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ORIGINAL RESEARCH

The efficient partition surplus Owen graph value

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Abstract
The Owen graph value for games with coalition structure and graph restricted communication was introduced by Vázquez-Brage et al. (Games Econ Behav 12: 42–53, 1996). It has been known that the value satisfies the axiom of component efficiency, requiring that the players of a component share the benefits generated by this component among themselves. In this paper we extend the Owen graph value to an efficient value and we provide axiomatic characterizations of this value.

Keywords TU-game · Coalition structure · Owen value · Graph · Owen graph value · Efficient extension

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1 Introduction

Cooperation among players can be depicted by a cooperative game with transferable utility (shortly, a TU-game). In this model, every subset of players can form a feasible coalition and each player in any coalition can obtain payoff from their cooperation. However, in many practical situations, the collection of feasible coalitions is restricted by some social, economical, communication, or technical structure. For this reason, one begins to consider the TU-games with cooperation structures.

Aumann and Drèze (1974) seem to be the first to consider TU-games with cooperation structure described by a coalition structure, i.e., a partition of the grand coalition. For this model, Owen (1977) introduced a modification of the Shapley value (Shapley, 1953),

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named the Owen value. Myerson (1977) introduced another class of TU-games with cooperation structures given by communication graphs on the player set and he proposed a famous allocation rule, called the Myerson value, applying the Shapley value to the so-called graph-restricted game. Inspired by Owen and Myerson's study, Vázquez-Brage et al. (1996) considered TU-games with coalition and graph structures by combining the two cooperation structures above and they introduced the Owen graph value for this model.

In a TU-game with graph structure, the Myerson value satisfies *component efficiency*, i.e., for each component of the graph, the total payoff to its players equals the worth of that component. This means that the worth of the grand coalition is not fully distributed among its members. Casajus (2007) developed the first efficient extension of the Myerson value to prevent the surplus. Later, Béal et al. (2015) suggested the interpretation that the communication graph does not necessarily affect the productivity but can influence the way the players distribute the worth. This interpretation supports values that are efficient, i.e., values which distribute the worth of the grand coalition among the players.

A variety of efficient extensions of the Myerson value and the other values with component efficiency have been developed in the literature. For example, the efficient egalitarian Myerson value (van den Brink et al., 2012; Béal et al., 2015), the efficient two-step egalitarian surplus Myerson value (Casajus, 2007; Hu et al., 2018), the efficient α-proportional Myerson value (Shan et al., 2019), the efficient quotient Myerson value (Li & Shan, 2020), the efficient average tree solution and the efficient compensation solution (Béal et al., 2018), the egalitarian efficient Aumann-Drèze value (Hu et al., 2019), the efficient egalitarian Owen graph value (Shan et al., 2020).

In practice, it might be not desirable that the Owen graph value satisfies component efficiency. To illustrate this, we provide an example of the research fund allocation that is similar to the one used in van den Brink et al. (2012) but now both coalition and graph structures are considered. A research fund that has money available to distribute amongst individual researchers and has the policy to stimulate interdisciplinary research, therefore a joint application of researchers from different disciplines will receive higher research grand. Moreover, it requires that only a group of researchers who can communicate can jointly submit a proposal. The set of researchers from the same discipline can be viewed as a (priori) union and two researchers are connected by a link if they can directly communicate. Hence, this situation can be described as a TU-game with coalition and graph structures (see Example 3.1). Although usually not all researchers can communicate, either directly or indirectly, still the full research budget is available and will be distributed. This requires a value to satisfy efficiency. In this paper we shall generalize the Owen graph value for a TU-game with coalition and graph structures as an efficient value.

The goal of this paper is to give an efficient extension of the Owen graph value for TU-games with coalition and graph structures, named the efficient partition surplus Owen graph value, by first assigning to each player his Owen graph value and then distributing the surplus by the two-step partition egalitarian surplus value. The two-step partition egalitarian surplus value requires that the surplus is equally distributed among all unions in the first step and then the payoff of each union in the first step is equally distributed among its players. We provide axiomatizations of this efficient value.

This article is organized as follows. In Sect. 2 we give preliminaries. Section 3 introduces the efficient partition surplus Owen graph value and we provide axiomatic characterizations of this efficient value. Section 4 contains some concluding remarks.

2 Preliminaries

2.1 TU-games and TU-games with coalition structures

A *TU-game* is a pair (N, v) consisting of a finite set of players $N = \{1, 2, \ldots, n\}$ and a *characteristic function* v defined on 2^N with that assigns to each *coalition* $S \subseteq N$ a real number $v(S)$, called the *worth* of S, and such that $v(\emptyset) = 0$. We denote by $\mathbb{G}(N)$ the set of TU-games with player set N. For any coalition $S \subseteq N$, the cardinality of S is denoted by $|S|$ or s. For simplicity, we shall write $S \cup i$, $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively.

For nonempty coalition $T \subseteq N$, the *subgame* (T, v_T) of (N, v) is defined by $v_T(S) = v(S)$ for any $S \subseteq T$.

An *allocation rule* or a *value* f on $\mathbb{G}(N)$ is a map that assigns a vector $f(N, v) \in \mathbb{R}^n$ to each game (N, v). The *Shapley value* (Shapley, 1953) is the value defined by

$$Sh_i(N, v) = \sum_{S:\, S \subseteq N \setminus i} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \big[v(S \cup i) - v(S)\big], \quad \text{for any } i \in N.$$

A *coalition structure* P on N is given by a partition $P = \{P_1, P_2, \ldots, P_m\}$ of player set N. Every element of a partition is called a *(priori) union*. Let $\mathbb{C}(N)$ denote the set of all coalition structures on N. In particular, we call $P^n = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$ and $P^N = \{N\}$ the *trivial coalition structures*. For any $i \in P_k$, $P_{-i} \in \mathbb{C}(N)$ is defined as $P_{-i} = \big\{P_1, \ldots, P_{k-1}, P_k \setminus \{i\}, P_{k+1}, \ldots, P_m, \{i\}\big\}$. For any nonempty $T \subseteq N$, *the restricted coalition structure* P_T to T is defined by $P_T = \{P_k \cap T \neq \emptyset \,|\, k \in M\}$, where $M = \{1, 2, \ldots, m\}$.

