}_S, z_S]$ and active set S.

There may be a tie for the entering point $\mathbf{p}_e$, in which case the next iteration of the Search Phase may be degenerate with zero change in the parameter α_k. Cycling can be avoided by an adaptation of Bland's rule that chooses the point with the smallest index among all points that are candidates for entering. That is, $e = \min\{k : \alpha_k = \alpha^*, \mathbf{p}_k \in P \setminus S\}$.

Property 8 If equations (3) and (4) of the KKT conditions, over the set $S \cup \{\mathbf{p}_e\}$, with $[\mathbf{x}^*, z^*] = [\mathbf{x}_S, z_S]$ have a solution with $\lambda_{i_l} < 0$, then the points in $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$ are affinely independent.

Proof Each column of the linear system over the set S determined by equations (3) and (4) from the KKT conditions with $[\mathbf{x}^*, z^*] = [\mathbf{x}_S, z_S]$ corresponds to a point in S. Since the points in S are affinely independent, the columns of this linear system are linearly independent. The linear system over the set $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$ has the same columns as the linear system over the set S, except that the column corresponding to $\mathbf{p}_e$ has replaced the column corresponding to $\mathbf{p}_{i_l}$. Therefore the columns of the linear system over the set $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$ are linearly independent since the column corresponding to $\mathbf{p}_e$ is a linear combination of the columns determined by the set S, with a non-zero multiplier for the column corresponding to $\mathbf{p}_{i_l}$. Therefore, the points in the set $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$ are affinely independent. □

The next property shows that if the point $\mathbf{p}_{i_l}$ leaves S, because $\lambda_{i_l} < 0$, then the point $\mathbf{p}_{i_l}$ is covered by the ball $[\mathbf{x}(\alpha), z(\alpha)]$ during the next search phase.

Property 9 In the Update Phase, suppose the point $\mathbf{p}_{i_l}$ is chosen to leave the set S because $\lambda_{i_l} < 0$ in the solution to equations (3) and (4) of the KKT conditions over the set $S \cup \{\mathbf{p}_e\}$. In the Search Phase, the active set is $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$, and the search path is $X_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} = \{\mathbf{x}(\alpha) : \alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0\}$. Then the ball $[\mathbf{x}(\alpha), z(\alpha)]$ remains feasible with respect to the leaving point $\mathbf{p}_{i_l}$, that is, $[\mathbf{p}_{i_l}, r_{i_l}] \subset [\mathbf{x}(\alpha), z(\alpha)]$ for $\alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0$.

Proof Let $z_{i_j}(\alpha) = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_j}\| + r_{i_j}$, for each $\mathbf{p}_{i_j} \in S$, and let $z'_{i_j}(\alpha) = \frac{(\mathbf{x}(\alpha)-\mathbf{p}_{i_j})}{\|\mathbf{x}(\alpha)-\mathbf{p}_{i_j}\|}\mathbf{x}'(\alpha)$, where $\mathbf{x}'(\alpha)$ is the normalized tangent vector to the search path at $\mathbf{x}(\alpha)$. By construction of the search path, with $\alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0$, $z_{i_1}(\alpha) = z_{i_j}(\alpha)$, for each $\mathbf{p}_{i_j} \in S \setminus \{\mathbf{p}_{i_l}\}$, which implies the derivatives are equal, that is, $z'_{i_1}(\alpha) = z'_{i_j}(\alpha)$, for each $\mathbf{p}_{i_j} \in S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$ and for $\alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0$. Property 24 shows that $z_{i_j}(\alpha)$ is decreasing along the search path, that is, $z'_{i_j}(\alpha) < 0$, for each $\mathbf{p}_{i_j} \in S \setminus \{\mathbf{p}_{i_l}\}$, and for $\alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0$.

In equation (4) substitute $\mathbf{x}(\alpha)$ for $\mathbf{x}_S$, and take the dot product of each summand with the tangent vector $\mathbf{x}'(\alpha)$, so that $\sum_{\mathbf{p}_{i_j}\in S\setminus\{\mathbf{p}_{i_l}\}} \frac{(\mathbf{x}(\alpha)-\mathbf{p}_{i_j})}{\|\mathbf{x}(\alpha)-\mathbf{p}_{i_j}\|}\mathbf{x}'(\alpha)\lambda_{i_j} + \frac{(\mathbf{x}(\alpha)-\mathbf{p}_{i_l})}{\|\mathbf{x}(\alpha)-\mathbf{p}_{i_l}\|}\mathbf{x}'(\alpha)(\lambda_{i_l}) = \sum_{\mathbf{p}_{i_j}\in S\setminus\{\mathbf{p}_{i_l}\}} z'_{i_j}(\alpha)\lambda_{i_j} + z'_{i_l}(\alpha)\lambda_{i_l} = 0$. Substitute $z'_{i_1}(\alpha) = z'_{i_j}(\alpha)$, for each $\mathbf{p}_{i_j} \in S \setminus \{\mathbf{p}_{i_l}\}$, and apply equation (3), to get $z'_{i_1}(\alpha)(1-\lambda_{i_l}) + z'_{i_l}(\alpha)(\lambda_{i_l}) = z'_{i_1}(\alpha) + (z'_{i_l}(\alpha) - z'_{i_1}(\alpha))\lambda_{i_l} = 0$, or $z'_{i_1}(\alpha) = -\lambda_{i_l}(z'_{i_l}(\alpha) - z'_{i_1}(\alpha))$. Since $z'_{i_1}(\alpha) < 0$ and $\lambda_{i_l} < 0$, then $z'_{i_l}(\alpha) < z'_{i_1}(\alpha) < 0$ so that $z_{i_l}(\alpha)$ is decreasing and decreasing at a faster rate than $z_{i_1}(\alpha)$. This shows that $[\mathbf{p}_{i_l}, r_{i_l}] \subset [\mathbf{x}(\alpha), z(\alpha)]$ for $\alpha_{S\cup\{\mathbf{p}_e\}\setminus\{\mathbf{p}_{i_l}\}} \leq \alpha \leq 0$. □

If there is more than one component $\lambda_{i_j} < 0$ in the solution to (3) and (4), an adaptation of Bland's rule is to choose the point to be deleted from S with the smallest index i_j such that $\lambda_{i_j} < 0$, that is, $i_l = \min\{i_j : \lambda_{i_j} < 0\}$.

Property 10 The primal algorithm solves $M(P)$ in a finite number of iterations.

Proof At each iteration of the search phase there is a primal feasible, but not dual feasible, ball $[\mathbf{x}_S, z_S]$, and an active set S, such that $\mathbf{x}_S \notin \text{aff}(S)$. If the search phase yields $\alpha^* = \alpha_m > \alpha_S$, then $z(\alpha^*) < z(\alpha_S)$ by Property 24, and $\mathbf{x}(\alpha^*) \in \text{aff}(S)$.

If the search phase yields $\alpha^* = \alpha_k > \alpha_S$, then $z(\alpha^*) < z(\alpha_S)$, and $S \leftarrow S \cup \{\mathbf{p}_k\}$.

The search phase yields $\alpha^* = \alpha_k = \alpha_S$ if and only if $\mathbf{p}_k$ is active for the ball $[\mathbf{x}_S, z_S]$. If $\mathbf{x}(\alpha^*) \in \text{conv}(S \cup \{\mathbf{p}_k\})$, then $[\mathbf{x}(\alpha^*), z(\alpha^*)]$ is optimal. Else, if $\mathbf{x}(\alpha^*) \in \text{aff}(S \cup \{\mathbf{p}_k\})$, some point $\mathbf{p}_{i_l}$ leaves S, and a new search is initiated with $[\mathbf{x}_S, z_S] = [\mathbf{x}(\alpha^*), z(\alpha^*)]$ and $S \setminus \{\mathbf{p}_{i_l}\} \cup \{\mathbf{p}_k\}$.

To prohibit an active set from being repeated in the sequence of degenerate steps, a list of points that have been eliminated from active sets occurring in the sequence of degenerate steps is generated, and these points are prohibited from consideration as entering points. After a finite number of degenerate iterations, either $\alpha^* = \alpha_m > \alpha_S$ or $\alpha^* = \alpha_k > \alpha_S$ and $z(\alpha^*)$ decreases. Since there are a finite number of active sets, and $z(\alpha)$ is bounded below, the algorithm finds a minimum ball $[\mathbf{x}_S, z_S]$ in a finite number of iterations. □

5 Dual algorithm

At each iteration of the dual algorithm there is a ball $[\mathbf{x}_S, z_S]$ and an active set S that satisfy expressions (3), (4), (5), and (6) of the KKT conditions, but do not satisfy expressions (2) of the KKT condition. That is, S and $[\mathbf{x}_S, z_S]$ are dual feasible but not primal feasible.

Since $[\mathbf{x}_S, z_S]$ is a feasible covering of the balls $[\mathbf{p}_i, r_i]$ for $\mathbf{p}_i \in S$, then $[\mathbf{x}_S, z_S]$ is an optimal solution to the problem $M(S)$, a relaxation of $M(P)$. Therefore, z_S is a lower bound on the optimal objective function value for $M(P)$. Assuming non-degeneracy, the radius z_S is shown to increase at each iteration of the dual algorithm.

The dual algorithm may be initialized by choosing the ball $[\mathbf{x}_S, z_S] = [\mathbf{p}_j, r_j]$ for any point $\mathbf{p}_j \in P$, and $S = \{\mathbf{p}_j\}$ as an active set. Then $[\mathbf{x}_S, z_S]$ and S are dual feasible.

Dual update phase

A dual feasible ball $[\mathbf{x}_S, z_S]$ and active set S are optimal to $M(P)$ if they are primal feasible to $M(P)$, that is, if $z_S \geq \|\mathbf{p}_i - \mathbf{x}_S\| + r_i$, for all $\mathbf{p}_i \in P$. If not optimal, then an infeasible point $\mathbf{p}_e \in P \setminus S$ is chosen to enter.

If the points in $S \cup \{\mathbf{p}_e\}$ are affinely independent, then $|S \cup \{\mathbf{p}_e\}| \leq n + 1$, and $\mathbf{x}_S \in \text{conv}(S \cup \{\mathbf{p}_e\})$, so that $[\mathbf{x}_S, z_S]$ remains dual feasible with respect to $S \cup \{\mathbf{p}_e\}$. The algorithm enters the Search Phase with the ball $[\mathbf{x}_S, z_S]$, active set S, and the entering point $\mathbf{p}_e$. The Search Phase searches for a new solution for which the set $S \cup \{\mathbf{p}_e\}$ is active.

