



Characterization of a value for games under restricted cooperation

M. Josune Albizuri¹ · Satoshi Masuya² · José M. Zarzuelo¹ 

Accepted: 2 May 2022 / Published online: 26 May 2022
© The Author(s) 2022

Abstract

The object of this paper is to study restricted cooperative games, that is, cooperative games for which the worth of some coalitions is unknown. We consider a value for these restricted cooperative games whose definition is based on the Harsanyi's dividends approach, and can therefore be seen as an extension of the Shapley value. We provide a characterization of this value with three axioms: Carrier, Symmetric-partnership and Additivity, which are similar to those proposed by Shapley (in: Kuhn and Tucker (eds) Contributions to the theory of games, Princeton University Press, Princeton, 1953). In addition, we characterize this value on the subclass of restricted cooperative simple games. Finally, we apply this value for restricted cooperative games to analyze the power distribution of the Catalanian Parliament in 1980 and compare the results with those of the coalitional value in Carreras and Owen (Math Soc Sci 15:87–92, 1988).

Keywords Restricted cooperative games · Shapley value · R-value · Conference structures

1 Introduction

When studying cooperative games with side-payments, it is generally assumed that the worths of all coalitions are known. However, in real world applications there are situations in which some of these worths are unknown or quite difficult to determine. This can happen when

The research reported in this paper has been partially supported by UPV/EHU (GIU17/0151 and GIU20/019), and Ministerio de Ciencia, Innovación y Universidades (Grant No. PID2019-105291GB-I00).

✉ José M. Zarzuelo
josemanuel.zarzuelo@ehu.es

M. Josune Albizuri
mj.albizuri@ehu.eus

Satoshi Masuya
masuya@ic.daito.ac.jp

¹ Faculty of Business Administration, Basque Country University (UPV/EHU), Lehendakari Aguirre 83, 48015 Bilbao, Spain

² Faculty of Business Administration, Daito Bunka University, 1-9-1, Takashimadaira, Itabashi-ku, Tokyo 175-8571, Japan

the amount of available information is very limited or extremely difficult to obtain, or some coalitions are not viable. For instance, in contexts such as water resources development, where estimates of benefits and costs are often unreliable (Loehman et al., 1979; Young et al., 1982). In other scenarios, some coalitions cannot be formed due to the impossibility of communication or incompatibilities between the agents. Some examples of this case are the study of neural information processing (Keinan et al., 2004), the analysis of reactions in metabolic networks (Sajitz-Hermstein & Nikoloski, 2012), or political contexts (Álvarez-Mozos et al., 2013) see also Sect. 5 later in this paper).

These situations can be modeled through restricted cooperative games (or R-games for short), also called partially defined games, which are defined only on a subset of feasible coalitions whose worth is known. To our knowledge, Faigle (1989) was the first to systematically study R-games, and more precisely the core and the balancedness property of these games. Although Faigle (1989) did not include any requirement on the set of feasible coalitions, in many subsequent works some kind of structure is usually assumed. Along this line several works can be mentioned. For instance, Algaba et al. (2000, 2001a) consider union stable systems that generalize the communication systems of Myerson (1977) and are related to Myerson (1980) conference structures. In Algaba et al. (2001b) the relationship between the line started by Faigle (1989) and the one introduced by Myerson (1977) was studied. Algaba et al. (2003) studied the Shapley value of cooperative games on antimatroids, in which the possibilities of coalition formation are determined by the positions of the players in a *antimatroid*. On the other hand van den Brink (2012) characterized the set of connected coalitions in a communication graph and compared the characterizing properties to those of a hierarchical structure represented by an antimatroid and Lange and Grabisch (2009) address a general framework leading to applications to games with communication graphs.

One of the most prominent solution concepts for side-payment games is the Shapley (1953) value,¹ that is endorsed by a well-known set of axioms characterizing it. The aim of this work is to study and characterize an extension of the Shapley value to the general class of R-games proposed by Calvo and Gutiérrez-López (2015).

Willson (1993) already proposed an extension of the Shapley value for a limited subfamily of R-games, together with an axiom system similar to that of Shapley (1953). More precisely, this author pays attention to R-games so that if the worth of a coalition is known, the worth of all coalitions with the same cardinality is also known. To extend the Shapley value to R-games, Willson (1993) constructs an auxiliary game in which the worth of all coalitions is known, and then the Shapley value of this auxiliary game is taken as the value of the original R-game. This auxiliary game is defined in such a way that it coincides with the original R-game on each feasible coalition, and assigns the worth zero to the unfeasible ones, which is not well justified. On their part, Aguilera et al. (2010) offered two different frameworks to extend the Shapley value to R-games. The first framework gives rise to the family of marginalist solutions. In the second one, the authors define the unanimity solution by matching it to the Shapley value on every unanimity game, without providing a justification, and then it is extended by linearity to the whole class of R-games. Subsequently, these authors state conditions under which the unanimity solution is marginalist.