A *game with coalition structure* is a triple (N, v, P) where (N, v) is a TU-game and P a coalition structure on N. We denote the set of all such games by $\mathbb{U}(N)$.

For any $(N, v, P) \in \mathbb{U}(N)$ with $P = \big\{P_k \,|\, k \in M = \{1, 2, \ldots, m\}\big\}$, the *quotient game* $(M, v^P) \in \mathbb{G}(M)$ is the TU-game in which every union is a player, concretely, $v^P(R) = v(\bigcup_{k \in R} P_k)$ for any $R \subseteq M$.

A value on $\mathbb{U}(N)$ is a map $f : \mathbb{U}(N) \to \mathbb{R}^n$ that assigns a payoff vector $f(N, v, P) \in \mathbb{R}^n$ to each $(N, v, P) \in \mathbb{U}(N)$. The *Owen value* (Owen, 1977) Ow is defined by

$$Ow_i(N, v, P) = \sum_{R \subseteq M \setminus h} \sum_{S \subseteq P_h \setminus i} \frac{|R|!(|M| - |R| - 1)!|S|!(|P_h| - |S| - 1)!}{|M|!|P_h|!} \Big[v\Big(\bigcup_{r \in R} P_r \cup S \cup i\Big) - v\Big(\bigcup_{r \in R} P_r \cup S\Big)\Big],$$

for any $(N, v, P) \in \mathbb{U}(N)$ and $i \in P_h \in P$. The Owen value can be characterized by the properties of *efficiency, null player, symmetry in the unions, symmetry in the quotient* and *additivity* below.

Efficiency (**E**). For any $(N, v, P) \in \mathbb{U}(N)$, $\sum_{i \in N} f_i(N, v, P) = v(N)$.

Null player (**NP**). For any $(N, v, P) \in \mathbb{U}(N)$ and any $i \in N$, if i is a *null player*, i.e., $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$, then $f_i(N, v, P) = 0$.

Symmetry in the unions (**SU**). For any $(N, v, P) \in \mathbb{U}(N)$, $P_k \in P$ and $i, j \in P_k$, if $v(S \cup i) = v(S \cup j)$ for any $S \subseteq N \setminus \{i, j\}$, then $f_i(N, v, P) = f_j(N, v, P)$.

Symmetry in the quotient (**SQ**). For any $(N, v, P) \in \mathbb{U}(N)$ and $P_k, P_s \in P = \{P_1, P_2, \ldots, P_m\}$, if $v^P(R \cup k) = v^P(R \cup s)$ for any $R \subseteq M \setminus \{k, s\}$, then $\sum_{i \in P_k} f_i(N, v, P) = \sum_{i \in P_s} f_i(N, v, P)$.

Additivity (**A**). For any $(N, v, P), (N, w, P) \in \mathbb{U}(N)$, $f(N, v + w, P) = f(N, v, P) + f(N, w, P)$ where $(v + w)(S) = v(S) + w(S)$ for any $S \subseteq N$.

2.2 TU-games with coalition and graph structures

A *graph* is a pair (N, L) consisting of a set N of *nodes* representing players and a set $L \subseteq L^N = \{\{i, j\} \mid i, j \in N, i \neq j\}$ of *links*. L^N is the *complete graph* and each *unordered* pair $\{i, j\} \in L$ with $i, j \in N$ a link. For simplicity of notation, we often write ij instead of $\{i, j\}$. We denote the set of all graphs on N by $g(N)$.

We say that two nodes i and j in L are *connected* if $ij \in L$ or there exists a sequence of nodes $i_1, i_2, \ldots, i_p$ with $i = i_1$, $j = i_p$ such that $i_k i_{k+1} \in L$ for $k = 1, 2, \ldots, p-1$. The graph (N, L) is connected if all $i, j \in N$ are connected in L. A nonempty set $S \subseteq N$ is connected if every pair of nodes in S is connected in the *subgraph* (S, L_S) induced by S in L, in which $L_S = \{ij \in L \mid i, j \in S\}$. We will assume that S is connected whenever $s = 1$. A set $T \subseteq N$ is called a *component* of the graph (N, L) if T is a maximal connected subset, i.e., T is connected in the graph and, for all $T \cup i$ is not connected for each $i \in N \setminus T$. Let $C(N, L)$ denote the set of all components in (N, L) and $C(S, L)$ the set of the components of S in (S, L_S). The component of N containing i is often denoted by C_i.

The set of all links incident with the node i is denoted by $L_i = \{l \in L \mid i \in l\}$. For any $\delta \subseteq L$, the graph $L_{-\delta} = L \setminus \delta$ is the graph obtained from (N, L) by deleting the links in δ. In particular, we shall write L_{-ij} and L_{-i} instead of $L_{-\{ij\}}$ and $L \setminus L_i$, respectively.

A triple (N, v, L) is a *game with graph structure*, or simply a *graph game*, consisting of a TU-game $(N, v) \in \mathbb{G}(N)$ and a graph $(N, L) \in g(N)$. A graph game is *connected* if the associated graph (N, L) is connected. The sets of graph games and connected graph games on N are denoted by $\mathcal{G}$ and $\mathcal{G}_C$, respectively. Clearly, $\mathcal{G}_C \subseteq \mathcal{G}$.