If the points in $S \cup \{\mathbf{p}_e\}$ are affinely dependent, then a point $\mathbf{p}_l \in S$ is chosen to leave S. Since $[\mathbf{x}_S, z_S]$ is dual feasible to $M(P)$, there exists a non-negative solution $(\pi_1, \ldots, \pi_s)$ to the linear system (7), (8), over the set S. Since the points in $S \cup \{\mathbf{p}_e\}$ are affinely dependent, the linear system (19) and (20) has a solution, and (19) implies $\lambda_j < 0$, for some $\mathbf{p}_j \in S$.

$$\sum_{\mathbf{p}_j \in S} \lambda_j = -1 \tag{19}$$

$$\sum_{\mathbf{p}_j \in S} (\mathbf{x}_S - \mathbf{p}_j)\lambda_j = -(\mathbf{x}_S - \mathbf{p}_e). \tag{20}$$

A leaving point $\mathbf{p}_l$ is chosen by the "minimum ratio rule":

$$\frac{\pi_l}{-\lambda_l} = \min_{\mathbf{p}_j \in S} \left\{ \frac{\pi_j}{-\lambda_j} : \lambda_j < 0 \right\}. \tag{21}$$

Algorithm PRIMAL($\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}, \{r_1, \ldots, r_m\}$)

Input: $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\} \subset I\!R^n$, $\{r_i \geq 0 : \text{for } \mathbf{p}_i \in P\}$
Output: The ball $[\mathbf{x}^*, z^*]$ with minimum z^* such that $[\mathbf{p}_i, r_i] \subset [\mathbf{x}^*, z^*]$ for all $\mathbf{p}_i \in P$

Initialize: Choose $\mathbf{x}_S$ arbitrarily
$z_S \leftarrow \max_{\mathbf{p}_i \in P} \|\mathbf{p}_i - \mathbf{x}_S\| + r_i = \|\mathbf{p}_{i_1} - \mathbf{x}_S\| + r_{i_1}$ and $S \leftarrow \{\mathbf{p}_{i_1}\}$
while $\mathbf{x}_S \notin \text{conv}(S)$
  For active set $S = \{\mathbf{p}_{i_1}, \ldots \mathbf{p}_{i_s}\}$ with $r_{i_1} \geq \cdots \geq r_{i_s}$ and primal feasible ball $[\mathbf{x}_S, z_S]$
  Primal Search Phase
  if $r_{i_1} = r_{i_s}$
    $\mathbf{d}_S \leftarrow$ PROJECTION($(\mathbf{p}_{i_1} - \mathbf{x}_S), R$), where $R = \{\mathbf{p}_{i_2} - \mathbf{p}_{i_1}, \ldots \mathbf{p}_{i_s} - \mathbf{p}_{i_1}\}$
    $\alpha_m \leftarrow [(\mathbf{p}_{i_1} - \mathbf{x}_S)\mathbf{d}_S]/\|\mathbf{d}_S\|^2$
    for $\mathbf{p}_k \in P \setminus S$
      if $r_{i_1} = r_k$ **then** solve for α_k using (11)
      else solve for α_k using (12)
  else
    $(\mathbf{v}_S, \mathbf{c}_S, \epsilon_S, a_S, b_S, c_S) \leftarrow$ INTERSECTIONS($S, \{r_{i_j} : \mathbf{p}_{i_j} \in S\}$)
    $\mathbf{u}_S \leftarrow$ PROJECTION($(\mathbf{c}_S - \mathbf{x}_S), \{\mathbf{v}_S\}$)
    $\phi_S \leftarrow \sin(\arctan\{(\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/b_S\})$
    $(\mathbf{h}_k, \mathbf{d}_k) \leftarrow$ HYPERPLANE($\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_k, r_{i_1}, r_{i_s}, r_k$)
    $\alpha_m \leftarrow 0$
    if $\epsilon_S > 1$
      for $\mathbf{p}_k \in P \setminus S$
        $\alpha_k \leftarrow \arcsin($COMPUTE$\phi(\phi_S, A, B, C))$ using (14)
    if $\epsilon_S < 1$
      for $\mathbf{p}_k \in P \setminus S$
        $\alpha_k \leftarrow \arcsin($COMPUTE$\phi(\phi_S, A, B, C))$ using (16)
    if $\epsilon_S = 1$
      for $\mathbf{p}_k \in P \setminus S$
        solve for α_k using (18)
  $\alpha^* \leftarrow \min\{\alpha_m, \min_{\mathbf{p}_k \in P \setminus S} \alpha_k\}$
  $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha^*), z(\alpha^*)]$ using (10), (13), (15), or (17)
  Primal Update Phase
  if $\alpha^* = \alpha_m$
    if $\mathbf{x}_S \in \text{conv}(S)$ **then** $[\mathbf{x}_S, z_S]$ is optimal
    else $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, where $\lambda_{i_l} < 0$ in (3), (4)
  else
    $S \leftarrow S \cup \{\mathbf{p}_e\}$, where $e = \min\{k : \alpha_k = \alpha^*, \mathbf{p}_k \in P \setminus S\}$
    if $\mathbf{x}_S \in \text{conv}(S)$ **then** $[\mathbf{x}_S, z_S]$ is optimal
    else $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, where $\lambda_{i_l} < 0$ in (3), (4)
end while

The next property states that the points in $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_l\}$ are affinely independent and that $\mathbf{x}_S$ remains dual feasible for the set $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_l\}$. The proof is given in Dearing and Thipwiwatpotjana (2006).

Property 11 Suppose that $[\mathbf{x}_S, z_S]$, and active set S, are dual feasible, but not optimal, to $M(P)$. Suppose that $\mathbf{p}_e$ is the point chosen to enter the set S and that the set $S \cup \{\mathbf{p}_e\}$ is affinely dependent. If the leaving point $\mathbf{p}_l$ is chosen by (21), then the points in $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_l\}$ are affinely independent, and $\mathbf{x}_S \in \text{conv}(S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_l\})$.

If the point $\mathbf{p}_l$ leaves S, then the set S is reset to $S \leftarrow S \setminus \{\mathbf{p}_l\}$. Thus $|S \cup \{\mathbf{p}_e\}| \leq n + 1$, and $\mathbf{x}_S \in \text{conv}(S \cup \{\mathbf{p}_e\})$, so that $[\mathbf{x}_S, z_S]$ remains dual feasible with respect to $S \cup \{\mathbf{p}_e\}$. The updated set S remains active for $[\mathbf{x}_S, z_S]$. The algorithm enters the Search Phase with the set S, the ball $[\mathbf{x}_S, z_S]$, and the entering point $\mathbf{p}_e$. The Search Phase searches for a new solution for which the set $S \cup \{\mathbf{p}_e\}$ will be active.

Dual search phase

Given a dual feasible ball $[\mathbf{x}_S, z_S]$, an active set S, and an entering point $\mathbf{p}_e$ as determined in the Update Phase, a search path $X_S = \{\mathbf{x}(\alpha) : \alpha \geq \alpha_S\}$ is constructed so that $\mathbf{x}(\alpha_S) = \mathbf{x}_S$ and $X_S \subset B_S$. If all the points in S have equal radii, the search path will be a ray, but if some points in S have unequal radii, the search path will be a two-dimensional conic section in $I\!R^n$.

For a search path X_S that is either a ray or a conic section, Property 23 shows that S is an active set for the ball $[\mathbf{x}(\alpha), z(\alpha)]$, and that S and $[\mathbf{x}(\alpha), z(\alpha)]$ satisfy the complementary slackness conditions (6) for $\alpha \geq \alpha_S$. Property 24 shows that $z(\alpha)$ is increasing for $\alpha \geq \alpha_S$.

The parameter $\alpha_e \geq \alpha_S$ is determined, if it exists, so that X_S intersects the bisector $B_{i_1,e}$ at $\mathbf{x}(\alpha_e)$. If $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_1,e}$, then $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_j,e}$ for each $\mathbf{p}_{i_j} \in S$. Thus it suffices to consider the intersection of X_S with only $B_{i_1,e}$.

Geometrically, the search moves the center $\mathbf{x}(\alpha)$ of the ball $[\mathbf{x}(\alpha), z(\alpha)]$ along the path X_S, while the radius $z(\alpha)$ increases. At the point $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_1,e}$, the constraint corresponding to $\mathbf{p}_e$ is active. If $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$, then the ball $[\mathbf{x}(\alpha_e), z(\alpha_e)]$ and S are checked for optimality. Otherwise, the parameter α^* is determined so that $\mathbf{x}(\alpha^*) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case some point $\mathbf{p}_{i_l} \in S$ is deleted from S, and the Update Phase is entered with the ball $[\mathbf{x}(\alpha^*), z(\alpha^*)]$, the set $S \setminus \{\mathbf{p}_{i_l}\}$, and entering point $\mathbf{p}_e$.

Case 1: All points in S have equal radii

The points in the active set are denoted by $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, where $r_{i_j} = r_{i_1}$, for $\mathbf{p}_{i_j} \in S$. In this case, the search path is the ray

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha \mathbf{d}_S, \alpha_S \leq \alpha\}, \tag{22}$$

where $\alpha_S = 0$ and $\mathbf{d}_S \leftarrow \text{PROJECTION}((\mathbf{p}_e - \mathbf{x}_S), R)$ with $R = \{(\mathbf{p}_{i_1} - \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})\}$. If $s = 1$, with $S = \{\mathbf{p}_{i_1}\}$, $\mathbf{d}_S = (\mathbf{p}_e - \mathbf{p}_{i_1})/\|\mathbf{p}_e - \mathbf{p}_{i_1}\|$.

There are two sub-cases to consider for computing α_e depending on whether the radius r_{i_1} equals the radius r_e.

Case 1a: $r_{i_1} = r_e$ Then $B_{i_1,e}$ is a hyperplane. The intersection of X_S and $B_{i_1,e}$ is determined by solving for α using the equation $(\mathbf{p}_{i_1} - \mathbf{p}_e)\mathbf{x}(\alpha) = (\mathbf{p}_{i_1} - \mathbf{p}_e)(\mathbf{p}_{i_1} + \mathbf{p}_e)/2$. If $(\mathbf{p}_{i_1} - \mathbf{p}_e)\mathbf{d}_S = 0$, $\alpha_e \leftarrow \infty$. Otherwise,

$$\alpha_e = \frac{(\mathbf{p}_{i_1} - \mathbf{p}_e)(\mathbf{p}_{i_1} + \mathbf{p}_e)/2 - (\mathbf{p}_{i_1} - \mathbf{p}_e)\mathbf{x}_S}{(\mathbf{p}_{i_1} - \mathbf{p}_e)\mathbf{d}_S}. \tag{23}$$

If $\alpha_e < 0$, $\alpha_e \leftarrow \infty$.

Case 1b: $r_{i_1} \neq r_e$ Since $z_S = \|\mathbf{p}_{i_j} - \mathbf{x}_S\| + r_j$ for each $\mathbf{p}_{i_j} \in S$, and $z_S < \|\mathbf{p}_e - \mathbf{x}_S\| + r_e$, then $\mathbf{p}_e$ is non-redundant to each $\mathbf{p}_{i_j} \in S$, and $B_{i_1,e}$ is a bisector. The intersection of X_S and

$B_{i_1,e}$ is determined by substituting $\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha\mathbf{d}_S$ for $\mathbf{x}$, and $c_{i_1,e}$, $\mathbf{c}_{i_1,e}$, $a_{i_1,e}$, $\mathbf{v}_{i_1,e}$, and $\epsilon_{i_1,e}$ for c, $\mathbf{c}$, a, $\mathbf{v}$, and ϵ, respectively, in the quadratic expression (36) for $B_{i_1,e}$. This gives the quadratic equation

$$A\alpha^2 + B\alpha + C = 0, \tag{24}$$

where $A = (\mathbf{d}_S)^2 - \epsilon_{i_1,e}^2(\mathbf{d}_S\mathbf{v}_{i_1,e})^2$, $B = 2(\mathbf{x}_S - \mathbf{c}_{i_1,e})\mathbf{d}_S - 2\epsilon_{i_1,e}^2[(\mathbf{x}_S - \mathbf{c}_{i_1,e})\mathbf{v}_{i_1,e}][\mathbf{d}_S\mathbf{v}_{i_1,e}]$, and $C = (\mathbf{x}_S - \mathbf{c}_{i_1,e})^2 - \epsilon_{i_1,e}^2[(\mathbf{x}_S - \mathbf{c}_{i_1,e})\mathbf{v}_{i_1,e}]^2 - a_{i_1,e}^2 + c_{i_1,e}^2$. The parameter α_e is chosen as the smallest positive real solution α to (24).