In this paper we pay attention to an alternative extension of the Shapley value to general R-games, proposed by Calvo and Gutiérrez-López (2015). This value for R-games is called here the R-value, and it is based on Harsanyi (1963) procedure for finding a solution for non-transferable utility games. Accordingly, each coalition is supposed to guarantee certain payments to its members: the Harsanyi dividends. The peculiarity is that coalitions whose

¹ In Algaba et al. (2019) it can be found a collection of recent works on the Shapley value.

worth is not known allocate zero as a dividend to their players. The R-value assigns to each player the sum of the dividends of the coalitions to which she belongs. It turns out that the R-value coincides with the unanimity solution defined by Aguilera et al. (2010). In this work we characterize the R-value by means of three axioms: *Carrier*, *Symmetric-partnership* and *Additivity*, which are closely related to those of Shapley (1953). This characterization offers alternative grounds for the R-value to the constructive definition by means of the Harsanyi dividends. The axioms together with the characterization result will be presented in Sect. 3.

Simple games are those for which the worth of a coalition can only be 0 or 1. These games are specially suitable for modeling voting systems in which a coalition can lose (worth 0) or win (worth 1). In Sect. 4 we characterize the restriction of the R-value to the family of simple R-games, similar to how Dubey (1975) did with the Shapley and Shubik (1954) index of simple games. In Sect. 5, we consider an application of simple R-games to assess the distribution of power in the Catalanian Parliament after the 1980 elections, that was already examined by Carreras and Owen (1988).

It is worth mentioning that Myerson (1980) also analyzes the outcome of a side-payments game when it depends on a subset of coalitions. However, Myerson's approach is completely different. Actually, this author deals with full games where the worth of all coalitions is known. From a formal point of view, the main distinction is that Myerson (1980) fixes the game and the value depends on the subset of permissible coalitions. In contrast, in our approach the subset of feasible coalitions is fixed and the value depends on the game. For more details the reader is referred to Sect. 6.

2 Restricted cooperative games. The R-value

In this section we present the general framework.

Let $N = \{1, 2, \dots, n\}$ be a fixed set of *players*. We assume that players can form coalitions, but not all coalitions are viable, except the total coalition that is always feasible. Accordingly we say that a family \mathcal{K} of subsets of N is a *partial collection* whenever $\emptyset, N \in \mathcal{K}$, and every member $S \in \mathcal{K}$ is called a *feasible coalition*.

A *restricted cooperative game*, or *R-game* for short, on a partial collection \mathcal{K} is a set-function v which maps every set $S \in \mathcal{K}$ to a real number $v(S)$, such that $v(\emptyset) = 0$.

As usual the number $v(S)$ represents what the players in coalition S can guarantee for themselves without cooperating with the other players. Traditionally, the characteristic function v is assumed to be defined on the the set of all coalitions, i.e. 2^N , but in the present work it is defined only for the elements of a partial collection \mathcal{K} . This can be interpreted as that only the worth of feasible coalitions is known, whereas the worth of the remaining coalitions is unknown. Occasionally, when $\mathcal{K} = 2^N$ we will refer to such a game as a *full game*.

The set of all R-games on a partial collection \mathcal{K} is denoted $\mathcal{G}^{\mathcal{K}}$.

Given a family of R-games $\mathcal{F} \subseteq \mathcal{G}^{\mathcal{K}}$, we define formally a *solution* for \mathcal{F} to be any function $\phi : \mathcal{F} \rightarrow \mathbf{R}^N$. The real number $\phi_i(v)$ represents an evaluation of player i of her "prospect that will arise as a result of a play" (Shapley, 1953).

Next we present a solution for R-games proposed by Calvo and Gutiérrez-López (2015) by means of the Harsanyi dividends. These dividends were introduced by Harsanyi (1963) to define a bargaining solution for non-transferable utility games, which is in turn a generalization of the Shapley value for side-payments games. According to Harsanyi's procedure all the members of a coalition S receive a dividend from S . Here two cases are considered, depending on whether the coalition is feasible or not: (i) if $S \notin \mathcal{K}$, the dividend of S is

zero; (ii) if $S \in \mathcal{K}$, i.e. S is feasible, the total amount of the dividends allocated by all the subcoalitions of S is $v(S)$, as in Harsanyi (1963).

Formally the procedure can be described as follows. If $v \in \mathcal{G}^{\mathcal{K}}$, define recursively a function $D : 2^N \rightarrow \mathbf{R}$ by:

$$D(v, \emptyset) = 0, \quad \text{and} \tag{1}$$

$$D(v, S) = \begin{cases} 0, & \text{if } S \notin \mathcal{K}; \\ v(S) - \sum_{T \subsetneq S} D(v, T), & \text{if } S \in \mathcal{K}. \end{cases} \tag{2}$$

The real number $D(v, S)$ represents the total amount that coalition S distributes among its members. Harsanyi (1963) proposed that a coalition S would share $D(v, S)$ equally among its members. So $\frac{D(v, S)}{|S|}$ is usually called the *Harsanyi dividend* of coalition S in v . And a player will be assigned the sum of all the dividends of the coalitions to which he or she belongs.

Consequently, we call *R-value* to the solution $\psi^{\mathcal{K}} : \mathcal{G}^{\mathcal{K}} \rightarrow \mathbf{R}^n$ defined by:

$$\psi_i^{\mathcal{K}}(v) = \sum_{\substack{S \subseteq N: \\ S \ni i}} \frac{D(v, S)}{|S|}, \quad \text{for all } i \in N. \tag{3}$$

3 Characterization of the R-value

Throughout this section we fix the partial collection \mathcal{K} , that is the family of feasible coalitions.

Next we characterize the R-value $\psi^{\mathcal{K}}$ by means of three axioms: Carrier, Symmetric-partnership, and Additivity. Before introducing these axioms some definitions are needed.