Myerson (1977) introduced the *graph-restricted game* (N, v^L) for a $(N, v, L) \in \mathcal{G}$ defined by

$$v^L(S) = \sum_{T \in C(S,L)} v(T), \quad \text{for all } S \subseteq N.$$

The Myerson value is the allocation rule that assigns to every graph game (N, v, L) the Shapley value of the graph-restricted game v^L, i.e., $\mu(N, v, L) = Sh(N, v^L)$, and it is characterized by *component efficiency* (**CE**) and *fairness* (**F**) in Myerson (1977).

A *game with coalition and graph structures*, or simply a CG-*game*, is a quadruple (N, v, L, P), where $(N, v) \in \mathbb{G}(N)$, $(N, L) \in g(N)$ and $(N, P) \in \mathbb{C}(N)$. Let $\mathcal{G}_{CG}$ denote the set of all CG-games on N. A CG-game (N, v, L, P) is connected if the graph (N, L) is connected, and the set of all connected CG-games is denoted by $\mathcal{G}^C_{CG}$. Clearly, $\mathcal{G}^C_{CG} \subseteq \mathcal{G}_{CG}$. For any $P_k, P_s \in P$, let $L_{(P_k,P_s)} = \{ij \in L \mid i \in P_k, j \in P_s\}$ and $L_{-(P_k,P_s)} = L \setminus L_{(P_k,P_s)}$.

For any $(N, v, L, P) \in \mathcal{G}_{CG}$, the *communication quotient game* $(M, v^{LP}) \in \mathbb{G}(M)$ is defined by

$$v^{LP}(R) = v^L(\cup_{r \in R} P_r) = \sum_{S \in C(\cup_{r \in R} P_r, L)} v(S), \quad \text{for all } R \subseteq M.$$

The communication quotient game says that the economic output of any coalition $R \in M$ is realized by the economic output of coalition $\cup_{r \in R} P_r$ restricted by L, that is, the sum of the economic output of all components in $C(\cup_{r \in R} P_r, L)$.

The *Owen graph value* ψ is defined in Vázquez-Brage et al. (1996) as the Owen value of the graph-restricted game v^L, that is,

$$\psi(N, v, L, P) = Ow(N, v^L, P).$$

Let $f : \mathcal{G}_{CG} \to \mathbb{R}^n$ be a value. We recall some axioms.

Component efficiency (**CE**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and any $C \in C(N, L)$,

$$\sum_{i \in C} f_i(N, v, L, P) = v(C).$$

Component efficiency for trivial coalition (**CETC**). For any subclasses $(N, v, L, P^n) \in \mathcal{G}_{CG}$ and any $C \in C(N, L)$, $\sum_{i \in C} f_i(N, v, L, P^n) = v(C)$.

Fairness in the graph (**FG**). For any $(N, v, L) \in \mathcal{G}$ and $ij \in L$ with $i, j \in N$,

$$f_i(N, v, L, P^n) - f_i(N, v, L_{-ij}, P^n) = f_j(N, v, L, P^n) - f_j(N, v, L_{-ij}, P^n).$$

Note that for any $(N, v, L, P^n) \in \mathcal{G}_{CG}$, $\psi(N, v, L, P^n) = \mu(N, v, L)$. Furthermore, fairness in the graph (**FG**) is equivalent to fairness (**F**) for the Myerson value μ when P is the trivial coalition structure, i.e., $P = P^n$. Fairness states that $f_i(N, v, L) - f_i(N, v, L_{-ij}) = f_j(N, v, L) - f_j(N, v, L_{-ij})$ for any $(N, v, L) \in \mathcal{G}$ and $ij \in L$.

Balanced contributions for the graph (**BCG**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i, j \in P_k$ with $P_k \in P$,

$$f_i(N, v, L, P) - f_i(N, v, L_{-j}, P) = f_j(N, v, L, P) - f_j(N, v, L_{-i}, P).$$

Balanced contributions for the unions (**BCU**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i, j \in P_k$ with $P_k \in P$,

$$f_i(N, v, L, P) - f_i(N, v, L, P_{-j}) = f_j(N, v, L, P) - f_j(N, v, L, P_{-i}).$$

Fairness in the quotient (**FQ**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $P_k, P_s \in P$,

$$\sum_{i \in P_k} f_i(N, v, L, P) - \sum_{i \in P_k} f_i(N, v, L_{-(P_k, P_s)}, P) = \sum_{i \in P_s} f_i(N, v, L, P) - \sum_{i \in P_s} f_i(N, v, L_{-(P_k, P_s)}, P).$$

Quotient game property (**QG**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $P_k \in P = \{P_1, P_2, \ldots, P_m\}$ with $M = \{1, 2, \ldots, m\}$,

$$\sum_{i \in P_k} f_i(N, v, L, P) = f_k(M, v^{LP}, L^M, P^m),$$

where the partition $P^m = \big\{\{P_1\}, \{P_2\}, \ldots, \{P_m\}\big\}$ and L^M is the complete graph on M.

The axiomatizations of the Owen graph value have been developed by Vázquez-Brage et al. (1996) and Alonso-Meijide et al. (2009)

Theorem 2.1 (Vázquez-Brage et al., 1996) *The Owen graph value ψ is the unique solution on $\mathcal{G}_{CG}$ satisfying component efficiency (**CE**), fairness in the quotient (**FQ**) and either balanced contributions for the unions (**BCU**) or balanced contributions for the graph (**BCG**).*

Theorem 2.2 (Alonso-Meijide et al., 2009) *The Owen graph value ψ is the unique solution on $\mathcal{G}_{CG}$ satisfying component efficiency for trivial coalition (**CETC**), fairness in the graph (**FG**), balanced contributions for the unions (**BCU**) and quotient game property (**QG**).*

3 The efficient partition surplus Owen graph value

In this section we shall introduce the efficient partition surplus Owen graph value for the CG-games. As mentioned above, the Owen graph value satisfies component efficiency, that is, $\sum_{i\in C} \psi_i(N, v, L, P) = v(C)$ for any $(N, v, L, P) \in \mathcal{G}_{CG}$, $C \in C(N, L)$, so $\sum_{i\in N} \psi_i(N, v, L, P) = v^L(N)$.