Algorithm CONVEXHULL($\epsilon_S, \mathbf{w}_S, \tilde{\mathbf{v}}_S, \tilde{\mathbf{u}}_S, a_S, b_S, \alpha_e, \phi_S$)

Input: $\epsilon_S, \mathbf{w}_S, \tilde{\mathbf{v}}_S, \tilde{\mathbf{u}}_S, a_S, b_S, \alpha_e, \phi_S$
Output: FLAG or (i_l, α_{i_l})

FLAG $\leftarrow$ FALSE
Solve $T\begin{bmatrix}\boldsymbol{\gamma} & \boldsymbol{\nu} & \boldsymbol{\mu}\end{bmatrix} = \left[\begin{bmatrix}1\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}0\\ \tilde{\mathbf{v}}_S\end{bmatrix}\begin{bmatrix}0\\ \tilde{\mathbf{u}}_S\end{bmatrix}\right]$,
  where $T = \begin{bmatrix}1 & \cdots & 1 & 1\\ \mathbf{w}_S - \mathbf{p}_{i_1} & \cdots & \mathbf{w}_S - \mathbf{p}_{i_s} & \mathbf{w}_S - \mathbf{p}_e\end{bmatrix}$
if $\epsilon_S = 0$ (Search path is a ray)
  if $(\boldsymbol{\gamma} - \alpha_e\boldsymbol{\nu}) \geq 0$ **then** FLAG $\leftarrow$ TRUE
  else
    for $\mathbf{p}_{i_j} \in S$
      if $\nu_{i_j} > 0$ **then** $\alpha_{i_j} \leftarrow \gamma_{i_j}/\nu_{i_j}$
      else $\alpha_{i_j} \leftarrow \infty$
    $\alpha_{i_l} \leftarrow \min_{\mathbf{p}_{i_j}\in S}\{\alpha_{i_j}\}$
else if $\epsilon_S > 1$ (Search path is a hyperbola)
  if $(\boldsymbol{\gamma} - a_S\sec(\alpha_e)\boldsymbol{\nu} - b_S\tan(\alpha_e)\boldsymbol{\mu}) \geq 0$ **then** FLAG $\leftarrow$ TRUE
  else
    for $\mathbf{p}_{i_j} \in S$
      $\phi_{i_j} \leftarrow$ COMPUTE$\phi(\phi_S, \gamma_{i_j}, -b_S\mu_{i_j}, a_S\nu_{i_j})$
      $\alpha_{i_j} \leftarrow \arcsin(\phi_{i_j})$
    $\alpha_{i_l} = \min_{\mathbf{p}_{i_j}\in S}\{\alpha_{i_j}\}$
else if $\epsilon_S < 1$ (Search path is an ellipse)
    if $(\boldsymbol{\gamma} - a_S\cos(\alpha_e)\boldsymbol{\nu} - b_S\sin(\alpha_e)\boldsymbol{\mu}) \geq 0$ **then** FLAG $\leftarrow$ TRUE
  else
    for $\mathbf{p}_{i_j} \in S$
      $\phi_{i_j} \leftarrow$ COMPUTE$\phi(\phi_S, a_S\nu_{i_j}, b_S\mu_{i_j}, \gamma_{i_j})$
      $\alpha_{i_j} = \arcsin(\phi_{i_j})$
    $\alpha_{i_l} = \min_{\mathbf{p}_{i_j}\in S}\{\alpha_{i_j}\}$
else if $\epsilon_S = 1$ (Search path is a parabola)
  if $(\boldsymbol{\gamma} - \tilde{c}_S\alpha_e^2\boldsymbol{\nu} - 2\tilde{c}_S\alpha_e\boldsymbol{\mu}) \geq 0$ **then** FLAG $\leftarrow$ TRUE
  else
    for $\mathbf{p}_{i_j} \in S$
      α_{i_j} is smallest real positive solution to $\tilde{c}_S\nu_{i_j}\alpha^2 + 2\tilde{c}_S\mu_{i_j}\alpha = \gamma_{i_j}$
    $\alpha_{i_l} = \min_{\mathbf{p}_{i_j}\in S}\{\alpha_{i_j}\}$
end if

The following procedure, implemented by Algorithm ConvexHull, is used to determine whether $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. Substitute $\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha \mathbf{d}_S$ for $\mathbf{x}_S$ in equation (8), expand, and simplify to obtain the linear system

$$T\boldsymbol{\pi}(\alpha) = \mathbf{e}_1 - \alpha[0, \tilde{\mathbf{v}}_S^T]^T, \tag{25}$$

where T is defined in Step 2 of Algorithm ConvexHull, with $\mathbf{w}_S = \mathbf{x}_S$, and the right-hand side of (25) is defined by $\tilde{\mathbf{v}}_S = \mathbf{d}_S$ and $\tilde{\mathbf{u}}_S = \mathbf{0}$. The matrix T is $(n+1)$-by-$(s+1)$ with rank $s+1$. The linear system (25) has a solution since $\mathbf{x}(\alpha) \in \text{aff}(S \cup \mathbf{p}_e)$ for $\alpha \geq 0$. To solve (25) for some value of α, solve $T\boldsymbol{\gamma} = \mathbf{e}_1$ for $\boldsymbol{\gamma}$, and solve $T\boldsymbol{\nu} = [0, \tilde{\mathbf{v}}_S^T]^T$ for $\boldsymbol{\nu}$. Then $\boldsymbol{\pi}(\alpha) = \boldsymbol{\gamma} - \alpha\boldsymbol{\nu}$.

If $\boldsymbol{\pi}(\alpha_e) \geq 0$, then $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. If some component of $\boldsymbol{\pi}(\alpha_e)$ is negative, then $\mathbf{x}(\alpha_e) \notin \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case, a new parameter α_{i_l} is determined so that $\alpha_S = 0 \leq \alpha_{i_l} < \alpha_e$, and $\mathbf{x}(\alpha_{i_l}) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. Since the $s+1$ points in $S \cup \{\mathbf{p}_e\}$ are affinely independent, $\text{conv}(S \cup \{\mathbf{p}_e\})$ is a simplex with $s+1$ vertices and $s+1$ facets. The $s+1$ vertices are the points in $S \cup \{\mathbf{p}_e\}$. The $s+1$ facets are denoted by $F_{i_j} = \text{conv}(S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_j}\})$, for each $\mathbf{p}_{i_j} \in S$, and by $F_e = \text{conv}(S)$, for $\mathbf{p}_e$. Each point $\mathbf{p}_{i_j} \in S$ corresponds to the component $\pi_{i_j}(\alpha)$ of $\boldsymbol{\pi}(\alpha)$ and to the facet F_{i_j}. The point $\mathbf{p}_e$ corresponds to the component $\pi_e(\alpha)$ of $\boldsymbol{\pi}(\alpha)$ and to the facet F_e.

Since $\mathbf{x}(0) \in \text{conv}(S \cup \{\mathbf{p}_e\})$, and $\mathbf{x}(\alpha_e) \notin \text{conv}(S \cup \{\mathbf{p}_e\})$, X_S must intersect some facet of $\text{conv}(S \cup \{\mathbf{p}_e\})$ between $\mathbf{x}(0)$ and $\mathbf{x}(\alpha_e)$. For each $\mathbf{p}_{i_j} \in S$, the parameter $\alpha_{i_j} \geq 0$ is computed, if it exists, so that $\mathbf{x}(\alpha_{i_j}) \in X_S \cap F_{i_j}$, which is equivalent to $\pi_{i_j}(\alpha_{i_j}) = \gamma_{i_j} - \alpha_{i_j}\delta_{i_j} = 0$. That is, for each $\mathbf{p}_{i_j} \in S$, with $\delta_{i_j} > 0$, set $\alpha_{i_j} = \gamma_{i_j}/\delta_{i_j}$. If $\delta_{i_j} \leq 0$, $\alpha_{i_j} \leftarrow \infty$. For the point $\mathbf{p}_e$, $\gamma_e = 0$, implies $\alpha_e = 0$, so $\alpha_e \leftarrow \infty$. Cavaleiro and Alizadeh (2018) present an equivalent procedure for finding α_{i_l} in their approach to the minimum covering ball problem of a set of points.

The intersection of X_S and a facet of $\text{conv}(S \cup \{\mathbf{p}_e\})$ first encountered along X_S occurs at α_{i_l}, where $\alpha_{i_l} = \min_{\mathbf{p}_{i_j} \in S}\{\alpha_{i_j}\}$. The point $\mathbf{p}_{i_l} \in S$ is chosen to leave S. Then the solution to the linear system (25) with $\mathbf{x}(\alpha_{i_l})$ substituted for $\mathbf{x}_S$, yields $\pi_{i_l}(\alpha_{i_l}) = 0$, and $\pi_{i_j}(\alpha_{i_l}) \geq 0$, for $\mathbf{p}_{i_j} \in S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$, so that $\mathbf{x}(\alpha_{i_l}) \in F_{i_l}$. This construction shows that $S \setminus \{\mathbf{p}_{i_l}\}$ is an active set for $[\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$ and that $[\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$ is dual feasible with respect to $S \cup \{\mathbf{p}_e\} \setminus \{\mathbf{p}_{i_l}\}$. However, the constraint corresponding to $\mathbf{p}_e$ is not active at $[\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$, and $\mathbf{p}_e$ is not added to S.

Compute $(\text{FLAG}, i_l, \alpha_{i_l}) \leftarrow$ ConvexHull$(\epsilon_S, \mathbf{x}_S, \mathbf{d}_S, \mathbf{0}, 0, 0, \alpha_e, 0)$, and if FLAG = TRUE, then $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case, $S \leftarrow S \cup \{\mathbf{p}_e\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$ are checked for optimality by the Update Phase. If FLAG = FALSE, delete the point $\mathbf{p}_{i_l}$ from S. Reset $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$. The Search Phase is re-entered with the active set S, the entering point $\mathbf{p}_e$, and the ball $[\mathbf{x}_S, z_S]$.

Case 2: At least two points in S have unequal radii

The points in the active set are denoted by $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, where $r_{i_1} \geq \cdots \geq r_{i_s}$. By assumption $r_{i_1} > r_{i_s}$, and by construction B_{i_1,i_s} is a bisector. The vectors and parameters of $B_S = B_{i_1,i_s} \cap_{k=2}^{s-1} H_k$ are computed using $(\mathbf{v}_S, \mathbf{c}_S, \epsilon_S, a_S, b_S, c_S) \leftarrow$ Intersections $(S, \{r_{i_j} : \mathbf{p}_{i_j} \in S\})$. If $\epsilon_S > 1$, then B_S is one branch of a hyperboloid. If $\epsilon_S < 1$, then B_S is an ellipsoid. If $\epsilon_S = 1$, then B_S is a paraboloid.

Case 2a: $\epsilon_S > 1$ Then B_S is a hyperboloid. Property 14 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S \sec(\alpha)\mathbf{v}_S + b_S \tan(\alpha)\mathbf{u} : -\pi < \alpha < \pi\}$ is one branch of a two-dimensional hyperbola with $\hat{X}_S \subset B_S \cap \text{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$. Also, $\hat{X}_S$ has the same vectors and parameters as B_S, with vertex $\mathbf{x}(0) = \mathbf{a}_S$.

