



Robust Pareto solutions for convex quadratic multiobjective optimization problems under data uncertainty

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Abstract

In this paper, we consider a convex quadratic multiobjective optimization problem, where both the objective and constraint functions involve data uncertainty. We employ a deterministic approach to examine robust optimality conditions and find robust (weak) Pareto solutions of the underlying uncertain multiobjective problem. We first present new necessary and sufficient conditions in terms of linear matrix inequalities for robust (weak) Pareto optimality of the multiobjective optimization problem. We then show that the obtained optimality conditions can be alternatively checked via other verifiable criteria including a robust Karush–Kuhn–Tucker condition. Moreover, we establish that a (scalar) relaxation problem of a robust weighted-sum optimization program of the multiobjective problem can be solved by using a semidefinite programming (SDP) problem. This provides us with a way to numerically calculate a robust (weak) Pareto solution of the uncertain multiobjective problem as an SDP problem that can be implemented using, e.g., MATLAB.

Keywords Multiobjective program · Robust optimization · Optimality condition · Semidefinite programming · Relaxation

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1 Introduction

Multiobjective (or multiple objective/multicriteria/vector) optimization problems refer to mathematical optimization models that can handle the multiple decision making problems and have been found in applications in various areas of real life and work such as defence, engineering, economics and logistics (Nguyen and Cao, 2019; Ehrgott, 2005; Jahn, 2004; Miettinen, 1999; Steuer, 1986; Luc, 1989). Multiobjective optimization problems are often involved in the presence of trade-offs between two or more conflicting objectives from a set of feasible choices. Consequently, identifying Pareto fronts or Pareto solutions that are trade-offs among the conflicting objectives of a nontrivial multiobjective optimization program is generally hard by its natural structure (Wiecek, 2007; Boğ et al. 2009.; Chinchuluun and Pardalos, 2007; Zhou et al., 2011; Niebling and Eichfelder, 2019).

A decision making problem that arises in defence, for example, may contain parameters that are unknown at the time when decision must be made, e.g., future technology, threats, costs and environments and these parameters are often obtained by prediction or estimation. Unknown environment factors of a defence force problem such as land surface, air and sea surroundings or weather would be shown up when a defence system is operating in a new terrain. The land combat vehicle (LCV) system (Nguyen and Cao, 2017, 2019) is formulated as a multiobjective optimization problem that contains *uncertainty data* caused by incomplete and noisy information or by unknown environment factors, and also by the interdependencies between LCV system configurations. Force Design (FD) is ultimately about selecting a portfolio of thousands of strategic investment projects in future Australian Defence Force (ADF) capabilities (Peacock et al., 2019). The challenge is to optimise the combination of invested projects that give government the overall capability it requires to achieve multiple effects (objectives) across a range of scenarios, whilst achieving the budget constraints.

This paper is the first step in extending some earlier approaches successfully used for evaluating and optimising defence problems such as the LCV system specifications and configurations for Australian Army, to more complex problems with higher dimensions that are relevant to force design (Nguyen and Cao, 2019; Nguyen et al., 2016; Nguyen and Cao, 2017) by considering a class of uncertain multiobjective optimization problems where the objectives and constraints are continuous functions and uncertain parameters also influence its search space. Robust approaches (or, robust optimization) (see e.g., Ben-Tal et al., 2009; Bertsimas et al., 2011; Ehrgott et al., 2014; Kuroiwa and Lee, 2012) have emerged as promising and efficient framework for solving mathematical programming problems under data uncertainty, and so a robust optimisation approach will be used instead of the heuristic methods from the previous work to handle this class of uncertain multiobjective optimization problems. It is worth mentioning that the robust approach of (scalar) *uncertain* programs was first considered in Soyster (1973). This is based on the principle that the robust counterpart of an uncertain program that enforced for all possible parameter realizations within the underlying uncertainty sets admits feasible solutions that are *immunized against* uncertainty data.

Nowadays, the robust approach has been deployed in real-world multiobjective optimization problems involving uncertain data (Ehrgott et al., 2014; Kuroiwa and Lee, 2012) as the robust multiobjective models better satisfy future performance demands in practical scenar-

ios, see e.g., Zhou-Kangas and Miettinen (2019), Ide and Schobel (2016), Doolittle et al. (2018). The concept of *minmax robustness* from Ben-Tal et al. (2009) was extended to the setting of multiobjective optimization in Ehrgott et al. (2014), Kuroiwa and Lee (2012). Moreover, one can see in Ehrgott et al. (2014) how robust solutions of an uncertain multiobjective optimization problem are calculated, and in Kuroiwa and Lee (2012) how optimality conditions of robust multiobjective optimization problems are established. Over the years, different and various research aspects of robust multiobjective optimization have been intensively studied and developed by many researchers. For example, Goberna et al. (2014) provided dual characterizations of robust weak Pareto solutions of a multiobjective linear semi-infinite programs under *constraint data uncertainty* (i.e, the problem involves uncertain parameters only in its constraints). Georgiev et al. (2013) established a necessary and sufficient condition and provided formulas to compute the *radius of robustness* of a linear multiobjective programming problem. In a *nonsmooth and nonlinear* setting, Zamani et al. (2015) provided formulas for calculating such radius of robustness. These results were further developed for semi-infinite multiobjective programs in Rahimi and Soleimani-damaneh (2018a) and for general robust vector optimization problems in Rahimi and Soleimani-damaneh (2018b).