In general, $v(N) \neq v^L(N)$. If that happens, this implies that the total Owen graph value obtained by the players in the grand coalition is $v^L(N)$ not $v(N)$, so this will yield a surplus $v(N) - v^L(N)$. A question we are facing is how to allocate the surplus reasonably. Various approaches to distribute the surplus $v(N) - v^L(N)$ have been developed in Casajus (2007), van den Brink et al. (2012), Shan et al. (2019), Li and Shan (2020) and elsewhere.

Here we introduce a new method to distribute the surplus $v(N) - v^L(N)$.

Definition 3.1 For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i \in P_k \in P$, the *efficient partition surplus Owen graph value* is defined by

$$EU\psi_i(N, v, L, P) := \psi_i(N, v, L, P) + \frac{v(N) - v^L(N)}{|P||P_k|}.$$

This value EUψ first assigns to each player his Owen graph value and then distributes the surplus $v(N) - v^L(N)$ (may be zero) with the two-step partition egalitarian surplus value, in which the surplus is equally distributed among all priori unions and then the payoff gained by each priori union in the first step is equally distributed among its players.

Note that, EU$\psi(N, v, L, P) = EE\mu(N, v, L)$ when $P^n = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$ or $P^N = \{N\}$ and EU$\psi(N, v, L, P) = Ow(N, v, P)$ when $L = L^N$, i.e., L is the complete graph.

3.1 Axiomatizations of the EUψ value

In this subsection we shall characterize the efficient partition surplus Owen graph value EUψ.

A value is *component decomposable* on $\mathcal{G}_{CG}$ if for any CG-game (N, v, L, P), the payoff of each player $i \in N$ is completely determined within the component C_i containing i.

Component decomposability (**CD**). A value f is *component decomposability*, if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i \in C_i \in C(N, L)$, $f_i(N, v, L, P) = f_i(C_i, v_{C_i}, L_{C_i}, P_{C_i})$.

One has observed that every value with component efficiency on $\mathcal{G}$ is component decomposable. This implies that the Owen graph value satisfies *component decomposability*.

To characterize the efficient partition surplus Owen graph value EUψ, we need the following properties.

Link-fairness in the quotient (**LFQ**). For any $(N, v, L, P) \in \mathcal{G}_{CG}$, any $P_k, P_s \in P$ and any $ij \in L_{(P_k, P_s)}$,

$$\sum_{h\in P_k} f_h(N, v, L, P) - \sum_{h\in P_k} f_h(N, v, L_{-ij}, P) = \sum_{h\in P_s} f_h(N, v, L, P) - \sum_{h\in P_s} f_h(N, v, L_{-ij}, P).$$

Link-fairness in the quotient states that the change of the total payoffs of any two unions is same when any one link joining them are severed.

Connected link-fairness in the quotient (**CLFQ**). For any $(N, v, L, P) \in \mathcal{G}^C_{CG}$, any $P_k, P_s \in P$ and any link $ij \in L_{(P_k, P_s)}$,

$$\sum_{h \in P_k} f_h(N, v, L, P) - \sum_{h \in P_k} f_h\big(C_h(N, L_{-ij}), v_{C_h(N, L_{-ij})}, (L_{-ij})_{C_h(N, L_{-ij})}, P_{C_h(N, L_{-ij})}\big)$$
$$= \sum_{h \in P_s} f_h(N, v, L, P) - \sum_{h \in P_s} f_h\big(C_h(N, L_{-ij}), v_{C_h(N, L_{-ij})}, (L_{-ij})_{C_h(N, L_{-ij})}, P_{C_h(N, L_{-ij})}\big).$$

Connected link-fairness in the quotient states that for any connected CG-game, the change of the total payoffs of any two unions P_k and P_s is the same if a link ij between P_k and P_s is removed. Connected link-fairness in the quotient compares the original payoffs of the union P_i $(i = k, s)$ with the payoffs P_i $(i = k, s)$ obtained if the CG-game is restricted to each player's component when the link ij is removed and imposes an equal payoff of the union variation. This axiom is motivated by connected fairness given in Béal et al. (2015, 2018).

We show that the Owen graph value satisfies link-fairness in the quotient (**LFQ**).

Lemma 3.1 *For $(N, v, L, P) \in \mathcal{G}_{CG}$, the Owen graph value $\psi(N, v, L, P)$ satisfies quasi-fairness in the quotient (***LFQ***).*

Proof For any $P_k, P_s \in P$ and any $ij \in L_{(P_k, P_s)}$, we let $w = v^L - v^{L_{-ij}}$. For all $R \subseteq M \setminus \{k, s\}$, clearly $w^P(R \cup k) = w^P(R \cup s) = 0$. By symmetry in the quotient (**SQ**) of the Owen value, $\sum_{i \in P_k} Ow_i(N, w, P) = \sum_{i \in P_s} Ow_i(N, w, P)$. Then, by additivity (**A**) of the Owen value, we obtain

$$\sum_{i \in P_k} Ow_i(N, v^L, P) - \sum_{i \in P_k} Ow_i(N, v^{L_{-ij}}, P) = \sum_{i \in P_s} Ow_i(N, v^L, P)$$
$$- \sum_{i \in P_s} Ow_i(N, v^{L_{-ij}}, P).$$

Since $\psi(N, v, L, P) = Ow(N, v^L, P)$, $\psi(N, v, L, P)$ satisfies **LFQ**. □

In order to give an characterization of the efficient partition surplus Owen graph value, we first give a characterization of the Owen graph value on the class of connected CG-games.