The search path X_S is constructed by computing the vector $\mathbf{u}_S \leftarrow$ PROJECTION$((\mathbf{p}_e - \mathbf{c}_S), R)$ where $R = \{(\mathbf{p}_{i_1} - \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})\}$, and by restricting the domain to $\{\alpha_S \leq \alpha\}$, where $\alpha_S = \arctan\{(\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/b_S\}$. If $\mathbf{x}_S \in \text{aff}(S)$, then $\alpha_S = 0$, else $\alpha_S > 0$. Then

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S \sec(\alpha)\mathbf{v}_S + b_S \tan(\alpha)\mathbf{u}_S : \alpha \geq \alpha_S\}. \tag{26}$$

The parameter α_e is determined, if it exists, so that $\mathbf{x}(\alpha_e) \in B_{i_1,e}$ and $\alpha_S \leq \alpha_e$. Property 20 shows that $X_S \cap B_{i_1,e} = X_S \cap H_e$ for the hyperplane $H_e = \{\mathbf{x} : \mathbf{h}_e\mathbf{x} = \mathbf{h}_e\mathbf{d}_e\}$, where $(\mathbf{h}_e, \mathbf{d}_e) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_e, r_{i_1}, r_{i_s}, r_e)$. The point $\mathbf{x}(\alpha_e) \in X_S \cap H_e$ is determined by solving for α using the equation $\mathbf{h}_e\mathbf{x}(\alpha) = \mathbf{h}_e\mathbf{d}_e$, which is equivalent to the equation $a_S\mathbf{h}_e\mathbf{v}_S \sec(\alpha) + b_S\mathbf{h}_e\mathbf{u}_S \tan(\alpha) = \mathbf{h}_e(\mathbf{d}_e - \mathbf{c}_S)$. Multiplying through by $\cos(\alpha)$, which is positive for $-\pi/2 < \alpha < \pi/2$, and rearranging gives

$$A\cos(\alpha) + B\sin(\alpha) = C \tag{27}$$

where $A = \mathbf{h}_e(\mathbf{d}_e - \mathbf{c}_S)$, $B = -b_S\mathbf{h}_e\mathbf{u}_S$, and $C = a_S\mathbf{h}_e\mathbf{v}_S$. Let $\phi = \sin(\alpha)$ for $-\pi/2 < \alpha < \pi/2$. Note that $\phi_S = \sin(\alpha_S)$. The solution α_e is determined by $\phi_e \leftarrow$ COMPUTE$\phi(\phi_S, A, B, C)$. and $\alpha_e = \arcsin(\phi_e)$.

Compute $(\text{FLAG}, i_l, \alpha_{i_l}) \leftarrow$ CONVEXHULL$(\epsilon_S, \mathbf{c}_S, \mathbf{v}_S, \mathbf{u}_S, a_S, b_S, \alpha_e, \phi_S)$. If FLAG = TRUE, $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case, $S \leftarrow S \cup \{\mathbf{p}_e\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$ are checked for optimality by the Update Phase. If FLAG = FALSE, delete the point $\mathbf{p}_{i_l}$ from S. Reset $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$. The set S is active with respect to the ball $[\mathbf{x}_S, z_S]$, and $[\mathbf{x}_S, z_S]$ is dual feasible with respect to $S \cup \{\mathbf{p}_e\}$. The Search Phase is re-entered with the active set S, the entering point $\mathbf{p}_e$, and the ball $[\mathbf{x}_S, z_S]$.

Case 2b: $\epsilon_S < 1$ Then B_S is an ellipsoid. Corollary 1 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S\cos(\alpha)\mathbf{v}_S + b_S\sin(\alpha)\mathbf{u} : -\pi \leq \alpha \leq \pi\}$ is a two-dimensional ellipse with $\hat{X}_S \subset B_S \cap \text{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$, and with the same vectors and parameters as B_S.

The search path X_S is constructed by computing $\mathbf{u}_S \leftarrow$ PROJECTION$((\mathbf{p}_e - \mathbf{c}_S), R)$, where $R = \{(\mathbf{p}_{i_1} - \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})\}$, and by restricting the domain to $\{\alpha_S \leq \alpha \leq \pi\}$, where $\alpha_S = \arcsin\{(\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/b_S\}$. If $\mathbf{x}_S \in \text{aff}(S)$, then $\alpha_S = 0$, else $\alpha_S > 0$. Then the search path X_S is defined by

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + a_S\cos(\alpha)\mathbf{v}_S + b_S\sin(\alpha)\mathbf{u} : \alpha_S \leq \alpha \leq \pi\}. \tag{28}$$

The parameter α_e is determined, if it exists, so that $\mathbf{x}(\alpha_e) \in B_{i_1,e}$ and $\alpha_S \leq \alpha_e$. Property 20 shows that $X_S \cap B_{i_1,e} = X_S \cap H_e$ for the hyperplane $H_e = \{\mathbf{x} : \mathbf{h}_e\mathbf{x} = \mathbf{h}_e\mathbf{d}_e\}$, where $(\mathbf{h}_e, \mathbf{d}_e) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_e, r_{i_1}, r_{i_s}, r_e)$. The point $\mathbf{x}(\alpha_e) \in X_S \cap H_e$ is determined by solving for α using the equation $\mathbf{h}_e\mathbf{x}(\alpha) = \mathbf{h}_e\mathbf{d}_e$, which is equivalent to the equation

$$A\cos(\alpha) + B\sin(\alpha) = C, \tag{29}$$

where $A = a_S\mathbf{h}_e\mathbf{v}_S$, $B = b_S\mathbf{h}_e\mathbf{u}_S$, and $C = \mathbf{h}_e(\mathbf{d}_e - \mathbf{c}_S)$. The solution α_e is determined by $\phi_e \leftarrow$ COMPUTE$\phi(\phi_S, A, B, C)$, and $\alpha_e = \arcsin(\phi_e)$.

Compute $(\text{FLAG}, i_l, \alpha_{i_l}) \leftarrow$ CONVEXHULL$(\epsilon_S, \mathbf{c}_S, \mathbf{v}_S, \mathbf{u}_S, a_S, b_S, \alpha_e, \phi_S)$. If FLAG = TRUE, $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case, $S \leftarrow S \cup \{\mathbf{p}_e\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$ are checked for optimality by the Update Phase. If FLAG = FALSE, delete the point $\mathbf{p}_{i_l}$ from S. Reset $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$. The set S is active with respect to the ball $[\mathbf{x}_S, z_S]$, and $[\mathbf{x}_S, z_S]$ is dual feasible with respect to $S \cup \{\mathbf{p}_e\}$. The Search Phase is re-entered with the active set S, the entering point $\mathbf{p}_e$, and the ball $[\mathbf{x}_S, z_S]$.

Case 2c: $\epsilon_S = 1$ Then B_S is a paraboloid. Property 17 shows that for any vector $\mathbf{u}$ orthogonal to $\mathbf{v}_S$, $\hat{X}_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + \tilde{c}_S\alpha^2\mathbf{v}_S + 2\tilde{c}_S\alpha\mathbf{u} : -\infty < \alpha < \infty\}$ is a two-dimensional parabola with $\hat{X}_S \subset B_S \cap \text{aff}(\mathbf{c}_S, \mathbf{v}_S, \mathbf{u})$, and $\hat{X}_S$ has the same vectors and parameters as B_S. The search path X_S is constructed by computing the vector $\mathbf{u}_S \leftarrow$ PROJECTION$((\mathbf{p}_e - \mathbf{c}_S), R)$, where $R = \{(\mathbf{p}_{i_1} - \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_1} - \mathbf{p}_{i_s})\}$. Also, the domain is restricted to $\alpha_S \leq \alpha$, where $\alpha_S = 0$ if $\mathbf{x}_S = \mathbf{c}_S$; otherwise $\alpha_S = (\mathbf{u}_S(\mathbf{x}_S - \mathbf{c}_S))/(2\tilde{c}_S) \geq 0$. Then the search path is defined by

$$X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + \tilde{c}_S\alpha^2\mathbf{v}_S + 2\tilde{c}_S\alpha\mathbf{u}_S : \alpha_S \leq \alpha\}. \tag{30}$$

The parameter $\alpha_e \geq \alpha_S$ is determined, if it exists, so that $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_1,e}$. Since $\mathbf{p}_e$ is infeasible, $\mathbf{p}_e$ is non-redundant to $\mathbf{p}_{i_1}$ so that $B_{i_1,e}$ is a bisector. If $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_1,e}$, then $\mathbf{x}(\alpha_e) \in X_S \cap B_{i_j,e}$, for all $\mathbf{p}_{i_j} \in S$. Thus, it suffices to determine the intersection of X_S with only $B_{i_1,e}$.

Property 20 shows that $X_S \cap B_{i_1,e} = X_S \cap H_e$ for the hyperplane $H_e = \{\mathbf{x} : \mathbf{h}_e\mathbf{x} = \mathbf{h}_e\mathbf{d}_e\}$, where $(\mathbf{h}_e, \mathbf{d}_e) \leftarrow$ HYPERPLANE $(\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_e, r_{i_1}, r_{i_s}, r_e)$. The point $\mathbf{x}(\alpha_e) \in X_S \cap H_e$ is determined by solving the equation $\mathbf{h}_e\mathbf{x}(\alpha) = \mathbf{h}_e\mathbf{d}_e$, for α, which gives the quadratic equation

$$A\alpha^2 + B\alpha = C, \tag{31}$$

where $A = \tilde{c}_S\mathbf{h}_e\mathbf{v}_S$, $B = 2\tilde{c}_S\mathbf{h}_e\mathbf{u}_S$, and $C = \mathbf{h}_e(\mathbf{d}_e - \mathbf{c}_S)$. Then α_e is chosen as the smallest real solution such that $\alpha_S \leq \alpha_e$, which must exist since $\mathbf{p}_e$ is infeasible and $z(\alpha)$ is increasing.

Compute $(\text{FLAG}, \alpha_{i_l}) \leftarrow$ CONVEXHULL$(\epsilon_S, \mathbf{c}_S, \mathbf{v}_S, \mathbf{u}_S, a_S, b_S, \alpha_e, \phi_S)$. If FLAG = TRUE, $\mathbf{x}(\alpha_e) \in \text{conv}(S \cup \{\mathbf{p}_e\})$. In this case, $S \leftarrow S \cup \{\mathbf{p}_e\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$ are checked for optimality by the Update Phase. If FLAG = FALSE, delete the point $\mathbf{p}_{i_l}$ from S. Reset $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, and $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$. The set S is active with respect to the ball $[\mathbf{x}_S, z_S]$, and $[\mathbf{x}_S, z_S]$ is dual feasible with respect to $S \cup \{\mathbf{p}_e\}$. The Search Phase is re-entered with the active set S, the entering point $\mathbf{p}_e$, and the ball $[\mathbf{x}_S, z_S]$.

Regardless of the type of search path: a ray, hyperbola, ellipse, or parabola, if there is more than one point $\mathbf{p}_{i_l} \in S \cup \{\mathbf{p}_e\}$ such that $\min_{\mathbf{p}_{i_j} \in S}\{\alpha_{i_j}\} = \alpha_{i_l}$, then X_S intersects the corresponding facets simultaneously, and there is a tie for the leaving point. In this case choose any candidate point to be deleted from S. The next Search Phase may lead to a degenerate iteration at the next step with $\alpha_{i_l} = 0$ for some $\mathbf{p}_{i_l}$. Cycling will not occur since at each degenerate iteration one point is deleted from the finite set S. After a finite number of points are deleted, S is reduced to two points, and the parameter α will be positive at the next iteration.

Property 12 The dual algorithm solves $M(P)$ in a finite number of iterations.

Proof Given a dual feasible ball $[\mathbf{x}_S, z_S]$, an active set S, and an entering point $\mathbf{p}_e \notin S$ such that $\|\mathbf{x}_S - \mathbf{p}_e\| + r_S > z_S$, the search phase finds a step size $\alpha_e > \alpha_S$, and a set $S' \subseteq S \cup \{\mathbf{p}_e\}$ such that the ball $[\mathbf{x}_{S'}, z_{S'}] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$ is dual feasible with active set S'. The search phase may require intermediate iterations, each identifying a step size $\alpha_{i_l} > \alpha_S$ and resulting in a point $\mathbf{p}_{i_l}$ leaving the current active set $S' \subseteq S$. At most $|S| - 1$ intermediate iterations are possible before $\mathbf{x}(\alpha_{i_l}) \in \text{conv}(S' \cup \{\mathbf{p}_e\})$. Property 24 shows that $z(\alpha_e) > z(\alpha_S)$ at each iteration, and $z(\alpha_{i_l}) > z(\alpha_S)$ at each intermediate iteration. Since the objective function value is bounded above and there are only a finite number of active sets, the algorithm finds the optimal solution in a finite number of iterations. □

6 Computational results

Both algorithms were implemented in MATLAB [Release R2017a (9.20.556344)] on a MacBook Pro with a 2.7 GHz Intel Core i5 processor running macOS Catalina Version 10.15.5. The primal algorithm code is roughly 700 lines long, and the dual algorithm code is roughly

800 lines long. Both algorithms were run to solve Problem $M(P)$ for problems with balls whose centers and radii were drawn randomly from the following distributions:

- Points drawn from the uniform distribution within a hypercube centered at the origin with each side of length 20. Test problems were created for each of the dimensions $n = 50, 100, 500, 1000$ and for each of the number of points $m = n/10, n/5, n/2, n, 2n, 5n, 10n$. Once a point was sampled, a value R was drawn from the uniform distribution on the interval [0, 1]. This value of R was then multiplied by each of the radius parameters $r = 0, 1, 2, 4, 6, 8$ to generate a problem instance with m points in n dimensions, with radii rR. This generated problem instances in which the balls were growing progressively larger for the same set of points. Forty values of R were sampled, yielding a suite of 6,720 different hypercube test problems.