By considering the objectives of an uncertain multiobjective optimization problem as set-valued functions, Eichfelder et al. (2020) proposed a numerical algorithm to calculate the optimal solutions of an unconstrained multiobjective optimization problem in which the objective functions are *twice continuously differentiable*. Engau and Sigler (2020) established a complete characterization of an alternative efficient set hierarchy based on different ordering relations with respect to the multiple objectives and scenarios. Lee and Lee (2018) examined optimality conditions and duality relations for weak efficient solutions of robust semi-infinite multiobjective optimization problems, where the related functions are *continuously differentiable*. Employing advanced techniques of variational analysis, Chuong (2020) obtained robust optimality conditions and robust duality for an uncertain *nonsmooth* multiobjective optimization problem under arbitrary uncertainty nonempty sets.

One of the recent research trends in robust multiobjective optimization has focused on exploiting special structures of the objective and constraint functions or the uncertainty sets to provide verifiable optimality conditions and calculate robust (weak) Pareto solutions by using semidefinite programming (SDP) (see e.g., Blekherman et al., 2012) problems. Chuong (2017) provided a characterization for weak Pareto solution of a robust *linear* multiobjective optimization problem based on a robust alternative theorem. In the setting of *sum of squares* (SOS)-convex polynomials (Helton and Nie, 2010), Chuong (2018) examined optimality conditions and duality by means of *linear matrix inequalities* for robust multiobjective SOS-convex polynomial programs, while Lee and Jiao (2021) employed an approximate scalarization technique to find robust Pareto solutions for a multiobjective optimization problem under the constraint data uncertainty. The interested reader is referred to a very recent work (Chuong and Jeyakumar, 2021) for calculating robust (weak) Pareto solutions for uncertain *linear* multiobjective optimization problems in terms of two-stage adjustable variables.

To the best of our knowledge, there are no results establishing verifiable optimality conditions and finding numerically robust (weak) Pareto solutions of an uncertain convex quadratic multiobjective optimization problem, where uncertain parameters involve in both the objective and constraint functions and reside in structured uncertainty sets such as spectrahedra (see e.g., Ramana and Goldman, 1995). The examination of such an uncertain multiobjective optimization problem is often complicated due to the challenges posed in dealing with the data uncertainty of both the objective and constraint functions. In this work, we consider a convex quadratic multiobjective optimization problem, where both the objective and constraint

functions involve data uncertainty. In our model, the uncertain parameters in the objective and constraint functions are allowed to vary affinely in the corresponding uncertainty sets, which are spectrahedra (cf. Ramana and Goldman, 1995).

To handle the proposed *uncertain* multiobjective optimization problem, we employ the *robust* deterministic approach (see e.g., Ben-Tal et al., 2009; Bertsimas et al., 2011; Ehrgott et al., 2014; Kuroiwa and Lee, 2012) to examine robust optimality conditions and find robust (weak) Pareto solutions of the underlying uncertain multiobjective problem. More precisely, we first present necessary and sufficient conditions in terms of linear matrix inequalities for robust (weak) Pareto optimality of the multiobjective optimization problem. We then show that the obtained optimality conditions can be alternatively checked via other verifiable criteria including a robust Karush-Kuhn-Tucker condition. To achieve these goals, we employ a special variable transformation combined with a classical minimax theorem (see e.g., Sion, 1958) and a commonly used alternative theorem in convex analysis (cf. Rockafellar, 1970). Moreover, we establish that a (scalar) relaxation problem of a robust weighted-sum optimization program of the multiobjective problem can be solved by using a semidefinite programming (SDP) problem. This provides us with a way to numerically calculate a robust (weak) Pareto solution of the uncertain multiobjective problem as an SDP problem that can be implemented using, e.g., MATLAB.

It is worth mentioning here that Chuong (2018) examined a robust multiobjective optimization problem with a general class of SOS-convex polynomials but it contained *uncertain parameters* only on the constraints (i.e., the objective functions of the problem do not have uncertain parameters). Our problem deals with convex quadratic functions, where both the objective functions and constrained functions involve uncertain parameters. Due to the uncertainty data of both the objective and constraint functions, optimality conditions for robust (weak) Pareto solutions obtained in this paper would not be derived from the corresponding conditions in Chuong (2018). Even in a special framework, where there is no uncertain parameters in the objective functions of the underlying problem, we would not obtain such necessary optimality conditions by applying the results in Chuong (2018). This is because Theorem 2.3(i) of Chuong (2018) is formulated under the (classical) Slater condition, while the necessary optimality condition of the current paper (see Theorem 3.1 below) is established under a more general qualification of *closedness of characteristic cone* which can be regarded as the *weakest* regularity conditions guaranteeing strong duality in robust convex programming (Jeyakumar and Li, 2010). To make use of the closedness of characteristic cone condition in establishing necessary optimality conditions, we employ novel dual techniques based on the epigraphical convex optimization approach. Moreover, our paper proposes a scheme (see Theorem 4.1 below) to calculate a robust (weak) Pareto solution of the uncertain multiobjective problem by solving an SDP reformulation. A scheme to identify robust (weak) Pareto solutions of an uncertain multiobjective optimization problem was *not available* in Chuong (2018), where one only found *weighted-sum efficient values* for the underlying multiobjective problem.