Theorem 3.1 *A value $f(N, v, L, P)$ on $\mathcal{G}^C_{CG}$ satisfies efficiency (***E***), connected link-fairness in the quotient (***CLFQ***) and balanced contributions for the unions (***BCU***) if and only if $f(N, v, L, P) = \psi(N, v, L, P)$ on $\mathcal{G}^C_{CG}$.*

Proof Existence. We show that the Owen graph value $\psi(N, v, L, P)$ satisfies the three properties. By Theorem 2.1, we have seen that $\psi(N, v, L, P)$ satisfies efficiency (**E**) and balanced contributions for the unions (**BCU**). We show that $\psi(N, v, L, P)$ satisfies connected link-fairness in the quotient (**CLFQ**). Let $ij \in L$.

If removing ij does not add new components, the property of link-fairness in the quotient (**LFQ**) directly implies connected link-fairness in the quotient (**CLFQ**).

If removing ij adds new components, then

$$\sum_{h \in P_k} \psi_h(N, v, L, P) - \sum_{h \in P_s} \psi_h(N, v, L, P) = \sum_{h \in P_k} \psi_h(N, v, L_{-ij}, P) - \sum_{h \in P_s} \psi_h(N, v, L_{-ij}, P)$$
$$= \sum_{h \in P_k} \psi_h\big(C_h(N, L_{-ij}), v_{C_h(N, L_{-ij})}, (L_{-ij})_{C_h(N, L_{-ij})}, P_{C_h(N, L_{-ij})}\big)$$
$$- \sum_{h \in P_s} \psi_h\big(C_h(N, L_{-ij}), v_{C_h(N, L_{-ij})}, (L_{-ij})_{C_h(N, L_{-ij})}, P_{C_h(N, L_{-ij})}\big),$$

where the first equation holds by link-fairness in the quotient (**LFQ**) and the second equation follows since the Owen graph value satisfies component decomposability (**CD**).

Uniqueness. Let f be a value on $\mathcal{G}^C_{CG}$ satisfying the above three properties. We have to prove that $f = \psi$. We first show that $\sum_{h\in P_k} f_h(N, v, L, P) = \sum_{h\in P_k} \psi_h(N, v, L, P)$ for any $k \in M$ by induction on $|N| + |L|$.

If $|N| = 1$, then clearly $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ by efficiency (**E**). We may assume that $|N| \geq 2$ and for any $(N', v, L', P_{N'}) \in \mathcal{G}^C_{CG}$ with $|N'| + |L'| < |N| + |L|$, $f(N', v, L', P_{N'}) = \psi(N', v, L', P_{N'})$ holds. Therefore, for $(N, v, L, P) \in \mathcal{G}^C_{CG}$ and any $P_k, P_s \in P$, there exists a link $ij \in L$ since L is connected. By connected link-fairness in the quotient (**CLFQ**) and the inductive hypothesis, we have

$$\begin{aligned}\sum_{h\in P_k} f_h(N, v, L, P) - \sum_{h\in P_s} f_h(N, v, L, P) &= \sum_{h\in P_k} f_h\big(C_h(N, L_{-ij}), v_{C_h(N,L_{-ij})}, (L_{-ij})_{C_h(N,L_{-ij})}, P_{C_h(N,L_{-ij})}\big)\\ &- \sum_{h\in P_s} f_h\big(C_h(N, L_{-ij}), v_{C_h(N,L_{-ij})}, (L_{-ij})_{C_h(N,L_{-ij})}, P_{C_h(N,L_{-ij})}\big)\\ &= \sum_{h\in P_k} \psi_h\big(C_h(N, L_{-ij}), v_{C_h(N,L_{-ij})}, (L_{-ij})_{C_h(N,L_{-ij})}, P_{C_h(N,L_{-ij})}\big)\\ &- \sum_{h\in P_s} \psi_h\big(C_h(N, L_{-ij}), v_{C_h(N,L_{-ij})}, (L_{-ij})_{C_h(N,L_{-ij})}, P_{C_h(N,L_{-ij})}\big)\\ &= \sum_{h\in P_k} \psi_h(N, v, L, P) - \sum_{h\in P_s} \psi_h(N, v, L, P).\end{aligned}$$

Thus, by connectedness of L, it is easy to see that there exists a constant d for all $k \in M$ such that

$$\sum_{h\in P_k} f_h(N, v, L, P) - \sum_{h\in P_k} \psi_h(N, v, L, P) = d.$$

By efficiency (**E**), we have

$$|M|d = \sum_{s\in M} \sum_{h\in P_s} \big(f_h(N, v, L, P) - \psi_h(N, v, L, P)\big) = v(N) - v(N) = 0,$$

and so $d = 0$. Hence, for any $k \in M$, we have

$$\sum_{h\in P_k} f_h(N, v, L, P) = \sum_{h\in P_k} \psi_h(N, v, L, P). \tag{1}$$