Algorithm DUAL($\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$, $\{r_1, \ldots, r_m\}$)

Input: $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\} \subset I\!R^n$, $\{r_i \geq 0 : \text{for } \mathbf{p}_i \in P\}$
Output: The ball $[\mathbf{x}^*, z^*]$ with minimum z^* such that $[\mathbf{p}_i, r_i] \subset [\mathbf{x}^*, z^*]$ for all $\mathbf{p}_i \in P$

1: **Initialize:** $[\mathbf{x}_S, z_S] = [\mathbf{p}_j, r_j]$ for some $\mathbf{p}_j \in P$, and $S \leftarrow \{\mathbf{p}_j\}$
2: **while** $\mathbf{x}_S$ is not primal feasible
3:     **Dual Update Phase**
4:     Select any $\mathbf{p}_e \in P \setminus S$ such that $\|\mathbf{x}_S - \mathbf{p}_e\| + r_e > z_S$
5:     **if** points in $S \cup \{\mathbf{p}_e\}$ are not affinely independent
6:         choose $\mathbf{p}_l$ to leave S using (19), (20), and (21)
7:         $S \leftarrow S \setminus \{\mathbf{p}_l\}$
8:     **Dual Search Phase**
9:     For active set $S = \{\mathbf{p}_{i_1}, \ldots \mathbf{p}_{i_s}\}$ with $r_{i_1} \geq \ldots \geq r_{i_s}$ and dual feasible ball $[\mathbf{x}_S, z_S]$
10:     **if** $r_{i_1} = r_{i_s}$
11:         $\mathbf{d}_S \leftarrow$ PROJECTION($(\mathbf{p}_e - \mathbf{x}_S)$, R), where $R = \{\mathbf{p}_{i_2} - \mathbf{p}_{i_1}, \ldots \mathbf{p}_{i_s} - \mathbf{p}_{i_1}\}$
12:         **if** $r_{i_1} = r_e$ **then** solve for α_e using (23)
13:         **else** solve for α_e using (24)
14:         (FLAG, α_{i_l}) $\leftarrow$ CONVEXHULL($\epsilon_S, \mathbf{x}_S, \mathbf{d}_S, \mathbf{0}, 0, 0, \alpha_e, 0$)
15:         **if** FLAG = TRUE **then** $S \leftarrow S \cup \{\mathbf{p}_e\}$, $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$
16:         **else** $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$, **Go To 10**
17:     **else**
18:         $(\mathbf{v}_S, \mathbf{c}_S, \epsilon_S, a_S, b_S, c_S) \leftarrow$ INTERSECTIONS(S, $\{r_{i_j} : \mathbf{p}_{i_j} \in S\}$).
16:         $\mathbf{u}_S \leftarrow$ PROJECTION($(\mathbf{p}_e - \mathbf{c}_S)$, R), where $R = \{\mathbf{p}_{i_1} - \mathbf{p}_{i_2}, \ldots, \mathbf{p}_{i_1} - \mathbf{p}_{i_s}\}$
17:         $(\mathbf{h}_e, \mathbf{d}_e) \leftarrow$ HYPERPLANE($\mathbf{p}_{i_1}, \mathbf{p}_{i_s}, \mathbf{p}_e, r_{i_1}, r_{i_s}, r_e$)
18:         **if** $\epsilon_S > 1$ **then** $\alpha_e \leftarrow \arcsin$(COMPUTE$\phi$($\phi_S, A, B, C$)) using (27)
19:         **else if** $\epsilon_S < 1$ **then** $\alpha_e \leftarrow \arcsin$(COMPUTE$\phi$($\phi_S, A, B, C$)) using (29)
20:         **else if** $\epsilon_S = 1$ **then** solve for α_e using (31)
21:         (FLAG, α_{i_l}) $\leftarrow$ CONVEXHULL($\epsilon_S, \mathbf{c}_S, \mathbf{v}_S, \mathbf{u}_S, a_S, b_S, \alpha_e, \phi_S$)
22:         **if** FLAG = TRUE **then** $S \leftarrow S \cup \{\mathbf{p}_e\}$, $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_e), z(\alpha_e)]$
23:         **else** $S \leftarrow S \setminus \{\mathbf{p}_{i_l}\}$, $[\mathbf{x}_S, z_S] \leftarrow [\mathbf{x}(\alpha_{i_l}), z(\alpha_{i_l})]$, **Go To 10**
24: **end while**

- In a similar manner, points drawn uniformly on the surface of a hypersphere of radius 10 centered at the origin. Test problems were created for each of the dimensions $n = 50, 100, 500, 1000$; for each of the number of points $m = n/10, n/5, n/2, n, 2n, 4n$; and for each of the radius parameters $r = 1, 1/2, 1/4, 1/16$. Again, R was drawn from the uniform distribution on the interval [0, 1], and forty values of R were sampled, yielding a suite of 3,840 different hypersphere test problems. Geometrically, the test problems resembled a bumpy sphere.

All test problems were solved to optimality by both the primal and dual algorithms. For the primal algorithm, the center of the initial covering ball was selected to be the arithmetic mean of the set of points P. For the dual algorithm, the initial set S was the ball whose center plus corresponding radius was farthest from the origin. For the dual algorithm, the entering point was selected to be the center of the ball that was most infeasible. To avoid redundancy, if there was a tie between two balls for being most infeasible, the ball with largest radius was selected to enter S.

A numerical tolerance was set to $8n\epsilon_{mach}/(1-8n\epsilon_{mach})$ for the dual algorithm and $100n\epsilon_{mach}/(1-100n\epsilon_{mach})$ for the primal algorithm. This quantity was used to detect primal feasibility, dual feasibility, eccentricity, etc. In solving all of the test problems, neither the primal nor the dual algorithms ever encountered an elliptic or a parabolic search path. Small test problems were constructed that did require elliptic or parabolic search paths, and were easily solved. These examples required careful placements of points and assignments of radius values.

For the primal algorithm, the computational complexity for each iteration is as follows:

1. For the Primal Search Phase, the most expensive operation is the calculation of the vectors $\mathbf{hp}_k$, $k = 2, \ldots, (|S|-1)$. This was done using the Gram-Schmidt algorithm, which has complexity of $O(2n|S|^2)$.
2. For the calculation of the parameter α^*, each point $\mathbf{p}_k$ in $P \setminus S$ must be examined and the associated parameter α_k must be calculated. This has a complexity of $O(53mn)$.
3. In the Primal Update Phase, the algorithm tests whether the KKT conditions are satisfied. This involves using least squares to calculate the solution to an n-by-$|S|$ system of linear equations, which has a complexity of $O(4n|S|^2 - 4|S|^3/3)$.

In conclusion, each iteration of the primal algorithm has complexity $O(6n|S|^2 - 4|S|^3/3 + 53mn)$.

For the dual algorithm, the computational complexity for each iteration is as follows:

1. In the Dual Update Phase, it first must be determined whether the points in $S \cup \{\mathbf{p}_e\}$ are affinely independent, which has complexity $O(4n|S|^2 - 4|S|^3/3)$. If they are affinely dependent, then solutions must be calculated for equations (7), (8), and (9), and for equations (19) and (20). These computations have complexity $O(2n|S|^2 - 2|S|^3/3)$ and $O(6n|S|^2 - 2|S|^3)$, respectively.
2. In the Dual Search Phase, vectors $\mathbf{hp}_k$, $k = 2, \ldots, (|S|-1)$ were calculated using the Gram-Schmidt algorithm, which has complexity of $O(2n|S|^2)$, and the calculation of the vector $\mathbf{u}_S$ has complexity $O(2n|S|^2 - 2|S|^3/3)$. Finally, the calculation of the parameter α^* requires $O(2n|S|^2 - 2|S|^3/3)$ operations.
3. Calculating whether the new dual feasible solution is also primal feasible requires $O(3mn)$ operations.

In conclusion, each iteration of the dual algorithm has complexity $O(18n|S|^2 - 4|S|^3 + 3mn)$.

Observations

1. Table 1 shows that for the Hypercube data, the average number of iterations for both the primal and dual closely tracts the average value of $|S|$ at optimality. This means that a point is added to S in most iterations, and there are very few iterations in which a point leaves S.

Table 1 Comparison of primal and dual algorithms for the hypercube dataset

(n, m)	Avg $\|S\|$ at	Avg Time(sec)		Avg Iterations	
	Optimality	Primal	Dual	Primal	Dual
(50, 5)	4.96	0.00103	0.00270	4.00	4.96
(50, 10)	8.40	0.00449	0.01154	8.27	8.40
(50, 25)	13.34	0.01089	0.01975	13.34	13.48
(50, 50)	16.50	0.01380	0.01673	16.55	16.95
(50, 100)	19.00	0.01912	0.01907	19.25	20.16
(50, 250)	21.33	0.03304	0.02062	21.88	23.28
(50, 500)	23.35	0.05858	0.02370	24.26	26.18
(100, 10)	9.49	0.00758	0.00998	8.89	9.49
(100, 20)	15.13	0.01236	0.01339	15.13	15.18
(100, 50)	22.42	0.01887	0.01817	22.42	22.63
(100, 100)	27.10	0.02880	0.02141	27.15	27.90
(100, 200)	30.03	0.04654	0.02412	30.19	31.60
(100, 500)	34.79	0.10227	0.03200	35.08	37.18
(100, 1000)	37.55	0.20142	0.04033	38.25	41.28
(500, 50)	37.07	0.04189	0.04178	37.07	37.11
(500, 100)	50.74	0.09324	0.07858	50.74	51.03
(500, 250)	65.46	0.23296	0.15783	65.47	66.41
(500, 500)	76.63	0.48139	0.19151	76.65	77.83
(500, 1000)	85.73	0.98211	0.27754	85.77	88.33
(500, 2500)	99.35	3.09055	0.49999	99.49	102.90
(500, 5000)	105.55	6.40139	0.73559	105.76	109.85
(1000, 100)	63.73	0.25193	0.18453	63.73	63.84
(1000, 200)	82.77	0.52855	0.51811	82.77	83.13
(1000, 500)	103.45	1.37670	1.01992	103.45	104.27
(1000, 1000)	120.10	3.07464	1.49074	120.10	121.74
(1000, 2000)	132.51	6.29294	1.92496	132.52	134.87
(1000, 5000)	150.80	17.25200	3.34911	150.85	154.46
(1000, 10,000)	162.98	36.36052	5.52762	163.05	167.53

2. Tables 1 and 2 show that the dual algorithm is faster than the primal algorithm over almost all problem classes (n, m). This can be explained by the computational complexity of the algorithms because at each iteration, the primal algorithm requires an operation for each point in $P \setminus S$, while the dual algorithm requires an operation for each point in S. Since $|S|$ is small initially and increases by at most one at each iteration, mn dominates $|S|$ and contributes more to the computational effort.
3. Except in the Hypersphere test cases in which $(n, m) = (50, 200)$ and $(n, m) = (100, 400)$, the algorithms use a similar number of iterations to solve the test problems to optimality.