The organization of the paper is as follows. Section 2 presents some notation and the definitions of *uncertain* and *robust* multiobjective optimization problems as well as their solution concepts that will be studied. Section 3 establishes necessary and sufficient optimality conditions along with other equivalent verifiable conditions for the underlying multiobjective optimization problem. In Sect. 4, we show that a (scalar) relaxation problem of a robust weighted-sum optimization program of the multiobjective problem can be solved by using an SDP problem. Section 5 summarizes the obtained results and provides some potential perspectives for a further study. Appendix presents numerical examples that show how we

can employ the obtained SDP reformulation scheme to find robust (weak) Pareto solutions of uncertain convex quadratic multiobjective optimization problems.

2 Preliminaries and robust multiobjective programs

Let us start with some notation and definitions. The notation \mathbb{R}^m signifies the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $m \in \mathbb{N} := \{1, 2, \dots\}$. The inner product in \mathbb{R}^m is defined by $\langle y, z \rangle := y^T z$ for all $y, z \in \mathbb{R}^m$. The origin of any space is denoted by 0 but we may use 0_m to denote the origin of \mathbb{R}^m for the sake of clarification. The nonnegative orthant of \mathbb{R}^m is denoted by $\mathbb{R}_+^m := \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, i = 1, \dots, m\}$. For a nonempty set $\Gamma \subset \mathbb{R}^m$, $\text{conv } \Gamma$ denotes the convex hull of Γ and $\text{cl } \Gamma$ (resp., $\text{int } \Gamma$) stands for the closure (resp., interior) of Γ , while $\text{coneco } \Gamma := \mathbb{R}_+ \text{conv } \Gamma$ stands for the convex conical hull of $\Gamma \cup \{0_m\}$, where $\mathbb{R}_+ := [0, +\infty) \subset \mathbb{R}$. As usual, the symbol $I_m \in \mathbb{R}^{m \times m}$ stands for the identity ($m \times m$) matrix. Let S^m be the space of all symmetric ($m \times m$) matrices. A matrix $A \in S^m$ is said to be positive semidefinite, denoted by $A \geq 0$, whenever $x^T A x \geq 0$ for all $x \in \mathbb{R}^m$. If $x^T A x > 0$ for all $x \in \mathbb{R}^m \setminus \{0_m\}$, then A is called positive definite, denoted by $A > 0$. The trace of a square ($m \times m$) matrix A is defined by $\text{Tr}(A) = \sum_{i=1}^m a_{ii}$, where a_{ii} is the entry in the i th row and i th column of A for $i = 1, \dots, m$.

For a closed convex subset $\Gamma \subset \mathbb{R}^m$, its indicator function $\delta_\Gamma : \mathbb{R}^m \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is defined as $\delta_\Gamma(x) := 0$ if $x \in \Gamma$ and $\delta_\Gamma(x) := +\infty$ if $x \notin \Gamma$. Given an extended real-valued function $\varphi : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, the domain and epigraph of φ are defined respectively by

$$\text{dom } \varphi := \{x \in \mathbb{R}^m \mid \varphi(x) < +\infty\}, \quad \text{epi } \varphi := \{(x, \mu) \in \mathbb{R}^m \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *conjugate function* of φ , denoted by $\varphi^* : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, is defined by

$$\varphi^*(v) = \sup \{v^T x - \varphi(x) \mid x \in \text{dom } \varphi\}, \quad v \in \mathbb{R}^m.$$

Letting φ_1 and φ_2 be proper lower semicontinuous convex functions, it is well known (see e.g., Zalinescu, 2002; Burachik and Jeyakumar, 2005; Rockafellar, 1970) that $\text{epi}(\lambda\varphi_1)^* = \lambda \text{epi}\varphi_1^*$ for any $\lambda > 0$ and

$$\text{epi}(\varphi_1 + \varphi_2)^* = \text{cl}(\text{epi}\varphi_1^* + \text{epi}\varphi_2^*), \quad (2.1)$$

where the closure operation is superfluous if one of φ_1, φ_2 is continuous at some point $x_0 \in \text{dom}\varphi_1 \cap \text{dom}\varphi_2$.

In this paper, we consider a convex quadratic multiobjective optimization problem under *data uncertainty* that can be captured as the following model:

$$\min_{x \in \mathbb{R}^n} \{(f_1(x, u^1), \dots, f_p(x, u^p)) \mid g_l(x, v^l) \leq 0, l = 1, \dots, k\}, \quad (\text{UP})$$

where $u^r, r = 1, \dots, p \in \mathbb{N}$ and $v^l, l = 1, \dots, k \in \mathbb{N}$ are *uncertain parameters* and they reside, respectively, in the *uncertainty sets* $\Omega_r \subset \mathbb{R}^m$ and $\Theta_l \subset \mathbb{R}^s$, and $f_r : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $g_l : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$ are bi-functions.