We next show that $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ for any $i \in N$ by induction on $|P|$. For any $(N, v, L, P) \in \mathcal{G}^C_{CG}$, if there is some $P_k \in P$ such that $|P_k| = 1$, say $P_k = \{i\}$, then $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ by (1). Suppose that $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ for $m \leq |P| \leq n$. We show that $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ for $|P| = m - 1$. Clearly, there exists a $P_k \in P$ such that $|P_k| \geq 2$. Let $i, j \in P_k$. Note that $|P_{-i}| = |P_{-j}| = m$, by balanced contributions for the unions (**BCU**) and the inductive hypothesis, we have

$$\begin{aligned}f_i(N, v, L, P) - f_j(N, v, L, P) &= f_i(N, v, L, P_{-j}) - f_j(N, v, L, P_{-i})\\ &= \psi_i(N, v, L, P_{-j}) - \psi_j(N, v, L, P_{-i})\\ &= \psi_i(N, v, L, P) - \psi_j(N, v, L, P).\end{aligned}$$

Hence, for each $i \in P_k \in P$, there exists a constant d_{P_k} such that

$$f_i(N, v, L, P) - \psi_i(N, v, L, P) = d_{P_k}. \tag{2}$$

Combining (1) and (2), we obtain

$$0 = \sum_{i \in P_k} \big(f_i(N, v, L, P) - \psi_i(N, v, L, P)\big) = |P_k| d_{P_k},$$

and so $d_{P_k} = 0$. Then, for any $P_k \in P$ with $|P_k| \geq 2$ and any $i, j \in P_k$, $f_i(N, v, L, P) = \psi_i(N, v, L, P)$. Therefore, $f_i(N, v, L, P) = \psi_i(N, v, L, P)$ for all $i \in N$. □

In Theorem 3.1, we characterize the Owen graph value on the class of connected CG-games. By Theorem 3.1, we are now ready to give an axiomatic characterization of the EUψ value on $\mathcal{G}_{CG}$.

Theorem 3.2 *The efficient partition surplus Owen graph value (EUψ) on $\mathcal{G}_{CG}$ is the unique value satisfying efficiency (**E**), link-fairness in the quotient (**LFQ**), connected link-fairness in the quotient (**CLFQ**) and balanced contributions for the unions (**BCU**).*

Proof Existence. That the value EU$\psi(N, v, L, P)$ satisfies the above four properties follows directly from the fact that the Owen graph value satisfies the axioms of component efficiency (**CE**), link-fairness in the quotient (**LFQ**) (by Lemma 3.1) and balanced contributions for the unions (**BCU**).

Uniqueness. Suppose that there exists a value f on $\mathcal{G}_{CG}$ satisfying the four properties. We have to show that $f =$EUψ. If $(N, v, L, P) \in \mathcal{G}^C_{CG}$, we immediately have $f = \psi =$EUψ by Theorem 3.1. We may therefore assume that $(N, v, L, P) \in \mathcal{G}_{CG} \backslash \mathcal{G}^C_{CG}$. Then $|C(N, L)| \geq 2$.

For any $k \in M$, we show that the following equality holds,

$$\sum_{h \in P_k} f_h(N, v, L, P) = \sum_{h \in P_k} EU\psi_h(N, v, L, P). \tag{3}$$

If $P = P^N$, then, by efficiency (**E**), we have

$$\sum_{h \in N} f_h(N, v, L, P) = v(N) = \sum_{h \in N} EU\psi_h(N, v, L, P).$$

If $P \neq P^N$, we prove that the equality (3) holds by contradiction. Suppose that $f \neq$EUψ. Then there must exist a $(N, v, L, P) \in \mathcal{G}_{CG} \setminus \mathcal{G}^C_{CG}$ such that $f(N, v, L, P) \neq$EU$\psi(N, v, L, P)$. Let L be a maximal graph such that there exists $f_i(N, v, L, P) \neq$EU$\psi_i(N, v, L, P)$ for some $i \in N$. Concretely, there exists $i \in C_i \in C(N, L)$ such that $f_i \neq$EUψ_i.

Since $|C(N, L)| \geq 2$ and $|P| \geq 2$, there exist $x, y \in N$ such that x and y are neither in the same component nor in the same priori union. For such a pair x, y, we assume that $x \in P_k \cap C$ and $y \in P_s \cap C'$ where $C \neq C'$, $C, C' \in C(N, L)$ and $s \neq k$, $s, k \in M$.

By the maximality of graph L and link-fairness in the quotient (**LFQ**), we have

$$\begin{aligned}
&\sum_{h \in P_k} f_h(N, v, L, P) - \sum_{h \in P_s} f_h(N, v, L, P) \\
&= \sum_{h \in P_k} f_h(N, v, L \cup \{xy\}, P) - \sum_{h \in P_s} f_h(N, v, L \cup \{xy\}, P) \\
&= \sum_{h \in P_k} EU\psi_h(N, v, L \cup \{xy\}, P) - \sum_{h \in P_s} EU\psi_h(N, v, L \cup \{xy\}, P)
\end{aligned}$$

$$= \sum_{h \in P_k} EU\psi_h(N, v, L, P) - \sum_{h \in P_s} EU\psi_h(N, v, L, P).$$

For any $k, s \in M$, the last expression can be rewritten as

$$\sum_{h \in P_k} f_h(N, v, L, P) - \sum_{h \in P_k} EU\psi_h(N, v, L, P) \\ = \sum_{h \in P_s} f_h(N, v, L, P) - \sum_{h \in P_s} EU\psi_h(N, v, L, P).$$

Next, fix k and sum up over all $s \in M$. By efficiency (**E**), we obtain

$$|M|\Big(\sum_{h \in P_k} f_h(N, v, L, P) - \sum_{h \in P_k} EU\psi_h(N, v, L, P)\Big) \\ = \sum_{s \in M} \sum_{h \in P_s} f_h(N, v, L, P) - \sum_{s \in M} \sum_{h \in P_s} EU\psi_h(N, v, L, P) \\ = v(N) - v(N) = 0.$$

Hence, for any $k \in M$, we have that

$$\sum_{h \in P_k} f_h(N, v, L, P) = \sum_{h \in P_k} EU\psi_h(N, v, L, P).$$

The remainder of the proof is similar to the proof of uniqueness in Theorem 3.1. By induction on $|P|$, we deduce that $f_i(N, v, L, P) =$EU$\psi_i(N, v, L, P)$ for all $i \in N$. □

Now we shall give an alternative characterization of the EUψ value. For this purpose, we need to generalize the axioms in Shan et al. (2019) to the setting of games with coalition and graph structures.