A coalition S is said to be a *zero-coalition* in $v \in \mathcal{G}^{\mathcal{K}}$ if

$$T \subseteq S \text{ and } T \in \mathcal{K} \implies v(T) = 0. \tag{4}$$

That is, S is a zero-coalition if all its feasible subcoalitions are powerless.

A coalition $M \in \mathcal{K}$ is said to be a *carrier* of $v \in \mathcal{G}^{\mathcal{K}}$ if for all $S \in \mathcal{K}$ it holds:

$$(i) S \cap M \in \mathcal{K} \text{ implies } v(S) = v(S \cap M), \text{ and} \tag{5}$$

$$(ii) S \cap M \notin \mathcal{K} \text{ implies } S \text{ is a zero-coalition.} \tag{6}$$

According to this definition, players outside a carrier M have no influence in the game. Indeed, the first condition above says that when a feasible coalition S is formed by joining coalition $S \setminus M$ outside of M with the feasible coalition $S \cap M$ inside of M , then the players outside of M do not make any contribution at all. While the second condition requires for M to be a carrier that in case $S \cap M$ is an unfeasible coalition, then all feasible coalitions included in S are powerless, so they do not make any contribution either.

Note that any feasible superset of a carrier of v is also a carrier of v , and consequently N is always a carrier. Furthermore, in the case of full game, i.e. when $\mathcal{K} = 2^N$, condition (ii) above does not apply, and a carrier coincides with the traditional definition of carrier for full games (Shapley, 1953).

Axiom 1 (Carrier) *If M is a carrier of $v \in \mathcal{F}$, then*

$$\sum_{i \in M} \phi_i(v) = v(M). \tag{7}$$

The Carrier axiom states that the players in a carrier distribute the full income of the carrier between them, since the rest of the players have no influence in the game.

A non-empty coalition $P \subseteq N$ is said to be a *coalition of partners*, or a *p-type coalition*, in $v \in \mathcal{G}^{\mathcal{K}}$ if for all $S \in \mathcal{K}$ such that $P \setminus S \neq \emptyset$ it holds:

$$(i) S \setminus P \in \mathcal{K} \text{ implies } v(S) = v(S \setminus P), \text{ and} \tag{8}$$

$$(ii) S \setminus P \notin \mathcal{K} \text{ implies } S \text{ is a zero-coalition.} \tag{9}$$

This definition can be interpreted as follows. If P is a p-type coalition in the R-game v , then all its proper subcoalitions cannot make any contribution to any coalition outside P . That is, coalition P behaves like one individual in the game, since all its proper subcoalitions are completely powerless.

Note that if v is a full game, i.e. $\mathcal{K} = 2^N$, condition (ii) above is empty, and this definition will coincide with the definition of partnership coalition, or p-type coalition, introduced by Kalai and Samet (1988), when they characterized the weighted Shapley values.

Axiom 2 (Symmetric-Partnership) *If P is a p-type coalition in $v \in \mathcal{F}$, then*

$$\phi_i(v) = \phi_j(v) \text{ for all } i, j \in P. \tag{10}$$

One can expect that a p-type coalition P , that behaves as one individual in the R-game v , will take its share and then its members will distribute this share equally among them. This is what Symmetric-partnership requires. Moreover, note that in the case of full games, the Symmetric-partnership axiom is weaker than the traditional Anonymity axiom [Axiom 1 in Shapley (1953)]. Actually the Anonymity axiom is of little use when dealing with R-games, since a partial collection may be not invariant under a permutation of N .

The sum of two R-games $v, w \in \mathcal{G}^{\mathcal{K}}$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in \mathcal{K}$.

Axiom 3 (Additivity) *If $v, w \in \mathcal{F}$, then*

$$\phi(v + w) = \phi(v) + \phi(w). \tag{11}$$

The axiom of Additivity is the adaptation of the corresponding axiom for conventional cooperative games.

The main result in this paper is the following:

Theorem 1 *There is a unique solution on $\mathcal{G}^{\mathcal{K}}$ that satisfies the Carrier, Symmetric-partnership and Additivity axioms, and it is the R-value $\psi^{\mathcal{K}}$.*

We shall prove it through some lemmas and propositions.

Lemma 1 *Let $v \in \mathcal{G}^{\mathcal{K}}$.*

- (i) *The R-value $\psi^{\mathcal{K}}$ satisfies $\sum_{i \in N} \psi_i^{\mathcal{K}}(v) = v(N)$.*
- (ii) *If S is a zero-coalition in v then $D(v, S) = 0$.*
- (iii) *If M is a carrier of v and S is a coalition such that $S \cap M = \emptyset$, then $D(v, S) = 0$.*
- (iv) *If M is a carrier of v and S is a coalition such that $S \setminus M \neq \emptyset$, then $D(v, S) = 0$.*

Proof (i) Observe first that $D(v, T) = 0$ whenever $T \notin \mathcal{K}$, and hence from expression (2) for the case $S = N$ we get

$$\sum_{i \in N} \psi_i^{\mathcal{K}}(v) = \sum_{i \in N} \sum_{\substack{T \in \mathcal{K}: \\ i \in T}} \frac{D(v, T)}{|T|} = \sum_{T \in \mathcal{K}} D(v, T) = v(N). \tag{12}$$

- (ii) It is straightforward.
- (iii) It follows immediately from the definition of carrier.
- (iv) Let M be a carrier of v , and S a coalition such that $S \setminus M \neq \emptyset$. If $S \notin \mathcal{K}$, then $D(v, S) = 0$ by definition.