Throughout this paper, the uncertainty sets $\Omega_r, r = 1, \dots, p$ and $\Theta_l, l = 1, \dots, k$ are assumed to be spectrahedra (see e.g., Ramana and Goldman, 1995) given by

$$\Omega_r := \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid A^r + \sum_{i=1}^m u_i^r A_i^r \geq 0\},$$

$$\Theta_l := \{v^l := (v_1^l, \dots, v_s^l) \in \mathbb{R}^s \mid B^l + \sum_{j=1}^s v_j^l B_j^l \succeq 0\} \tag{2.2}$$

with given symmetric matrices $A^r, A_i^r, i = 1, \dots, m \in \mathbb{N}$ and $B^l, B_j^l, j = 1, \dots, s \in \mathbb{N}$, while the bi-functions $f_r, r = 1, \dots, p$ and $g_l, l = 1, \dots, k$ are convex quadratic functions given by

$$\begin{aligned} f_r(x, u^r) &:= x^T Q^r x + (\xi^r)^T x + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T x + \beta_i^r), x \in \mathbb{R}^n, \\ u^r &:= (u_1^r, \dots, u_m^r) \in \Omega_r, \\ g_l(x, v^l) &:= x^T M^l x + (\theta^l)^T x + \gamma^l + \sum_{j=1}^s v_j^l ((\theta_j^l)^T x + \gamma_j^l), x \in \mathbb{R}^n, \\ v^l &:= (v_1^l, \dots, v_s^l) \in \Theta_l \end{aligned} \tag{2.3}$$

with $Q^r \succeq 0, \xi^r \in \mathbb{R}^n, \xi_i^r \in \mathbb{R}^n, \beta^r \in \mathbb{R}, \beta_i^r \in \mathbb{R}, i = 1, \dots, m$ and $M^l \succeq 0, \theta^l \in \mathbb{R}^n, \theta_j^l \in \mathbb{R}^n, \gamma^l \in \mathbb{R}, \gamma_j^l \in \mathbb{R}, j = 1, \dots, s$ fixed.

The spectrahedral data uncertainty sets in (2.2) encompass all important popular data uncertainty sets used in practical robust models such as box uncertainty set, ball uncertainty set or ellipsoidal uncertainty set (see e.g., Vintant, 2014). Moreover, spectrahedra admit the special structures of *linear matrix inequalities* that empower us to employ variable transformations and express optimality conditions in terms of linear matrix inequalities. We assume that the uncertainty sets in (2.2) are nonempty and compact.

To treat the uncertain multiobjective problem (UP), we associate with it a *robust* counterpart that can be formulated as

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u^1 \in \Omega_1} f_1(x, u^1), \dots, \max_{u^p \in \Omega_p} f_p(x, u^p) \right) \mid g_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k \right\}. \tag{RP}$$

In this model, the uncertain parameters of the objective and constraint functions are enforced for all possible realizations within the corresponding uncertainty sets, and so the robust counterpart problem (RP) aims at locating a (weak) Pareto solution which is *immunized* against the data uncertainty. It is worth mentioning that by taking the maximum on each uncertainty set to all possible realizations of the corresponding objective, the underlying problem becomes a multiobjective problem with *nonsmooth* objective functions. The transformed problem is a *nonsmooth* multiobjective problem involving *an infinite number of constraints*, and so solving this problem is in general computationally *intractable* for *general compact convex* uncertainty sets by using the duality approach as in Chuong (2020).

The following solution concepts are in the sense of *minmax robustness* (see e.g., Ehrgott et al., 2014; Kuroiwa and Lee, 2012) for multiobjective optimization.

Definition 2.1 (*Robust Pareto and weak Pareto solutions*) Let $\bar{x} \in C := \{x \in \mathbb{R}^n \mid g_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k\}$.

(i) We say that \bar{x} is a *robust Pareto solution* of problem (UP) if it is a Pareto solution of problem (RP), i.e., there is no other point $x \in C$ such that

$$\begin{aligned} \max_{u^r \in \Omega_r} f_r(x, u^r) &\leq \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r), r = 1, \dots, p \text{ and} \\ \max_{u^r \in \Omega_r} f_r(x, u^r) &< \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r) \text{ for some } r \in \{1, \dots, p\}. \end{aligned}$$

(ii) We say that \bar{x} is a *robust weak Pareto solution* of problem (UP) if it is a weak Pareto solution of problem (RP), i.e., there is no other point $x \in C$ such that

$$\max_{u^r \in \Omega_r} f_r(x, u^r) < \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r), \quad r = 1, \dots, p.$$

3 Robust optimality conditions

In this section, we present necessary as well as sufficient conditions for robust (weak) Pareto optimality of the uncertain multiobjective problem (UP).

Theorem 3.1 *Let $\bar{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP), i.e., $\bar{x} \in C := \{x \in \mathbb{R}^n \mid g_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k\}$.*

(i) **(Necessary optimality)** *Assume that the characteristic cone $K := \text{coneco}\{(0_n, 1) \cup \text{epi } g_l^*(\cdot, v^l), v^l \in \Theta_l, l = 1, \dots, k\}$ is closed. If \bar{x} is a robust weak Pareto solution of problem (UP), then there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that*

$$\left(\begin{array}{c} \sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l \qquad \qquad \qquad \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right) \\ \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \end{array} \right) \geq 0, \tag{3.1}$$

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0, \quad r = 1, \dots, p, \quad \lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l \geq 0, \quad l = 1, \dots, k, \tag{3.2}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r)$ for $r = 1, \dots, p$.