Proportional fair distribution of surplus between unions (**PFDSU**). There exists a constant c such that for each $i \in P_k \in P$ with $i \in C_i \in C(N, L)$,

$$f_i(N, v, L, P) - f_i(C_i, v_{C_i}, L_{C_i}, P_{C_i}) = \frac{1}{|P_k|}c.$$

Coherence with the Owen graph value for connected graphs (**COC**). For any $(N, v, L, P) \in \mathcal{G}^C_{CG}$, it holds that $f(N, v, L, P) = \psi(N, v, L, P)$.

Coherence with the Owen graph value for connected graphs requires that allocation rule $f(N, v, L, P)$ equals the Owen graph value $\psi(N, v, L, P)$ for any $(N, v, L, P) \in \mathcal{G}^C_{CG}$. Specially, $\psi(N, v, L, P^n) = \mu(N, v, L)$ when $(N, v, L, P^n) \in \mathcal{G}_{CG}$. Hence, coherence with the Owen graph value for connected graphs (**COC**) implies coherence with the Myerson value for connected graphs (**CMC**).

By **E**, **COC** and **PFDSU**, we give another characterization of the EUψ value. We omit its proof since it follows along the same lines as the one in Shan et al. (2019).

Theorem 3.3 *The efficient partition surplus Owen graph value (EUψ) on $\mathcal{G}_{CG}$ is the unique value satisfying efficiency (**E**), coherence with the Owen graph value for connected graphs (**COC**) and proportional fair distribution of surplus between unions (**PFDSU**).*

3.2 Independence of the axioms in axiomatizations

In this subsection we show that the independence of the axioms invoked in Theorems 3.1, 3.2 and 3.3.

- The *null value* $\phi^1(N, v, L, P) = 0$ satisfies **CLFQ**, **LFQ** and **BCU**, but not **E**.
- The value $\phi^2(N, v, L, P) = \frac{v(N)}{|N|}$ satisfies **E**, **LFQ** and **BCU**, but not **CLFQ**.
- For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i \in P_k \in P$, the value

$$\phi_i^3(N, v, L, P) = \begin{cases} \psi_i(N, v, L, P) + \frac{v(N)-v^L(N)}{3|P||P_k \cap \mathscr{N}(N,v)|}, & i \in \mathscr{N}(N, v) \\ \psi_i(N, v, L, P) + \frac{2\left(v(N)-v^L(N)\right)}{3|P||P_k \setminus \mathscr{N}(N,v)|}, & i \notin \mathscr{N}(N, v) \end{cases}$$

  where $\mathscr{N}(N, v) = \{i \in N \mid v(S \cup i) = v(S), S \subseteq N \setminus i\}$ represents the set of null players in (N, v). It is easily verified that ϕ_i^3 satisfies **E**, **LFQ** and **CLFQ**, but not **BCU**.
- For any $(N, v, L, P) \in \mathcal{G}_{CG}$ and $i \in N$, the value $\phi_i^4(N, v, L, P) = \psi_i(N, v, L, P) + \frac{v(N)-v^L(N)}{|N|}$ satisfies **E**, **CLFQ** and **BCU**, but not **LFQ**.
- ϕ_i^4 satisfies **E** and **COC**, but not **PFDSU**.
- The value $\phi_i^5(N, v, L, P) = \psi_i(N, v, L, P) + \frac{v(N)-v^L(N)}{2|P||P_k|}$ satisfies **COC** and **PFDSU**, but not **E**.
- The value $\phi_i^6(N, v, L, P) = \frac{v(C_i)}{|C_i|} + \frac{v(N)-v^L(N)}{|P||P_k|}$ satisfies **E** and **PFDSU**, but not **COC**.

3.3 An example

To illustrate the efficient partition surplus Owen graph value, we give an example of a fund allocation problem as described in the introduction.

Example 3.1 In a fund allocation problem, suppose that the budget of the fund is 90 and there are five researchers, say 1, 2, 3, 4, 5, from three different disciplines, named P_1, P_2 and P_3, where 1 is in P_1, 2 and 3 in P_2, 4 and 5 in P_3. An individual proposal just gives access to the fund, but does not secure any amount of money. In order to stimulate interdisciplinary cooperation, a joint application of researchers from different disciplines can receive higher research grand. Two researchers in the same discipline can secure themselves a grant of 5 when writing a proposal together, while two researchers in the different disciplines a grant of 20, three researchers a grant of 40, four researchers a grant of 60. However, researchers 1 and 2, 3 and 4, 4 and 5 are able to directly communicate, thus 1 and 2 can negotiate with each other for a proposal, while 3, 4 and 5 can negotiate with each other for a proposal.

This situation can be described as a TU-game with coalition and graph structures (N, v, L, C), where $N = \{1, 2, 3, 4, 5\}$, $L = \{12, 34, 45\}$, $P = \{P_1, P_2, P_3\} = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ and v is defined by

$$v(S) = \begin{cases} 0, & |S| = 1 \\ 5, & \text{if } S = P_2 \text{ or } P_3, \\ 20, & \text{if } S = 2 \text{ and } S \neq P_2, P_3, \\ 40, & |S| = 3, \\ 60, & |S| = 4, \\ 90, & S = N. \end{cases}$$

Next we provide different schemes to allocate the fund by using the Myerson value $\mu(N, v, L)$, the Owen value $Ow(N, v, P)$, the Owen graph value $\psi(N, v, L, P)$ and the efficient partition surplus Owen graph value $\text{EU}\psi(N, v, L, P)$, which are shown in Table 1.