Assume now that $S \in \mathcal{K}$. We show that $D(v, S) = 0$ by induction on the cardinality of S . If $|S| \leq 1$ it is immediate. So let $|S| \geq 2$, and consider two possibilities:

1. $S \cap M \notin \mathcal{K}$. Since M is a carrier then S is a zero-coalition and part (ii) of this lemma applies.
2. $S \cap M \in \mathcal{K}$. By expression (2) we get

$$D(v, S) = v(S) - \sum_{T \subsetneq S} D(v, T) = v(S) - \sum_{\substack{T \subsetneq S \\ T \setminus M \neq \emptyset}} D(v, T) - \sum_{\substack{T \subsetneq S \\ T \subseteq S \cap M}} D(v, T). \tag{13}$$

By the induction hypothesis if $T \subsetneq S$ and $T \setminus M \neq \emptyset$ then $D(v, T) = 0$. Moreover, by expression (2), $\sum_{T \subseteq S \cap M} D(v, T) = v(S \cap M)$. So expression (13) becomes

$$D(v, S) = v(S) - v(S \cap M). \tag{14}$$

Since M is a carrier and $S \cap M \in \mathcal{K}$, we have $v(S) = v(S \cap M)$, and this concludes the proof. □

Lemma 2 *The R-value $\psi^{\mathcal{K}}$ satisfies the Carrier axiom on $\mathcal{G}^{\mathcal{K}}$.*

Proof Let $v \in \mathcal{G}^{\mathcal{K}}$ and M a carrier of v .

If $i \notin M$ and $i \in S$, by parts (iii) and (iv) of Lemma 1, it must be $D(v, S) = 0$. So $\psi_i^{\mathcal{K}}(v) = 0$ for all $i \notin M$. Consequently by part (i) of the same lemma, and the fact that M is a carrier we have

$$\sum_{i \in M} \psi_i^{\mathcal{K}}(v) = \sum_{i \in N} \psi_i^{\mathcal{K}}(v) = v(N) = v(M \cap N) = v(M), \tag{15}$$

as desired. □

Lemma 3 *Let S be a p-type coalition of $v \in \mathcal{G}^{\mathcal{K}}$. If $T \in \mathcal{K}$ is such that $S \cap T \neq \emptyset$ and $S \setminus T \neq \emptyset$, then $D(v, T) = 0$.*

Proof By induction on $|T|$. If $|T| = 1$, then $T \setminus S = \emptyset \in \mathcal{K}$, and by definition of p-type coalition $v(T) = v(T \setminus S) = v(\emptyset) = 0$, and hence $D(v, T) = 0$.

Now assume $|T| \geq 2$. From expression (2) we have

$$D(v, T) = v(T) - \sum_{R \subsetneq T} D(v, R) = v(T) - \sum_{\substack{R \subsetneq T \\ R \cap S \neq \emptyset}} D(v, R) - \sum_{R \subseteq T \setminus S} D(v, R), \tag{16}$$

If $R \subsetneq T$, then $S \setminus T \neq \emptyset$ implies $S \setminus R \neq \emptyset$. And by the induction hypothesis we get $\sum_{\substack{R \subsetneq T \\ R \cap S \neq \emptyset}} D(v, R) = 0$. Thus expression (16) becomes

$$D(v, T) = v(T) - \sum_{R \subseteq T \setminus S} D(v, R). \tag{17}$$

Now consider two cases. If $T \setminus S \in \mathcal{K}$, since S is a p-type coalition we get $v(T) = v(T \setminus S)$, and the second term in the equality above is zero by expression (2). Otherwise, if $T \setminus S \notin \mathcal{K}$, then T is a zero-coalition and part (ii) of Lemma 1 applies. This completes the proof. □

Lemma 4 *The R-value ψ^K satisfies the Symmetric-partnership axiom on \mathcal{G}^K .*

Proof Let S be a p-type coalition in $v \in \mathcal{G}^K$, and $i \in S$. If $i \in T$ and $S \not\subseteq T$, from Lemma 3 it holds $D(v, T) = 0$. Consequently

$$\psi_i^K(v) = \sum_{\substack{T \in \mathcal{K}: \\ i \in T}} \frac{D(v, T)}{|T|} = \sum_{\substack{T \in \mathcal{K}: \\ S \subseteq T}} \frac{D(v, T)}{|T|}. \tag{18}$$

But this expression is the same for every player in S , thus the lemma is established. □

Lemma 5 *The R-value ψ^K satisfies the Additivity axiom on \mathcal{G}^K .*

Proof Observe that $D(v + w, S) = D(v, S) + D(w, S)$ for all $S \subseteq N$. Then the lemma follows immediately. □

We shall prove the uniqueness through the following lemmas and propositions.