(ii) **(Sufficiency for robust weak Pareto solutions)** *If there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.1) and (3.2) hold, then \bar{x} is a robust weak Pareto solution of problem (UP).*

(iii) **(Sufficiency for robust Pareto solutions)** *If there exist $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.1) and (3.2) hold, then \bar{x} is a robust Pareto solution of problem (UP).*

Proof Denote $F_r(x) := \max_{u^r \in \Omega_r} f_r(x, u^r), r = 1, \dots, p$ and $G_l(x) := \max_{v^l \in \Theta_l} g_l(x, v^l), l = 1, \dots, k$ for $x \in \mathbb{R}^n$. It is easy to see that $F_r, r = 1, \dots, p$ and $G_l, l = 1, \dots, k$ are convex functions finite on \mathbb{R}^n .

(i) Let the cone K be closed, and let \bar{x} be a robust weak Pareto solution of problem (UP). We see that

$$\{x \in C \mid F_r(x) - F_r(\bar{x}) < 0, r = 1, \dots, p\} = \emptyset,$$

where $C := \{x \in \mathbb{R}^n \mid g_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k\}$ is the robust feasible set of problem (UP). Invoking a classical alternative theorem in convex analysis (cf. Rockafellar, 1970, Theorem 21.1), we find $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$ such that

$$\sum_{r=1}^p \alpha_r F_r(x) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}) \quad \text{for all } x \in C.$$

This shows that

$$\inf_{x \in C} \left\{ \sum_{r=1}^p \alpha_r \max_{u^r \in \Omega_r} \left\{ x^T Q^r x + (\xi^r)^T x + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T x + \beta_i^r) \right\} \right\} \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}), \tag{3.3}$$

where $u^r := (u_1^r, \dots, u_m^r)$, $r = 1, \dots, p$. Let $\Omega := \prod_{r=1}^p \Omega_r \subset \mathbb{R}^{pm}$. We see that Ω is a convex compact set and (3.3) becomes the following inequality

$$\inf_{x \in C} \max_{(u^1, \dots, u^p) \in \Omega} \left\{ \sum_{r=1}^p \alpha_r \left(x^T Q^r x + (\xi^r)^T x + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T x + \beta_i^r) \right) \right\} \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}). \tag{3.4}$$

Let $H : \mathbb{R}^n \times \mathbb{R}^{pm} \rightarrow \mathbb{R}$ be defined by $H(x, u) := \sum_{r=1}^p \alpha_r (x^T Q^r x + (\xi^r)^T x + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T x + \beta_i^r))$ for $x \in \mathbb{R}^n$ and $u := (u^1, \dots, u^p) \in \mathbb{R}^{pm}$. Then, H is an affine function in variable u and a convex function in variable x and so, we invoke a classical minimax theorem (see e.g., Sion, 1958, Theorem 4.2) and (3.4) to claim that

$$\max_{u \in \Omega} \inf_{x \in C} H(x, u) = \inf_{x \in C} \max_{u \in \Omega} H(x, u) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}).$$

Therefore, we can find $\bar{u} := (\bar{u}^1, \dots, \bar{u}^p)$, where $\bar{u}^r := (\bar{u}_1^r, \dots, \bar{u}_m^r) \in \Omega_r$, $r = 1, \dots, p$, such that $\inf_{x \in C} H(x, \bar{u}) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x})$. This shows that

$$\left(0_n, - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \right) \in \text{epi} \left(H(\cdot, \bar{u}) + \delta_C \right)^*.$$

Due to the continuity of $H(\cdot, \bar{u})$ on \mathbb{R}^n , by taking (2.1) into account, we arrive at

$$\left(0_n, - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \right) \in \text{epi} H^*(\cdot, \bar{u}) + \text{epi} \delta_C^*.$$

Moreover, we have (cf. Jeyakumar and Li, 2010, Page 3390 or Jeyakumar, 2003, p. 951) that $\text{epi} \delta_C^* = \text{cl} K = K$, where the last equality holds as the cone $K := \text{coneco}\{(0_n, 1)\} \cup \text{epi} g_l^*(\cdot, v^l)$, $v^l \in \Theta_l$, $l = 1, \dots, k$ is assumed to be closed. Consequently,

$$\left(0_n, - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \right) \in \text{epi} H^*(\cdot, \bar{u}) + K,$$

which means that there exist $(w, \lambda) \in \text{epi} H^*(\cdot, \bar{u})$ and $\bar{v}^{lt} := (\bar{v}_1^{lt}, \dots, \bar{v}_s^{lt}) \in \Theta_l$, $(\bar{w}^{lt}, \bar{\lambda}^{lt}) \in \text{epi} g_l^*(\cdot, \bar{v}^{lt})$, $\bar{\alpha} \geq 0$, $\bar{\alpha}^{lt} \geq 0$, $l = 1, \dots, k$, $t = 1, \dots, s_l$ such that

$$0_n = w + \sum_{l=1}^k \sum_{t=1}^{s_l} \bar{\alpha}^{lt} \bar{w}^{lt}, \quad - \sum_{r=1}^p \alpha_r F_r(\bar{x}) = \lambda + \bar{\alpha} + \sum_{l=1}^k \sum_{t=1}^{s_l} \bar{\alpha}^{lt} \bar{\lambda}^{lt}. \tag{3.5}$$