In Table 1,

(i) The Owen value Ow assigns a higher funding 21.667 to researcher 1 and the same funding 17.083 to the other researchers. This shows that the coalition structure affects

Table 1 The four values for the games

Player	$Ow(N, v, P)$	$\mu(N, v, L)$	$\psi(N, v, L, P)$	EU$\psi(N, v, L, P)$
1	21.667	10.000	10.000	20.000
2	17.083	10.000	10.000	15.000
3	17.083	15.000	17.500	22.500
4	17.083	17.500	16.250	21.500
5	17.083	7.500	6.250	11.250

the allocation of funds and plays a protective role for priori union P_1 which contains only one researcher.

(ii) When the communication graph structure is considered, the Myerson value μ and Owen graph value ψ both give a low funding 10 to researchers 1 and 2, but they all assign high funding 17.5 and 16.25 to researcher 4, respectively. This illustrates the important bargaining power of 4 in the graph game. But, note that $\psi(N, v, L, P)$ gives a slightly higher funding 17.5 to researcher 3 than $\mu_3(N, v, L) = 15$, this shows that interdisciplinary cooperation increases the bargaining power of researcher 3.

(iii) The total funding for μ and ψ is $v^L(N) = 60$ not $v(N) = 90$.

(iv) The efficient partition surplus Owen graph value EUψ distributes the budget 90 to all researchers.

4 Concluding remarks

In this paper we introduce a new efficient extension of the Owen graph value and provide three axiomatizations of the value.

We give an overview of the values that appeared in this paper.

(1) The Owen value

$$Ow(N, v, P) : (N, v, P) \to (M, v^P) \to (P_k, v^{P_k}) \to Sh(P_k, v^{P_k})$$

where (P_k, v^{P_k}) is the internal TU-game defined via the quotient game, $v^{P_k}(S) = Sh_{P_k}(M \setminus P_k \cup S, v^q_{-P_k,+S})$ with $v^q_{-P_k,+S}$ being the quotient game in which P_k is replaced by S.

(2) The Myerson value $\mu(N, v, L) : (N, v, L) \to (N, v^L) \to Sh(N, v^L)$, where v^L is the graph-restricted game.

(3) The Owen graph value $\psi(N, v, L, P) : (N, v, L, P) \to (N, v^L, P) \to Ow(N, v^L, P)$.

(4) The efficient partition surplus Owen graph value

$$\text{EU}\psi(N, v, L, P) : \psi(N, v, L, P) + \frac{v(N) - v^L(N)}{|P||P_k|} \to \text{EU}\psi(N, v, L, P).$$

Actually, the distributions of the surplus $v(N) - v^L(N)$ can be modified in different fashions. For instance, Shan et al. (2020) distribute the surplus by $(v(N) - v^L(N))/n$ as in van den Brink et al. (2012). Furthermore, we can also obtain the other efficient extensions of the Owen graph value by replacing $(v(N) - v^L(N))/|P||P_k|$ in EUψ by the distribution ways of the surplus $v(N) - v^L(N)$ developed in Casajus (2007); Shan et al. (2019). These

Table 2 The characterizations of the four values: Ow, μ, ψ and EUψ

	Ow	μ	ψ	EUψ
Efficiency (**E**)	$\oplus$	$-$	$-$	$\oplus^{\{1,2\}}$
Component efficiency (**CE**)	$+$	$-$	$\oplus^{\{1,2\}}$	$-$
Component efficiency for trivial coalition (**CETC**)	$+$	$\oplus$	$\oplus^{\{3\}}$	$-$
Null player (**NP**)	$\oplus$	$-$	$-$	$-$
Symmetry in the unions (**SU**)	$\oplus$	$-$	$-$	$-$
Symmetry in the quotient (**SQ**)	$\oplus$	$-$	$-$	$-$
Additivity (**A**)	$\oplus$	$+$	$+$	$-$
Fairness in the graph (**FG**)	$+$	$\oplus$	$\oplus^{\{3\}}$	$+$
Balanced contributions for the graph (**BCG**)	$+$	$+$	$\oplus^{\{1\}}$	$+$
Balanced contributions for the unions (**BCU**)	$+$	$+$	$\oplus^{\{2,3\}}$	$\oplus^{\{1\}}$
Fairness in the quotient (**FQ**)	$+$	$+$	$\oplus^{\{1,2\}}$	$+$
Quotient game property (**QG**)	$+$	$+$	$\oplus^{\{3\}}$	$-$
Link-fairness in the quotient (**LFQ**)	$+$	$+$	$+$	$\oplus^{\{1\}}$
Connected link-fairness in the quotient (**CLFQ**)	$+$	$+$	$+$	$\oplus^{\{1\}}$
Proportional fair distribution of surplus between unions (**PFDSU**)	$+$	$+$	$+$	$\oplus^{\{2\}}$
Coherence with the Owen graph value for connected graphs (**COC**)	$+$	$+$	$+$	$\oplus^{\{2\}}$

efficient Owen graph values can be similarly characterized by modifying the axioms for the EUψ value.

Finally, we summarize the axiomatizations of the Owen value Ow, the Myerson value μ, the Owen graph value ψ and the efficient partition surplus Owen graph value EUψ in Table 2. In Table 2, '+' represents that the value satisfies the axiom, '–' has the converse meaning and '$\oplus$' symbols indicate the sets of axioms used for characterizations of the value. The superscripts 1, 2, 3 represent three different characterizations of the values, respectively.

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