Lemma 6 *If M is a carrier on $v \in \mathcal{G}^K$, then $N \setminus M$ is a p-type coalition in v .*

Proof Just notice that if T is a coalition, then $T \setminus (N \setminus M) = M \cap T$, and the lemma immediately follows from the definitions of carrier and p-type coalition. □

Lemma 7 *Let $\phi : \mathcal{G}^K \rightarrow \mathbf{R}^N$ be a solution that satisfies the Carrier and Symmetric-partnership axioms, and $v \in \mathcal{G}^K$. If M is a carrier of v , then $\phi_i(v) = 0$ for every $i \in N \setminus M$.*

Proof Since M and N are both carriers we have

$$\sum_{i \in M} \phi_i(v) = v(M) = v(M \cap N) = v(N) = \sum_{i \in N} \phi_i(v), \tag{19}$$

and consequently

$$\sum_{i \in N \setminus M} \phi_i(v) = 0. \tag{20}$$

On the other hand, if M is a carrier then Lemma 6 together with the Symmetric-partnership axiom imply $\phi_i(v) = \phi_j(v)$ for every $i, j \in N \setminus M$. This together with equality (20) completes the proof. □

For every $T \in \mathcal{K}$, $T \neq \emptyset$ define the unanimity R-game u_T^K for every $S \in \mathcal{K}$ by

$$u_T^K(S) = \begin{cases} 1, & \text{if } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Proposition 1 *The set $\{u_T^K : T \in \mathcal{K}\}$ is a basis of the vector space \mathcal{G}^K .*

Proof There are $|\mathcal{K}| - 1$ unanimity R-games on \mathcal{K} and the dimension of \mathcal{G}^K is also $|\mathcal{K}| - 1$. So it is enough to prove that these R-games are linearly independent. By contradiction, assume that $\sum_{T \in \mathcal{K}} \alpha_T u_T^K = 0$, with $\alpha_T \in \mathbf{R}$ not all zero. Let T_0 be a minimal set in $\{T \in \mathcal{K} : T \neq \emptyset, \alpha_T \neq 0\}$. Then $(\sum_{T \in \mathcal{K}} \alpha_T u_T^K)(T_0) = \alpha_{T_0} \neq 0$, which is the desired contradiction. □

Proposition 2 *Let $\phi : \mathcal{G}^K \rightarrow \mathbf{R}^N$ be a solution that satisfies the Carrier and Symmetric-partnership axioms, $c \in \mathbf{R}$, and $T \in \mathcal{K}$, then it holds*

$$\phi_i(c \cdot u_T^K) = \begin{cases} c/|T|, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases} \tag{22}$$

Proof First notice that T is a carrier on $c \cdot u_T^K$, thus from Lemmas 6 and 7, we have $\phi_i(c \cdot u_T^K) = 0$ for every $i \in N \setminus T$. On the other hand T is also a p-type coalition for u_T^K , so $\phi_i(c \cdot u_T^K) = \phi_j(c \cdot u_T^K)$ for every $i, j \in T$. Then the result easily follows from the Carrier axiom. \square

Proof of Theorem 1 Lemmas 2, 4 and 5 show that ψ^K satisfies the three axioms. Uniqueness follows from propositions 1 and 2 together with the Additivity axiom. \square

Remark 1 As mentioned in the Introduction, Aguilera et al. (2010) propose two approaches to extend the Shapley value to R-games. The first one does not guarantee uniqueness and gives rise to the family of marginalist solutions. In the second one, Aguilera et al. (2010) define a particular solution on the unanimity games u_T^K and extend this solution by linearity to the whole vector space \mathcal{G}^K . It turns out that this particular solution coincides with ψ^K on every unanimity game. Therefore by Proposition 2, it is clear that the R-value ψ^K coincides with the unanimity solution of Aguilera et al. (2010) on \mathcal{G}^K .

4 Simple R-games

Von Neumann and Morgenstern (1944) pioneered the use of simple games to study the distribution of power in voting systems in their classic “Theory of Games and Economic Behavior”. Later, Shapley and Shubik (1954) suggested to use the restriction of the Shapley value to the domain of simple games to measure the voting power of the players. This traditional way of measuring power, however, does not take into account that some coalitions may not be feasible, possibly because of the issues at stake or the individual positions of the players concerning these issues. An alternative to analyze these voting situations is to consider simple R-games and make use of the restriction of the R-value ψ^K to simple R-games.

An R-game v on a partial collection \mathcal{K} is called *simple* if: (1) it assumes only the values 0 and 1; (2) it is monotonic; that is $S, T \in \mathcal{K}$ together with $S \subseteq T$ imply $v(S) \leq v(T)$; and (3) it is not identical to zero.

Coalitions whose worth is 1 are called winning, and losing otherwise.

The set of all simple R-games on \mathcal{K} is denoted S^K .

A *power index* on S^K is any function $\eta : S^K \rightarrow \mathbf{R}^N$.

Let $v, w \in S^K$, define the operations $v \wedge w$ and $v \vee w$ by

$$(v \wedge w)(S) = \min \{v(S), w(S)\}, \quad (v \vee w)(S) = \max \{v(S), w(S)\} \tag{23}$$

Let η be a power index on S^K . Consider the following axiom.

Axiom 4 (Transfer) *If $v, w \in S^K$, then*

$$\eta(v \wedge w) + \eta(v \vee w) = \eta(v) + \eta(w). \tag{24}$$

Dubey (1975) characterized on the class of full simple games the Shapley–Shubik index by replacing the Additivity axiom in Shapley (1953) system with the Transfer axiom above. One may wonder if it is possible to derive also the value ψ^K on S^K by using the Transfer axiom in Theorem 1, paralleling Dubey (1975) result. The answer is positive as it is shown in the following theorem.²

² It turns out that Dubey (1975) characterization for full simple games relies on the fact that every coalition is feasible, and consequently his proof cannot be directly applied to the case of R-games.