Areas of continued research for improving the computational efficiency of the algorithms include the following:

Table 2 Comparison of primal and dual algorithms for the hypersphere dataset

(n, m)	Avg $\|S\|$ at	Avg Time(sec)		Avg Iterations	
	Optimality	Primal	Dual	Primal	Dual
(50, 5)	5.00	0.00200	0.00157	4.00	5.00
(50, 10)	9.83	0.00650	0.00418	8.98	9.84
(50, 25)	19.79	0.01727	0.01273	19.79	20.68
(50, 50)	28.61	0.02161	0.01417	29.58	31.68
(50, 100)	35.84	0.04910	0.02501	52.78	48.05
(50, 200)	39.73	0.18351	0.03760	131.22	64.72
(100, 10)	9.94	0.00757	0.00535	9.00	9.94
(100, 20)	18.60	0.01488	0.01224	18.22	18.60
(100, 50)	35.52	0.02954	0.02223	35.58	36.58
(100, 100)	49.59	0.06062	0.04161	51.15	55.03
(100, 200)	61.05	0.16333	0.07442	83.36	77.89
(100, 400)	70.36	0.78824	0.12047	210.79	104.66
(500, 50)	42.88	0.05627	0.05854	42.57	42.90
(500, 100)	68.59	0.16850	0.15592	68.59	68.93
(500, 250)	114.22	0.67362	0.55144	114.29	117.54
(500, 500)	152.79	1.58460	1.18185	155.44	163.84
(500, 1000)	195.76	4.31656	2.54152	209.83	219.93
(500, 2000)	239.21	12.80964	5.46031	294.93	288.14
(1000, 100)	72.08	0.37321	0.30619	71.93	72.19
(1000, 200)	111.60	1.19929	1.14259	111.60	112.10
(1000, 500)	176.94	4.74913	4.67487	177.04	180.97
(1000, 1000)	235.04	11.62543	10.56394	236.73	246.53
(1000, 2000)	304.95	29.58509	21.92983	315.08	330.30
(1000, 4000)	386.62	85.54698	44.87798	431.09	439.17

- Use column update methods to solve sequential linear systems that differ by only one column.
- Investigate alternative rules for starting solutions and for choosing candidate points to enter an active set.
- For the primal algorithm, investigate optimizing sequentially over subsets of a partition of P.
- For the primal algorithm, investigate whether there are structures that allow points covered by a feasible ball to remain covered by subsequent iterations.
- Investigate a primal-dual approach using the bisector search paths.
- Investigate the use of search paths generated by bisectors to other optimization problems.

Appendix: Results on conic sections and applications to the primal and dual algorithms

Appendix A.1–A.4 present properties on the intersection of hyperplanes with conic sections in $\mathbb{R}^n$, and on the intersection of a sequence of hyperboloids with a common focal point. The proofs are in Dearing (2017). These properties provide the basis for expressions in algorithm INTERSECTIONS used to compute the vectors and parameters of two-dimensional conic sections that define the search paths of the primal and dual algorithms. Appendix A.5 presents properties used to prove that the primal and dual algorithms compute the optimal solution in a finite number for iterations.

A.1 Hyperboloids and ellipsoids

An n-dimensional hyperboloid of two sheets, symmetric about its major axis, is the set

$$H = \{\mathbf{x} \in I\!R^n : |\|\mathbf{x} - \mathbf{p}_1\| - \|\mathbf{x} - \mathbf{p}_2\|| = 2a\}, \tag{32}$$

where $\mathbf{p}_1, \mathbf{p}_2 \in I\!R^n$ are the focal points, and a is a positive constant such that $2a < \|\mathbf{p}_1 - \mathbf{p}_1\|$. H is specified by the following vectors and parameters, all of which are determined by the focal points and the constant a. The center $\mathbf{c} = (\mathbf{p}_1 + \mathbf{p}_2)/2$ is the midpoint of the focal points. The axis vector $\mathbf{v} = (\mathbf{p}_1 - \mathbf{p}_2)/\|\mathbf{p}_1 - \mathbf{p}_2\|$ orients the hyperboloid and is parallel to the major axis, which is the line through $\mathbf{p}_1$ and $\mathbf{p}_2$. The focal distance $c = \|\mathbf{p}_1 - \mathbf{p}_2\|/2$ is the distance from the center to each to the focal points. The eccentricity $\epsilon = c/a$ specifies the shape of the hyperboloid.

The sheet of H closest to $\mathbf{p}_1$ is the set H_1, and the sheet closest to $\mathbf{p}_2$ is the set H_2, where

$$H_1 = \{\mathbf{x} : \|\mathbf{p}_2 - \mathbf{x}\| - \|\mathbf{p}_1 - \mathbf{x}\| = 2a\} \qquad = \{\mathbf{x} : \|\mathbf{p}_1 - \mathbf{x}\| = \epsilon\mathbf{v}(\mathbf{x} - \mathbf{d}_1)\} \tag{33}$$

$$H_2 = \{\mathbf{x} : \|\mathbf{p}_2 - \mathbf{x}\| - \|\mathbf{p}_1 - \mathbf{x}\| = -2a\} \qquad = \{\mathbf{x} : \|\mathbf{p}_2 - \mathbf{x}\| = \epsilon\mathbf{v}(\mathbf{d}_2 - \mathbf{x})\}. \tag{34}$$

The directrix of H_1 is $\{\mathbf{x} : \mathbf{v}\mathbf{x} = \mathbf{v}\mathbf{d}_1\}$, with directrix vector $\mathbf{d}_1 = \mathbf{c} + d\mathbf{v}$ and $d = a^2/c$. The directrix of H_2 is $\{\mathbf{x} : \mathbf{v}\mathbf{x} = \mathbf{v}\mathbf{d}_2\}$, with directrix vector $\mathbf{d}_2 = \mathbf{c} - d\mathbf{v}$. The vertex $\mathbf{a}_1 = \mathbf{c} + a\mathbf{v}$ of H_1 is the point of intersection between the major axis and H_1. The vertex of H_2 is $\mathbf{a}_2 = \mathbf{c} - a\mathbf{v}$. Observe that $H = H_1 \cup H_2$.

The triangle inequality implies $2a = |\|\mathbf{p}_2 - \mathbf{x}\| - \|\mathbf{p}_1 - \mathbf{x}\|| \leq \|\mathbf{p}_2 - \mathbf{p}_1\| = 2c$, so that $a \leq c$ and $\epsilon \geq 1$. By definition, H is a hyperboloid if $a < c$, or $\epsilon > 1$. If $a = c$, so that $\epsilon = 1$, H is a degenerate hyperboloid where $H_1 = \{\mathbf{x} = \mathbf{p}_1 + \alpha\mathbf{v}, \alpha \geq 0\}$, the ray along the major axis from $\mathbf{p}_1$ in the direction $\mathbf{v}$, and $H_2 = \{\mathbf{x} = \mathbf{p}_2 - \alpha\mathbf{v}, \alpha \geq 0\}$, the ray from $\mathbf{p}_2$ in the direction $-\mathbf{v}$. If $a > c$, $H = \emptyset$.

An n-dimensional ellipsoid, symmetric about its major axis, is the set of points $\mathbf{x} \in I\!R^n$ such that the sum of the distances from $\mathbf{x}$ to two given points $\mathbf{p}_1$ and $\mathbf{p}_2$ equals a positive constant $2a$. That is,

$$E = \{\mathbf{x} : \| \mathbf{p}_2 - \mathbf{x} \| + \| \mathbf{p}_1 - \mathbf{x} \| = 2a\}. \tag{35}$$

An ellipsoid E is specified by the same vectors and parameters that specify a hyperboloid, all of which are determined by the focal points $\mathbf{p}_1$ and $\mathbf{p}_2$ and the positive constant a.

The triangle inequality implies $2c = \| \mathbf{p}_1 - \mathbf{p}_2 \| \leq \| \mathbf{p}_1 - \mathbf{x} \| + \| \mathbf{p}_2 - \mathbf{x} \| = 2a$, so that $c \leq a$. If $a = c$, E is the line segment between $\mathbf{p}_1$ and $\mathbf{p}_2$, and is a degenerate ellipsoid. The application to finding a minimum covering ball is concerned only with the case $c < a$, or $\epsilon < 1$.

The next property gives an equivalent representation of a hyperboloid or an ellipsoid as a quadratic form. A version of this representation for hyperboloids in $I\!R^3$ is reported in Leva (1996). The proof for hyperboloids (or ellipsoids) expands expressions (32), (33), and (34) (or (35) for ellipsoids) and applies the definition of related vectors and parameters to obtain (36).

Property 13 Given the focal points $\mathbf{p}_1$ and $\mathbf{p}_2$ and a positive constant a, with corresponding axis vector $\mathbf{v}$, center point $\mathbf{c}$, and eccentricity $\epsilon = c/a$, let H be the hyperboloid determined by these vectors and parameters if $c > a$, and let E be the ellipsoid determined by these vectors and parameters if $c < a$. Define the set Q by

$$Q = \{\mathbf{x} : (\mathbf{x}-\mathbf{c})^T[I-\epsilon^2\mathbf{v}\mathbf{v}^T](\mathbf{x}-\mathbf{c}) = a^2-c^2\} = \{\mathbf{x} : (\mathbf{x}-\mathbf{c})^2-\epsilon^2(\mathbf{v}(\mathbf{x}-\mathbf{c}))^2 = a^2-c^2\}. \tag{36}$$

Then $Q = H$ if $c > a$, and $Q = E$ if $c < a$.

Property 14 Suppose H is a hyperboloid in $I\!R^n$ with focal points $\mathbf{p}_1$ and $\mathbf{p}_2$, center $\mathbf{c}$, axis vector $\mathbf{v}$, eccentricity ϵ, and sheets H_1 and H_2. If $\mathbf{u}$ is a unit vector orthogonal to the axis vector $\mathbf{v}$, then $H_1 \cap \text{aff}(\mathbf{v}, \mathbf{u}, \mathbf{c}) = \{\mathbf{x}_1(\alpha) = \mathbf{c} + a\sec(\alpha)\mathbf{v} + b\tan(\alpha)\mathbf{u}, \frac{-\pi}{2} < \alpha < \frac{\pi}{2}\}$, where $b = \sqrt{c^2 - a^2}$, is one branch of a two-dimensional hyperbola with the same vectors and parameters as H_1. Furthermore, $H_2 \cap \text{aff}(\mathbf{v}, \mathbf{u}, \mathbf{c}) = \{\mathbf{x}_2(\alpha) = \mathbf{c} - a\sec(\alpha)\mathbf{v} + b\tan(\alpha)\mathbf{u}, \frac{-\pi}{2} < \alpha < \frac{\pi}{2}\}$ is one branch of a two-dimensional hyperbola with the same vectors and parameters as H_2.

Corollary 1 *Suppose E is an ellipsoid in $I\!R^n$ with focal points $\mathbf{p}_1$ and $\mathbf{p}_2$, center $\mathbf{c}$, axis vector $\mathbf{v}$, eccentricity ϵ. If $\mathbf{u}$ is a unit vector orthogonal to the axis vector $\mathbf{v}$, then $E \cap \text{aff}(\mathbf{v}, \mathbf{u}, \mathbf{c}) = \{\mathbf{x}(\alpha) = \mathbf{c} + a\cos(\alpha)\mathbf{v} + b\sin(\alpha)\mathbf{u}, -\pi < \alpha < \pi\}$, where $b = \sqrt{a^2 - c^2}$, is a two-dimensional ellipse with the same vectors and parameters as E.*

An n-dimensional right circular cone C, with center $\mathbf{c}$, axis vector $\mathbf{v}$, and eccentricity ϵ, is also expressed in terms of the quadratic form Q with $\epsilon > 1$, but with a right hand side value of zero. That is,

$$C = \{\mathbf{x} : (\mathbf{x} - \mathbf{c})^T[I - \epsilon^2\mathbf{v}\mathbf{v}^T](\mathbf{x} - \mathbf{c}) = 0\}. \tag{37}$$

The cone C has two "sheets": $C_1 = \{\mathbf{x} : \| \mathbf{x} - \mathbf{c} \| = \epsilon\mathbf{v}(\mathbf{x} - \mathbf{c})\}$ is the subset of C that is closest to the focal point $\mathbf{p}_1$, and $C_2 = \{\mathbf{x} : \| \mathbf{x} - \mathbf{c} \| = \epsilon\mathbf{v}(\mathbf{c} - \mathbf{x})\}$ is the subset of C closest to the focal point $\mathbf{p}_2$. Observe that $C = C_1 \cup C_2$.