Observe by $(\bar{w}^{lt}, \bar{\lambda}^{lt}) \in \text{epi } g_l^*(\cdot, \bar{v}^{lt}), l = 1, \dots, k, t = 1, \dots, s_l$ that, for each $x \in \mathbb{R}^n$, one has $\bar{\lambda}^{lt} \geq (\bar{w}^{lt})^T x - g_l(x, \bar{v}^{lt})$. This together with (3.5) and the relation $(w, \lambda) \in \text{epi } H^*(\cdot, \bar{u})$ implies that, for each $x \in \mathbb{R}^n$,

$$H(x, \bar{u}) \geq w^T x - \lambda \geq - \sum_{l=1}^k \sum_{t=1}^{s_l} \bar{\alpha}^{lt} g_l(x, \bar{v}^{lt}) + \sum_{r=1}^p \alpha_r F_r(\bar{x}). \tag{3.6}$$

Letting $\alpha_r^i := \alpha_r \bar{u}_i^r, r = 1, \dots, p, i = 1, \dots, m$, we assert that

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r = \alpha_r \left(A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \right) \geq 0, r = 1, \dots, p. \tag{3.7}$$

To see this, recall that $\alpha_r \geq 0$ and $\bar{u}^r \in \Omega_r$ (and so $A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \geq 0$) for $r = 1, \dots, p$.

Similarly, by letting $\lambda_l := \sum_{t=1}^{s_l} \bar{\alpha}^{lt} \geq 0, \lambda_l^j := \sum_{t=1}^{s_l} \bar{\alpha}^{lt} \bar{v}_j^{lt}, l = 1, \dots, k, j = 1, \dots, s$, we see that

$$\lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l = \sum_{t=1}^{s_l} [\bar{\alpha}^{lt} (B^l + \sum_{j=1}^s \bar{v}_j^{lt} B_j^l)] \geq 0, l = 1, \dots, k$$

due to $\bar{\alpha}^{lt} (B^l + \sum_{j=1}^s \bar{v}_j^{lt} B_j^l) \geq 0, l = 1, \dots, k, t = 1, \dots, s_l$. Consequently, (3.2) holds.

Now, we deduce from (3.6) that

$$\begin{aligned} &x^T \left(\sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l \right) x + \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T x \\ &+ \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^n. \end{aligned}$$

This is equivalent to (cf. Ben-Tal and Nemirovski, 2001, Simple lemma, p. 163) the following matrix inequality

$$\left(\begin{array}{cc} \sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l & \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right) \\ \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T & \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \end{array} \right) \geq 0,$$

showing that (3.1) is also established and so the assertion in (i) has been justified.

(ii) Let $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ be such that (3.1) and (3.2) hold.

Consider any $r \in \{1, \dots, p\}$. By the compactness of Ω_r , we assert by (3.2) that if $\alpha_r = 0$, then $\alpha_r^i = 0$ for all $i = 1, \dots, m$. Suppose, on the contrary, that $\alpha_r = 0$ but there exists $\tilde{i} \in \{1, \dots, m\}$ with $\alpha_r^{\tilde{i}} \neq 0$. In this case, we get by (3.2) that $\sum_{i=1}^m \alpha_r^i A_i^r \geq 0$. Let $\bar{u}^r :=$

$(\bar{u}_1^r, \dots, \bar{u}_m^r) \in \Omega_r$. It follows by definition that $A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \geq 0$ and

$$A^r + \sum_{i=1}^m (\bar{u}_i^r + \lambda \alpha_r^i) A_i^r = \left(A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \right) + \lambda \sum_{i=1}^m \alpha_r^i A_i^r \geq 0 \text{ for all } \lambda > 0.$$

This guarantees that $\bar{u}^r + \lambda(\alpha_r^1, \dots, \alpha_r^m) \in \Omega_r$ for all $\lambda > 0$, which contradicts the fact that $(\alpha_r^1, \dots, \alpha_r^m) \neq 0_m$ and Ω_r is a bounded set. Thus, our claim must be true.

Now, take $\hat{u}^r := (\hat{u}_1^r, \dots, \hat{u}_m^r) \in \Omega_r$ and define $\tilde{u}^r := (\tilde{u}_1^r, \dots, \tilde{u}_m^r)$ with

$$\tilde{u}_i^r := \begin{cases} \hat{u}_i^r & \text{if } \alpha_r = 0, \\ \frac{\alpha_r^i}{\alpha_r} & \text{if } \alpha_r \neq 0, \end{cases} \quad i = 1, \dots, m.$$

This shows that $\tilde{u}^r \in \Omega_r$. Then, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} & \alpha_r x^T Q^r x + (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r)^T x + \alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r \\ &= \alpha_r \left(x^T Q^r x + (\xi^r)^T x + \beta^r + \sum_{i=1}^m \tilde{u}_i^r ((\xi_i^r)^T x + \beta_i^r) \right) = \alpha_r f_r(x, \tilde{u}^r), \end{aligned} \tag{3.8}$$

where we remind that if $\alpha_r = 0$, then $\alpha_r^i = 0$ for all $i = 1, \dots, m$ as said above. Similarly, by (3.2), we can find $\tilde{v}^l := (\tilde{v}_1^l, \dots, \tilde{v}_s^l) \in \Theta_l$, $l = 1, \dots, k$ such that

$$\begin{aligned} & \lambda_l x^T M^l x + (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l)^T x + \lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l \\ &= \lambda_l \left(x^T M^l x + (\theta^l)^T x + \gamma^l + \sum_{j=1}^s \tilde{v}_j^l ((\theta_j^l)^T x + \gamma_j^l) \right) = \lambda_l g_l(x, \tilde{v}^l), \quad l = 1, \dots, k \end{aligned} \tag{3.9}$$

for each $x \in \mathbb{R}^n$.