Theorem 2 *Let \mathcal{K} be a partial collection, then there is a unique index on $\mathcal{S}^{\mathcal{K}}$ that satisfies the Carrier, Symmetric-partnership and Transfer axioms, and it is the restriction of $\psi^{\mathcal{K}}$ to this subclass of R-games.*

Proof Let \mathcal{K} be a partial collection. By Lemmas 2 and 4 it is clear that $\psi^{\mathcal{K}}$ satisfies Carrier and Symmetric-partnership on $\mathcal{S}^{\mathcal{K}}$. The fact that $\psi^{\mathcal{K}}$ satisfies Transfer on $\mathcal{S}^{\mathcal{K}}$ follows from Lemma 5 and the equality

$$(v \wedge w) + (v \vee w) = v + w. \tag{25}$$

Next, we turn to prove uniqueness. Let η be a power index on $\mathcal{S}^{\mathcal{K}}$ that satisfies Carrier, Symmetric-partnership and Transfer axioms. We will prove that $\eta(v) = \psi^{\mathcal{K}}(v)$ for every $v \in \mathcal{S}^{\mathcal{K}}$ by a double induction as follows. First notice that v has a finite number of minimal winning coalitions, S_1, S_2, \dots, S_m , such that $v = u_{S_1}^{\mathcal{K}} \vee u_{S_2}^{\mathcal{K}} \vee \dots \vee u_{S_m}^{\mathcal{K}}$. Define the 1st-indicator I of v to be the minimum cardinality of the minimal winning coalitions of v . Define the 2nd-indicator J of v the number of minimal winning coalitions of v whose cardinality is the 1st-indicator I of v . We prove that $\eta(v) = \psi^{\mathcal{K}}(v)$ by backwards induction on the 1st-indicator I and forward induction on the 2nd-indicator J .

If $I = n$ then $v = u_N^{\mathcal{K}}$, and similarly to Proposition 2 we get $\eta(u_N^{\mathcal{K}}) = \psi^{\mathcal{K}}(u_N^{\mathcal{K}})$. Assume now that $\eta(v) = \psi^{\mathcal{K}}(v)$ whenever the 1st-indicator of v is at least I .

Now let v be any simple R-game that has 1st-indicator $I - 1$. We shall prove that $\eta(v) = \psi^{\mathcal{K}}(v)$ by induction on the 2nd-indicator J of v .

If $J = 1$, then v has a unique minimal winning coalition of cardinality $I - 1$, say S_1 . Denote $v_1 = u_{S_1}^{\mathcal{K}}$ and $v_2 = u_{S_2}^{\mathcal{K}} \vee \dots \vee u_{S_m}^{\mathcal{K}}$. By Proposition 2 we have $\eta(v_1) = \psi^{\mathcal{K}}(v_1)$. Moreover, the 1st-indicators of v_2 and $v_1 \wedge v_2$ are clearly greater or equal than I , so by the induction hypothesis on I , it holds $\eta(v_2) = \psi^{\mathcal{K}}(v_2)$ and $\eta(v_1 \wedge v_2) = \psi^{\mathcal{K}}(v_1 \wedge v_2)$. Hence by the Transfer axiom, equality (25), and additivity of $\psi^{\mathcal{K}}$ we obtain

$$\begin{aligned} \eta(v) &= \eta(v_1 \vee v_2) = \eta(v_1) + \eta(v_2) - \eta(v_1 \wedge v_2) \\ &= \psi^{\mathcal{K}}(v_1) + \psi^{\mathcal{K}}(v_2) - \psi^{\mathcal{K}}(v_1 \wedge v_2) = \psi^{\mathcal{K}}(v_1 \vee v_2) = \psi^{\mathcal{K}}(v). \end{aligned} \tag{26}$$

And therefore the index η coincides with the R-value whenever $J = 1$.

Assume now that η coincides with the R-value for every game such that the number of minimal winning coalitions of cardinality $I - 1$ is at most J . Now let v have 1st-indicator $I - 1$ and $J + 1$ coalitions of cardinality $I - 1$. Then v can be written as $v_1 \vee v_2$, where $v_1, v_2 \in \mathcal{S}^{\mathcal{K}}$ have 1st-indicator at least $I - 1$ and 2nd-indicator at most J (for instance, taking $v_1 = u_{S_1}^{\mathcal{K}}$ and $v_2 = u_{S_2}^{\mathcal{K}} \vee \dots \vee u_{S_m}^{\mathcal{K}}$ as before). Since the game $v_1 \wedge v_2$ has also a 1st-indicator at least I . Using the same argument as in (26) we get $\eta(v) = \psi^{\mathcal{K}}(v)$. And the proof is complete. \square

5 One case study: the Catalanian Parliament, 1980–1984

In this section we will apply the R-value in a real application: The Catalanian Parliament after the 1980 elections. This was also analyzed in Carreras and Owen (1988), where they used the Shapley value as a measure of the power. However, the ideological location of the parties suggested that some coalitions were more likely than others. Consequently, these authors also employed a modification of the Shapley value, the coalitional value (Owen, 1977), more suitable for studying games with a priori unions of players. In this section, we will make use of the R-value as a power index. That is, we model the Catalanian Parliament as a simple

R-game, by considering that some coalitions might not actually form, because of the location of the parties on the ideological map, and then will use the R-value as a power index.

First a brief description of the situation provided by Carreras and Owen (1988) is given. The Catalanian Parliament consists of 135 seats, so any coalition of 68 members or more wins. The results in the 1980 elections are in Table 1.