For any point $\mathbf{x} \in C$, let γ be the angle between the vector $\mathbf{x}-\mathbf{c}$ and the axis vector $\mathbf{v}$. Then the expression for C_1 shows that $\mathbf{v}(\mathbf{x} - \mathbf{c})/\|\mathbf{x} - \mathbf{c}\| = \cos(\gamma) = 1/\epsilon$, or $\sec(\gamma) = \epsilon = c/a$. The next Property and its proof are analogous to Property 14.

Property 15 Given a cone C in $I\!R^n$ with center $\mathbf{c}$, axis vector $\mathbf{v}$, and sheets C_1 and C_2, if $\mathbf{u}$ is a unit vector orthogonal to $\mathbf{v}$, then $C_1 \cap \text{aff}(\mathbf{u}, \mathbf{v}, \mathbf{c}) = \{\bar{\mathbf{x}}_1(\beta) = \mathbf{c} + a|\beta|\mathbf{v} + b\beta\mathbf{u}, -\infty < \beta < \infty\}$, where $b = \sqrt{c^2 - a^2}$, is one branch of a two-dimensional cone with the same center and axis vector as C. Furthermore, $C_2 \cap \text{aff}(\mathbf{u}, \mathbf{v}, \mathbf{c}) = \{\bar{\mathbf{x}}_2(\beta) = \mathbf{c} - a|\beta|\mathbf{v} + b\beta\mathbf{u}, -\infty < \beta < \infty\}$ is one branch of a two-dimensional cone with the same vectors and parameters as C.

Property 16 If a cone C and a hyperboloid H have the same axis vector $\mathbf{v}$, center $\mathbf{c}$, and eccentricity ϵ, then C is the asymptotic approximation of H.

A.2 Paraboloids

An n-dimensional paraboloid, symmetric about its major axis, is the set P of all points $\mathbf{x} \in I\!R^n$ such that the distance from $\mathbf{x}$ to a given point $\mathbf{p}_1$ on the major axis, equals the distance from $\mathbf{x}$ to the hyperplane that is orthogonal to the major axis and contains a point $\mathbf{p}_2$ on the major axis. A paraboloid is specified by the two points $\mathbf{p}_1$ and $\mathbf{p}_2$ only. The point $\mathbf{p}_1$ is the focal point of the paraboloid. The major axis is the line through the points $\mathbf{p}_1$ and $\mathbf{p}_2$. The axis vector $\mathbf{v} = (\mathbf{p}_1 - \mathbf{p}_2)/\|\mathbf{p}_1 - \mathbf{p}_2\|$ is the unit vector parallel to the major axis. A paraboloid P, defined by $\mathbf{p}_1, \mathbf{p}_2 \in I\!R^n$, is the set

$$P = \{\mathbf{x} : \|\mathbf{p}_1 - \mathbf{x}\| = \mathbf{v}(\mathbf{x} - \mathbf{p}_2)\}. \tag{38}$$

The vertex of a paraboloid is the center $\mathbf{c} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, and is the intersection of the paraboloid with the major axis. The focal distance $c = \|\mathbf{p}_1 - \mathbf{p}_2\|/2$. The directrix of a paraboloid P is the hyperplane with normal vector $\mathbf{v}$ containing the point $\mathbf{p}_2$. Observe that $\mathbf{c} - \mathbf{p}_2 = c\mathbf{v}$, and $\mathbf{c} - \mathbf{p}_1 = -c\mathbf{v}$. All paraboloids have the same shape, so there is no parameter ϵ or a. The next property characterizes a two-dimensional parabola that is a subset of a given paraboloid P with the same parameters and vectors.

Property 17 Given a paraboloid P in $\mathbb{R}^n$, with focal points $\mathbf{p}_1$ and $\mathbf{p}_2$, if $\mathbf{u}$ is a unit vector orthogonal to the axis vector $\mathbf{v}$, then $P \cap \text{aff}(\mathbf{v}, \mathbf{c}, \mathbf{u}) = \{\mathbf{x}(\alpha) = \mathbf{c} + c\alpha^2\mathbf{v} + 2c\alpha\mathbf{u}, -\infty < \alpha < \infty\}$ is a two-dimensional parabola that is a subset of P and has the same vectors and parameters as P.

Paraboloids may also be expressed in terms of a quadratic form similar to (36). However, for a paraboloid, there is no eccentricity, and the right hand side is a linear expression of $\mathbf{x}$.

Property 18 The paraboloid P has the equivalent expression

$$P = \{\mathbf{x} : (\mathbf{x} - \mathbf{c})^T [I - \mathbf{v}\mathbf{v}^T](\mathbf{x} - \mathbf{c}) = 4c\mathbf{v}(\mathbf{x} - \mathbf{c})\}. \tag{39}$$

A.3 Intersections of hyperplanes with conic sections in $\mathbb{R}^n$

From the classical studies of conic sections in $\mathbb{R}^3$, it is well known that if a plane and a cone intersect at an appropriate angle, measured between the axis vector of the cone and the normal vector of the plane, the intersection is either a two-dimensional hyperbola, ellipse, or parabola. These results extend to the intersection of a hyperplane with each conic section in $\mathbb{R}^n$, and are reported below. For the intersection of a hyperplane with a hyperboloid, or a cone, conditions are given for identifying the resulting intersection as a hyperboloid, an ellipsoid or a paraboloid of dimension $n-1$. For the intersection of a hyperplane and an ellipsoid, the resulting intersection is always an ellipsoid of dimension $n-1$. For the intersection of a hyperplane and paraboloid, conditions are given for identifying the resulting intersection as a paraboloid or an ellipsoid of dimension $n-1$. In each case, expressions are given for computing the vectors and parameters of the resulting intersection. These expressions are used in algorithm INTERSECTIONS to compute the search paths for the primal and dual algorithms.

Property 19 Suppose $Q = \{\mathbf{x} : (\mathbf{x} - \mathbf{c})^T [I - \epsilon^2\mathbf{v}\mathbf{v}^T](\mathbf{x} - \mathbf{c}) = a^2 - c^2\}$ is a hyperboloid in $\mathbb{R}^n$, centered at $\mathbf{c} = (c_1, \ldots, c_n)^T$, with axis vector $\mathbf{v}$ of unit length, eccentricity $\epsilon > 1$, and parameters a and c, and suppose $HP = \{\mathbf{x} : \mathbf{h}\mathbf{x} = \mathbf{h}(\mathbf{c} + \hat{h}\mathbf{h})\}$ is a hyperplane with $\|\mathbf{h}\| = 1$. Let $\rho = \sqrt{1 - (\mathbf{h}\mathbf{v})^2}$. Then $Q \cap HP$ is a hyperboloid of dimension $n-1$ iff $\epsilon\rho > 1$, or an ellipsoid of dimension $n-1$ iff $\epsilon\rho < 1$ and $\hat{h}^2 \geq a^2(1 - \rho^2\epsilon^2)$, or a paraboloid of dimension $n-1$ iff $\epsilon\rho = 1$.

A.4 Applications to problem $M(P)$

Problem $M(P)$ assumes a given a set $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\} \subset \mathbb{R}^n$, of distinct points and the set of balls $\{[\mathbf{p}_i, r_i], \mathbf{p}_i \in P\}$, where r_i is a non-negative radius corresponding to each $\mathbf{p}_i \in P$. The bisector $B_{j,k}$ of points $\mathbf{p}_j, \mathbf{p}_k \in P$ is defined by (1). If $\mathbf{p}_j, \mathbf{p}_k \in P$ are non-redundant and $r_j > r_k$, then the bisector $B_{j,k}$ is one sheet of a hyperboloid in $\mathbb{R}^n$.

For problem $M(P)$, the next result characterizes the intersection of three bisectors corresponding to a set of three non-redundant and affinely independent points in P. Using this result, the intersection of any two bisectors may be determined by the intersection of either bisectors in the pair with the hyperplane H_T. This result leads to a procedure for finding the intersection of bisectors determined by all pairs of points in a subset of P.

Property 20 extends a result in Leva (1996) that assumes only two hyperboloids with a common focal point.

Property 20 Suppose that $T = \{\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l\}$ is a subset of three affinely independent and non-redundant points from P, ordered so that $r_j \geq r_k \geq r_l$, with $r_j > r_l$, and suppose $\mathcal{B} = \{B_{j,k}, B_{j,l}, B_{k,l}\}$ is the set of bisectors defined by each pair of points in T. Let $H_T = \{\mathbf{h}_T\mathbf{x} = \mathbf{h}_T\mathbf{d}_T\}$, where $\mathbf{h}_T = \epsilon_{j,k}\mathbf{v}_{j,k} - \epsilon_{j,l}\mathbf{v}_{j,l}$, $\mathbf{d}_T = \mathbf{d}_{j,l} + \frac{(\mathbf{v}_{j,k}(\mathbf{d}_{j,k}-\mathbf{d}_{j,k})}{\mathbf{v}_{j,k}\mathbf{u}_T}\mathbf{u}_T$, and $\mathbf{u}_T \leftarrow$ PROJECTION $(\mathbf{h}_T, \{\mathbf{v}_{j,l}\})$. Then the intersection of each pair of bisectors in $\mathcal{B}$ equals the intersection of H_T with either bisector of the pair. If any of the bisectors in $\mathcal{B}$ is a hyperplane, it is identical to H_T.

Given an active set $S = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s}\}$, the following results show how to compute the vectors and parameters of $B_S = \cap_{1\leq j<k\leq s} B_{i_j,i_k}$. If all the radii of the points in S are equal, each bisector B_{i_j,i_k} is a hyperplane, so that B_S is a linear manifold of dimension $n - s + 1$. If some of the radii are unequal, B_S will be a conic section of dimension $n - s + 2$.

The next two properties give alternative representations for B_S.

Property 21 $B_S = \cap_{j=2}^{s} B_{i_1,i_j}$.

Property 22 $B_S = B_{i_1,i_s} \cap_{j=2}^{s-1} H_j$, where for $j = 2, \ldots, s-1$, $H_j = \{\mathbf{h}_j\mathbf{x} = \mathbf{h}_j\mathbf{d}_j\}$, is determined by the set $T_j = \{\mathbf{p}_{i_1}, \mathbf{p}_{i_j}, \mathbf{p}_{i_s}\}$, with $r_{i_1} \geq r_{i_j} \geq r_{i_s}$, and $r_{i_1} > r_{i_s}$.

A.5 Properties of search paths

Property 23 At each iteration of either the primal or dual algorithm with active set S, and for either a linear, hyperbolic, elliptic, or parabolic search path $X_S = \{\mathbf{x}(\alpha) : \alpha \in D\}$, S is active for the ball $[\mathbf{x}(\alpha), z(\alpha)]$, for $\alpha \in D$. Furthermore, the complementary slackness conditions (6) of the KKT conditions are satisfied by S and the ball $[\mathbf{x}(\alpha), z(\alpha)]$, for $\alpha \in D$, where $z(\alpha) = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| + r_{i_1}$.