Arguing as in the proof of (i), (3.1) can be equivalently rewritten as

$$\begin{aligned} & \sum_{r=1}^p \alpha_r (x^T Q^r x + (\xi^r)^T x + \beta^r) + \sum_{r=1}^p \sum_{i=1}^m \alpha_r^i ((\xi_i^r)^T x + \beta_i^r) \\ &+ \sum_{l=1}^k \lambda_l (x^T M^l x + (\theta^l)^T x + \gamma^l) + \sum_{l=1}^k \sum_{j=1}^s \lambda_l^j ((\theta_j^l)^T x + \gamma_j^l) \\ &- \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^n, \end{aligned}$$

and so, by (3.8) and (3.9), we arrive at

$$\sum_{r=1}^p \alpha_r f_r(x, \tilde{u}_r) + \sum_{l=1}^k \lambda_l g_l(x, \tilde{v}^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^n. \tag{3.10}$$

Next, let $\hat{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP). Then, it shows that $g_l(\hat{x}, \tilde{v}^l) \leq 0$ for $\tilde{v}^l \in \Theta_l, l = 1, \dots, k$. Estimating (3.10) at \hat{x} , we obtain that $\sum_{r=1}^p \alpha_r f_r(\hat{x}, \tilde{u}_r) \geq$

$\sum_{r=1}^p \alpha_r F_r(\bar{x})$ and so,

$$\sum_{r=1}^p \alpha_r F_r(\hat{x}) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}) \tag{3.11}$$

due to the fact that $\alpha_r \geq 0$ and $F_r(\hat{x}) \geq f_r(\hat{x}, \tilde{u}^r)$ for $r = 1, \dots, p$.

By $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p \setminus \{0\}$, (3.11) guarantees that there is no other robust feasible point $\hat{x} \in \mathbb{R}^n$ such that

$$F_r(\hat{x}) < F_r(\bar{x}), \quad r = 1, \dots, p,$$

and so \bar{x} is a robust a weak Pareto solution of problem (UP).

(iii) Let $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ be such that (3.1) and (3.2) are valid. Proceeding similarly as in the proof of (ii), we arrive at the conclusion in (3.11). This, together with $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$, ensures that \bar{x} is a robust a Pareto solution of problem (UP). The proof of the theorem is complete. \square

Remark 3.2 Note that the closedness of the characteristic cone K in Theorem 3.1 is fulfilled if at least one of the following conditions is valid: a) (see e.g., Chuong and Jeyakumar, 2017) $g_l(\cdot, v^l), v^l \in \Theta_l, l = 1, \dots, k$ are linear functions and $\Theta_l, l = 1, \dots, k$ are polytopes; b) (see e.g., Jeyakumar and Li, 2010, Proposition 3.2) the Slater constraint qualification holds, i.e., there exists $\tilde{x} \in \mathbb{R}^n$ such that

$$g_l(\tilde{x}, v^l) < 0, \forall v^l \in \Theta_l, l = 1, \dots, k. \tag{3.12}$$

The following example illustrates that the optimality conditions in (3.1) and (3.2) may go awry if the closedness of the characteristic cone K imposed in Theorem 3.1 is violated.

Example 3.3 (The importance of the closedness regularity) Consider a robust convex quadratic multiobjective optimization problem of the form:

$$\begin{aligned} \min_{x := (x_1, x_2) \in \mathbb{R}^2} \{ & (\max_{u := (u_1, u_2) \in \Omega} \{x_2^2 + x_1 + u_1\}, \max_{u := (u_1, u_2) \in \Omega} \{x_2^2 + 3x_1 - u_2\}) \mid v_1 x_1 \\ & + (v_2 + 1)x_2 \leq 0, \forall v := (v_1, v_2) \in \Theta \}, \end{aligned} \tag{EP1}$$

where $\Omega := \{u := (u_1, u_2) \in \mathbb{R}^2 \mid \frac{u_1^2}{2} + \frac{u_2^2}{3} \leq 1\}$ and $\Theta := \{v := (v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \leq 1\}$. The problem (EP1) can be expressed in terms of problem (RP), where $\Omega_1 := \Omega_2 := \Omega$ and $\Theta_1 := \Theta$ are described respectively by

$$\begin{aligned} A^r &:= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1^r := B_1^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2^r := B_2^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ B^1 &:= I_3, r = 1, 2, \end{aligned}$$

and $f_r, r = 1, 2, g_1$, are given respectively by $Q^r := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \xi^1 := (1, 0), \xi^2 := (3, 0), \xi_1^r := \xi_2^r := 0_2, \beta^r := \beta_2^1 := \beta_1^2 := 0, \beta_1^1 := 1, \beta_2^2 := -1, r = 1, 2$ and $M^1 := 0_{2 \times 2}, \theta^1 := (0, 1), \theta_1^1 := (1, 0), \theta_2^1 := (0, 1), \gamma^1 := \gamma_1^1 := \gamma_2^1 := 0$.