In Table 2 it is included a concise explanation of the parties ideology at that time taken from Carreras and Owen (1988).

The minimal winning coalitions of the corresponding simple game are five: namely, {CiU, PSC}, {CiU, PSUC}, {CiU, CC, ERC}, {CiU,ERC,PSUC} and {PSC, PSUC, ERC}. Carreras and Owen (1988) calculated the Shapley value of this simple game, which can be found in the first column of Table 3 as a first approximation to measure the influence of the parties involved. More interestingly, they also calculated the coalitional Shapley value for several coalition structures. The PSA party was a dummy player and was not taken into account in the analysis. According to Carreras and Owen (1988), two proto-coalitions could form representing the right-center and the left-center of the map: {CiU,CC} and {PSC,PSUC} respectively, thus leaving to ERC an enviable position. The indices associated with these two cases of a priori unions are in the second and third columns of Table 3.

Instead, in this paper we have considered the case where not all coalitions can be formed by incompatibilities or different attitudes towards the issues at stake. Consequently, coalitions of parties at the ends of the political spectrum have been considered non-feasible, unless some other intermediate party is also present in the coalition. So we have discarded as non-viable the coalitions to which PSUC (left) belongs but neither PSC nor ERC (both centered left) belong. That is, we have considered as non-feasible the following coalitions: {CiU, PSUC},

Table 1 Results in the 1980 elections of the Catalanian Parliament

Party	CiU	PSC	PSUC	CC	ERC	PSA
Members	43	33	25	18	14	2

Table 2 Ideological description of the parties at the Catalanian Parliament in 1980

CiU	Nationalist right centered coalition
PSC	Left centered party associated with the PSOE, the main opposition party in Spain
PSUC	Communist party with similar policies to the PCE, the communist party at national level
CC	Right centered party associated with the UCD, the governing party in Spain
ERC	Moderate left centered nationalist party
PSA	Radical left-wing party trying to represent the Andalusian immigrants in Catalonia

Table 3 Different power indices in the Catalanian Parliament in 1980

	Initial game	{{CiU, CC}, ERC}	{{PSC, PSUC}, ERC}	R-value
CiU	0.4000	0.5000	0	0.3667
PSC	0.2333	0	0.3333	0.2833
ERC	0.0667	0.3333	0.3333	0.1167
PSUC	0.2333	0	0.3333	0.0333
CC	0.0667	0.1667	0	0.2
PSA	0	0	0	0

{PSUC, CC}, {CiU, PSUC, CC}. Similarly we have excluded also the coalitions to which CC (right) belongs, but CiU (centered right) does not. Thus we have also considered the following coalitions as non-viable: {PSC, CC}, {CC, ERC}, {PSC, PSUC, CC}, {PSC, CC, ERC}, and {PSC, PSUC, CC, ERC}. Furthermore, the PSA has not any influence in the game,³ so for simplicity we have excluded all coalitions to which this party belongs.⁴

With the viable coalitions we have constructed a R-game, in which the winning coalitions get 1 and the losing one get 0. In this R-game there are only four minimal winning coalitions: {CiU, PSC}, {CiU, PSUC, ERC}, {CiU, CC, ERC} and {PSC, PSUC, ERC}. The R-value for this partial collection is on the last column of Table 3.

By comparing the Shapley value of the initial game with the R-value, one can observe that the more centered parties, i.e. CiU, PSC and ERC in those years, are assigned a higher power index by the R-value than the less centered parties PSUC and CC. Furthermore, note that despite of the fact that PSUC and CC had considerably more seats than ERC, the R-value assigns a higher index to ERC, since its ideological location was less extreme. This seems more reasonable than the Shapley value of the initial game.

The coalitional values (second and third columns) obtained by Carreras and Owen (1988) refer to considerations in which the final coalitions have already been formed. In order to compare the coalitional values with the R-value, it would be interesting to define the “coalitional R-value”, but this is out of the scope of this work.

6 Conclusions

In this paper we have characterized a value proposed by Calvo and Gutiérrez-López (2015) for games with restricted cooperation. We called this value the R-value, and its definition is based on Harsanyi (1963) procedure for finding a solution in the context of non-transferable utility games.

Some remarks are in order. Myerson (1977) characterized a value in the context of graph communication situations by means of Component-Efficiency and Fairness. Later, Myerson (1980) generalized this result to more general situations in which players are organized in *conference structures*, that is non-empty coalitions that are not singletons. It is worth to note that Algaba et al. (2001a) extended the Myerson value with graph communication situations to *union stable systems*, and characterized this value also by means of Component-Efficiency and Fairness. Union stable systems refer to situations in which the cooperation among the players is restricted, but in such a way that if two feasible coalitions have some players in common, they can act as intermediaries between these two coalitions, and hence the union of two feasible coalitions will be also feasible. The relation between conference structures and union stable systems is based on the relation of *connectedness* between players. According to Myerson (1980), given a family Q of permissible coalitions, “two players are connected by Q if they can be coordinated either by meeting together in some permissible coalition to which they both belong, or by meeting in separate coalitions which have some members in common to serve as intermediaries, or by some longer sequence of overlapping conferences” (see also Definition 2.5 and Theorem 2.3 in Algaba et al. (2001a)).