Proof For either the primal or dual algorithm, consider a search path that is the ray $X_S = \{\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha\mathbf{d}_S, \alpha \in D\}$. By construction, $\alpha_S \in D$, and $\mathbf{x}_S = \mathbf{x}(\alpha_S) \in X_S \subset B_S = \cap_{\mathbf{p}_j,\mathbf{p}_k\in S} B_{j,k}$. Then for $\alpha \in D$, and each pair of points $\{\mathbf{p}_{i_j}, \mathbf{p}_{i_k}\} \subset S$, $B_{i_j.i_k} = \{\mathbf{x} : (\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{x} = (\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{x}_S\}$, and $(\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{x}(\alpha) = (\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{x}_S + \alpha(\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{d}_S = (\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{x}_S$ since by construction $(\mathbf{p}_{i_j} - \mathbf{p}_{i_k})\mathbf{d}_S = 0$. Thus for each $\{\mathbf{p}_{i_j}, \mathbf{p}_{i_k}\} \subset S$, $\mathbf{x}(\alpha) \in B_{i_j,i_k}$. That is, $\|\mathbf{x}(\alpha) - \mathbf{p}_{i_j}\| + r_{i_j} = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_k}\| + r_{i_k}$ which implies S is active for $[\mathbf{x}(\alpha), z(\alpha)]$.

For either the primal or dual algorithm consider a search path X_S that is either a hyperbola, specified by (13), an ellipse specified by (15), or a parabola specified by (17). In each case, for $\alpha \in D$, substitution of $\mathbf{x}(\alpha)$ for $\mathbf{x}_S$ satisfies the expression (36) for B_S which shows that $X_S \subset B_S$. Thus, for each $\{\mathbf{p}_{i_j}, \mathbf{p}_{i_k}\} \subset S$, $\mathbf{x}(\alpha) \in B_{i_1,i_j}$ which implies $\|\mathbf{x}(\alpha) - \mathbf{p}_{i_j}\| + r_{i_j} = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_k}\| + r_{i_k}$ so that S is active for $[\mathbf{x}(\alpha), z(\alpha)]$ for $\alpha \in D$.

Since the points in S are affinely independent, there exists a solution π_{i_j} for $\mathbf{p}_{i_j} \in S$ to the linear system (7) and (8) with $\mathbf{x}(\alpha)$ replacing $\mathbf{x}$. This result along with the result that S is active for $[\mathbf{x}(\alpha), z(\alpha)]$ for all cases of the search path and for $\alpha \in D$, shows that $(z(\alpha) - \|\mathbf{x}(\alpha) - \mathbf{p}_{i_j}\| - r_{i_j})\pi_{i_j} = 0$ for $\mathbf{p}_{i_j} \in S$. For $\mathbf{p}_i \in P \setminus S$, choose $\pi_i = 0$ so that $(z(\alpha) - \|\mathbf{x}(\alpha) - \mathbf{p}_i\| - r_i)\pi_i = 0$. Thus, the complementary slackness conditions (6) of the KKT are satisfied at each point on the search path. □

Property 24 At each iteration of either the primal or dual algorithm with active set S, and a search path $X_S = \{\mathbf{x}(\alpha) : \alpha \in D\}$ that is either a ray, hyperbola, ellipse, or parabola, the objective function $z(\alpha)$ is decreasing for the primal algorithm and increasing for the dual algorithm.

Proof For either the primal or dual algorithm the linear search path is the ray $X_S = \{\mathbf{x}(\alpha) = \mathbf{x}_S + \alpha \mathbf{d}_S, \alpha \in D\}$. The objective function is $z(\alpha) = \|\mathbf{x}_S + \alpha \mathbf{d}_S - \mathbf{p}_{i_1}\| + r_{i_1}$ and $z'(\alpha) = [(\mathbf{x}_S - \mathbf{p}_{i_1})\mathbf{d}_S + \alpha \mathbf{d}_S^2]/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\|$.

For the primal algorithm $D = \{0 \le \alpha \le \alpha_m\}$, $\alpha_m = (\mathbf{p}_{i_1} - \mathbf{x}_S)\mathbf{d}_S/\|\mathbf{d}_S\|^2 > 0$, and $\mathbf{d}_S \leftarrow$ PROJECTION$((\mathbf{p}_{i_1} - \mathbf{x}_S), R)$, with $R = \{\mathbf{p}_{i_2} - \mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_s} - \mathbf{p}_{i_1}\}$. By construction $(\mathbf{x}_S - \mathbf{p}_{i_1})\mathbf{d}_S < 0$, so that $z'(0) < 0$, for $0 \le \alpha < \alpha_m$, $z'(\alpha) = 0$, and $z'(\alpha_m) = 0$,

For the dual algorithm $D = \{0 \le \alpha\}$, and $\mathbf{d}_S \leftarrow$ PROJECTION$((\mathbf{p}_e - \mathbf{x}_S), R)$. Then $z'(\alpha) = \alpha \mathbf{d}_S^2]/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\|$, since $(\mathbf{x}_S - \mathbf{p}_{i_1})\mathbf{d}_S = 0$. Thus for $0 < \alpha$, $z'(\alpha) > 0$, and $z'(0) = 0$,

For the primal or dual algorithm with a hyperbolic search path, $X_S = \{\mathbf{x}_S(\alpha) = \mathbf{c}_S + a_S \sec(\alpha)\mathbf{v}_S + b_S \tan(\alpha)\mathbf{u}_S, \alpha \in D\}$, with $\mathbf{x}'(\alpha) = a_S \sec(\alpha)\tan(\alpha)\mathbf{v}_S + b_S \sec^2(\alpha)\mathbf{u}_S$. The objective function is $z(\alpha) = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| + r_{i_1}$, and $z'(\alpha) = (\mathbf{x}(\alpha) - \mathbf{p}_{i_1})\mathbf{x}'(\alpha)/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| = a_S \sec(\alpha)\tan(\alpha)[(\mathbf{c}_S - \mathbf{p}_{i_1})\mathbf{v}_S + (c_S^2/a_S)\sec(\alpha)]/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\|$.

Since $\mathbf{x}(0) = \mathbf{x}_S = \mathbf{c}_S + a_S\mathbf{v}_S \in \text{conv}(S)$, there exists $\pi_j \ge 0$, for each $\mathbf{p}_{i_j} \in S$ such that $\sum_{\mathbf{p}_{i_j} \in S} \pi_j = 1$ and $\sum_{\mathbf{p}_{i_j} \in S}(\mathbf{x}(0) - \mathbf{p}_{i_j})\pi_j = \sum_{\mathbf{p}_{i_j} \in S}(\mathbf{c}_S - \mathbf{p}_{i_j} + a_S\mathbf{v}_S)\pi_j = \mathbf{0}$. Multiply the last equation by $\mathbf{v}_S$ to get $\sum_{\mathbf{p}_{i_j} \in S}((\mathbf{c}_S - \mathbf{p}_{i_j})\mathbf{v}_S + a_S)\pi_j = 0$.

If $(\mathbf{c}_S - \mathbf{p}_{i_j})\mathbf{v}_S + a_S = 0$ for each $\mathbf{p}_{i_j} \in S$, then the points in S are on the hyperplane containing the point $\mathbf{c}_S + a_S\mathbf{v}_S$ with normal vector $\mathbf{v}_S$. Thus, the points in S are affinely dependent, which is a contradiction. If the summands $(\mathbf{c}_S - \mathbf{p}_{i_j})\mathbf{v}_S + a_S$, for each $\mathbf{p}_{i_j} \in S$, are either all positive or all negative, the sum is not zero. Therefore, there exists some $\mathbf{p}_{i_k} \in S$, such that $(\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S + a_S > 0$. For the hyperbolic search path, $\epsilon_S^2 > 1$, or $c_S^2/a_S > a_S$. Then $0 < (\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S + a_S < (\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S + c_S^2/a_S \le (\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S + (c_S^2/a_S)\sec(\alpha)$.

For the primal algorithm $D = \{\alpha_S \le \alpha < 0\}$. For $\alpha_S < \alpha < 0$, $\tan(\alpha) < 0$ and $z'(\alpha) < 0$. For $\alpha = 0$, $z'(\alpha) = 0$. For the dual algorithm $D = \{0 \le \alpha\}$. For $0 < \alpha$, $\tan(\alpha) > 0$ and $z'(\alpha) > 0$. For $\alpha = 0$, $z'(\alpha) = 0$.

For the primal or dual algorithm with a elliptlic search path, $X_S = \{\mathbf{x}_S(\alpha) = \mathbf{c}_S + a_S \cos(\alpha)\mathbf{v}_S + b_S \sin(\alpha)\mathbf{u}_S, \alpha \in D\}$ with $\mathbf{x}'(\alpha) = -a_S \sin(\alpha)\mathbf{v}_S + b_S \cos(\alpha)\mathbf{u}_S$. The objective function is $z(\alpha) = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| + r_i$, and $z'(\alpha) = (\mathbf{x}(\alpha) - \mathbf{p}_{i_1})\mathbf{x}'(\alpha)/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| = a_S \sin(\alpha)[(\mathbf{p}_{i_1} - \mathbf{c}_S)\mathbf{v}_S - (c_S^2/a_S)\cos(\alpha)]/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\|$.

By the same argument used for the hyperbolic case, since $\mathbf{x}(0) = \mathbf{c}_S + a_S\mathbf{v}_S \in \text{conv}(S)$, there exists some $\mathbf{p}_{i_k} \in S$, such that $(\mathbf{p}_{i_k} - \mathbf{c}_S)\mathbf{v}_S - a_S > 0$. For the elliptic search path, $\epsilon_S^2 < 1$, so that $c_S^2/a_S < a_S$. Then $0 < (\mathbf{p}_{i_k} - \mathbf{c}_S)\mathbf{v}_S - a_S < (\mathbf{p}_{i_k} - \mathbf{c}_S)\mathbf{v}_S - c_S^2/a_S \le (\mathbf{p}_{i_k} - \mathbf{c}_S)\mathbf{v}_S - (c_S^2/a_S)\cos(\alpha)$.

For the primal algorithm, $D = \{\alpha_S \le \alpha < 0\}$. For $\alpha_S < \alpha < 0$, $\sin(\alpha) < 0$, and $z'(\alpha) < 0$. For $\alpha = 0$, $z'(\alpha) = 0$. For the dual algorithm with $D = \{0 \le \alpha\}$. For $0 < \alpha$, $\sin(\alpha) > 0$ and $z'(\alpha) > 0$. For $\alpha = 0$, $z'(\alpha) = 0$.

For the primal or dual algorithm with a parabolic search path, $X_S = \{\mathbf{x}(\alpha) = \mathbf{c}_S + \tilde{c}_S\alpha^2\mathbf{v}_S + 2\tilde{c}_S\alpha\mathbf{u}_S, \alpha \in D\}$ with $\mathbf{x}'(\alpha) = 2\tilde{c}_S\alpha\mathbf{v}_S + 2\tilde{c}_S\mathbf{u}_S$. The objective function is $z(\alpha) = \|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| + r_{i_1}$ and $z'(\alpha) = (\mathbf{x}(\alpha) - \mathbf{p}_{i_1})\mathbf{x}'(\alpha)/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\| = 2\tilde{c}\alpha[(\mathbf{c}_S - \mathbf{p}_{i_1})\mathbf{v}_S + \tilde{c}(\alpha^2 + 2)]/\|\mathbf{x}(\alpha) - \mathbf{p}_{i_1}\|$.

Using the same argument as used for the hyperboloid and ellipsoid cases, $\mathbf{x}(0) = \mathbf{c}_S \in \text{conv}(S)$ implies there exists some $\mathbf{p}_{i_k} \in S$, such that $(\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S > 0$, which implies $(\mathbf{c}_S - \mathbf{p}_{i_k})\mathbf{v}_S + \tilde{c}(\alpha^2 + 2) > 0$.

For the primal algorithm $D = \{-\infty < \alpha \leq 0\}$, so that $z'(\alpha) < 0$, and for $\alpha = 0$, $z'(\alpha) = 0$. For the dual algorithm $D = \{0 \leq \alpha < \infty\}$, so that $z'(\alpha) > 0$, and for $\alpha = 0$, $z'(\alpha) = 0$. □

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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