³ Note that the fact that the PSA party was dummy in the full game does not imply that the PSA party is dummy in any R-game obtained by restricting the full game to a partial collection of coalitions. The reason is that the PSA party may have a non-zero marginal contribution by turning a non-feasible coalition in a feasible coalition with its presence.

⁴ The exclusion of this party does not affect any calculation in this example.

The aim of Myerson (1980) was to understand how players organize themselves in conferences to negotiate and make joint plans that will ultimately determine the final result. Consequently, in the definition of an allocation rule, this author fixes a full game and the player's payoffs depend on each conference structure. Unlike Myerson's approach, in ours the payoffs depends on the game at stake, whereas the feasible family of coalitions \mathcal{K} is fixed; that is, the value represents an evaluation of player i of her prospect for playing a particular game.

It is worthwhile also to compare the characterization offered in this paper with that of Calvo and Gutiérrez-López (2015). These authors characterize the value using two axioms. The first axiom is a modification of the Component Efficiency axiom where the dividends appear explicitly in its definition. On the other hand, the axiom of Fairness requires that the value will be fair similarly as in Myerson (1980), that is, two players should obtain the same benefits (or losses) from their joint cooperation. Fairness is a complex requirement that compares the value on different sets of feasible coalitions. The axiomatization presented in this article offers the advantage of being formed by a more elementary and characteristic set of axioms when the family of feasible coalitions is fixed.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. Funding was provided by UPV/EHU (GIU17/0151 and GIU20/019), and Ministerio de Ciencia, Innovación y Universidades (Grant No. PID2019-105291GB-I00).

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Aguilera, N. E., Di Marco, S. C., & Escalante, M. S. (2010). The Shapley value for arbitrary families of coalitions. *European Journal of Operational Research*, 204, 125–138.
- Algaba, E., Bilbao, J. M., Borm, P., & López, J. J. (2000). The position value for union stable systems. *Mathematical Methods of Operational Research*, 52, 221–236.
- Algaba, E., Bilbao, J. M., Borm, P., & López, J. J. (2001a). The Myerson value for union stable structures. *Mathematical Methods of Operational Research*, 54, 359–371.
- Algaba, E., Bilbao, J. M., & López, J. J. (2001b). A unified approach to restricted games. *Theory and Decision*, 50, 333–345.
- Algaba, E., Bilbao, J. M., van den Brink, R., & Jiménez-Losada, A. (2003). Axiomatizations of the Shapley value for cooperative games on antimatroids. *Mathematical Methods of Operations Research*, 57, 49–65.
- Algaba, E., Fragnelli, V., & Sanchez-Soriano, J. (2019). Handbook of the Shapley Value. CRC Press. <https://doi.org/10.1201/9781351241410>.
- Álvarez-Mozos, M., Hellman, Z., & Winter, E. (2013). Spectrum value for coalitional games. *Games and Economic Behavior*, 82, 132–142.
- Calvo, E., & Gutiérrez-López, E. (2015). The value in games with restricted cooperation. Discussion Papers in Economic Behavior 01/15. University of Valencia.
- Carreras, F., & Owen, G. (1988). Evaluation of the Catalanian Parliament, 1980–1984. *Mathematical Social Sciences*, 15, 87–92.
- Dubey, P. (1975). On the uniqueness of the Shapley value. *International Journal of Game Theory*, 4, 131–139.
- Faigle, U. (1989). Cores of games with restricted cooperation. *Zeitschrift für Operations Research*, 33, 405–422.

- Harsanyi, J. C. (1963). A simplified bargaining model for the n -person cooperative game. *International Economic Review*, 4, 194–220.
- Kalai, E., & Samet, D. (1988). Weighted Shapley values. In A. E. Roth (Ed.), *The Shapley value: Essays in honor of Lloyd S. Shapley* (pp. 83–99). Cambridge University Press.
- Keinan, A., Sandbank, B., Hilgetag, C. C., Meilijson, I., & Ruppin, E. (2004). Fair attribution of functional contribution in artificial and biological networks. *Neural Computation*, 16, 1887–1915.
- Lange, F., & Grabisch, M. (2009). Values on regular games under Kirchoff's laws. *Mathematical Social Sciences*, 58, 322–340.
- Loehman, E., Orlando, J., Tschirhart, J., & Whinston, A. (1979). Cost allocation for a regional waste water treatment system. *Water Resources Research*, 15, 193–202.
- Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2, 225–229.
- Myerson, R. B. (1980). Conference structures and fair allocation rules. *International Journal of Game Theory*, 9, 169–182.
- Owen, G. (1977). Values of games with a priori unions. In *Mathematical economics and game theory* (pp. 76–88). Springer.
- Sajitz-Hermstein, M., & Nikoloski, Z. (2012). Restricted cooperative games on metabolic networks reveal functionally important reactions. *Journal of Theoretical Biology*, 314, 192–203.
- Shapley, L. S. (1953). A Value for n -Person Games. In H. W. Kuhn & A. W. Tucker (Eds.), *Contributions to the theory of games* (Vol. 2, pp. 307–317). Princeton University Press.
- Shapley, L. S., & Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48, 787–792.
- van den Brink, R. (2012). On hierarchies and communication. *Social Choice and Welfare*, 39, 721–735.
- Von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton University Press.
- Willson, S. J. (1993). A value for partially defined cooperative games. *International Journal of Game Theory*, 21, 371–384.
- Young, H. P., Okada, N., & Hashimoto, N. (1982). Cost allocation in water resources development. *Water Resources Research*, 18, 453–475.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.