By direct computation, we see that the feasible set of problem (EP1) is given by

$$C := \{x \in \mathbb{R}^2 \mid g_1(x, v) \leq 0, \forall v \in \Theta\} = \{0\} \times (-\infty, 0].$$

Moreover, it can be checked that $\bar{x} := (0, 0)$ is a Pareto solution of the robust problem (EP1). However, we assert that the conditions in (3.1) and (3.2) are not valid at \bar{x} for this setting. To

see the conclusion, assume on the contrary that there exist $(\alpha_1, \alpha_2) \in \mathbb{R}_+^2 \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}$, $r = 1, 2$, $i = 1, 2$, $\lambda_1 \in \mathbb{R}_+$, $\lambda_1^j \in \mathbb{R}$, $j = 1, 2$ such that

$$\left(\begin{array}{c} \sum_{r=1}^2 \alpha_r Q^r + \lambda_1 M^1 \qquad \qquad \qquad \frac{1}{2} \left(\sum_{r=1}^2 (\alpha_r \xi^r + \sum_{i=1}^2 \alpha_r^i \xi_i^r) + \lambda_1 \theta^1 + \sum_{j=1}^2 \lambda_1^j \theta_j^1 \right) \\ \frac{1}{2} \left(\sum_{r=1}^2 (\alpha_r \xi^r + \sum_{i=1}^2 \alpha_r^i \xi_i^r) + \lambda_1 \theta^1 + \sum_{j=1}^2 \lambda_1^j \theta_j^1 \right)^T \quad \sum_{r=1}^2 (\alpha_r \beta^r + \sum_{i=1}^2 \alpha_r^i \beta_i^r) + \lambda_1 \gamma^1 + \sum_{j=1}^2 \lambda_1^j \gamma_j^1 - \sum_{r=1}^2 \alpha_r F_r(\bar{x}) \end{array} \right) \geq 0, \tag{3.13}$$

$$\alpha_r A^r + \sum_{i=1}^2 \alpha_r^i A_i^r \geq 0, r = 1, 2, \quad \lambda_1 B^1 + \sum_{j=1}^2 \lambda_1^j B_j^1 \geq 0, \tag{3.14}$$

where $F_1(\bar{x}) := \max_{u \in \Omega} f_1(\bar{x}, u) = \sqrt{2}$ and $F_2(\bar{x}) := \max_{u \in \Omega} f_2(\bar{x}, u) = \sqrt{3}$. Note that (3.14) amounts to the following inequalities

$$\frac{(\alpha_r^1)^2}{2} + \frac{(\alpha_r^2)^2}{3} \leq (\alpha_r)^2, r = 1, 2, \tag{3.15}$$

$$(\lambda_1^1)^2 + (\lambda_1^2)^2 \leq (\lambda_1)^2. \tag{3.16}$$

We deduce from (3.13) that

$$(\alpha_1 + \alpha_2)x_2^2 + (\alpha_1 + 3\alpha_2 + \lambda_1^1)x_1 + (\lambda_1 + \lambda_1^2)x_2 + \alpha_1^1 - \alpha_2^2 - \sqrt{2}\alpha_1 - \sqrt{3}\alpha_2 \geq 0$$

for all $x_1 \in \mathbb{R}$ and all $x_2 \in \mathbb{R}$. This entails that

$$(\alpha_1 + 3\alpha_2 + \lambda_1^1)x_1 + \alpha_1^1 - \alpha_2^2 - \sqrt{2}\alpha_1 - \sqrt{3}\alpha_2 \geq 0 \text{ for all } x_1 \in \mathbb{R}, \tag{3.17}$$

$$(\alpha_1 + \alpha_2)x_2^2 + (\lambda_1 + \lambda_1^2)x_2 + \alpha_1^1 - \alpha_2^2 - \sqrt{2}\alpha_1 - \sqrt{3}\alpha_2 \geq 0 \text{ for all } x_2 \in \mathbb{R}, \tag{3.18}$$

On the one hand, we get by (3.17) that $\alpha_1 + 3\alpha_2 + \lambda_1^1 = 0$ and

$$\alpha_1^1 - \alpha_2^2 - \sqrt{2}\alpha_1 - \sqrt{3}\alpha_2 \geq 0.$$

On the other hand, it holds that $\alpha_1^1 - \sqrt{2}\alpha_1 \leq |\alpha_1^1| - \sqrt{2}\alpha_1 \leq 0$, where the last inequality holds by virtue of (3.15). Similarly, we have $-\alpha_2^2 - \sqrt{3}\alpha_2 \leq |\alpha_2^2| - \sqrt{3}\alpha_2 \leq 0$, and so

$$\alpha_1^1 - \sqrt{2}\alpha_1 - \alpha_2^2 - \sqrt{3}\alpha_2 \leq 0.$$

Therefore, $\alpha_1^1 - \alpha_2^2 - \sqrt{2}\alpha_1 - \sqrt{3}\alpha_2 = 0$. This and (3.18) guarantee that $\lambda_1 + \lambda_1^2 = 0$.

Substituting now $\lambda_1^1 = -\alpha_1 - 3\alpha_2$ and $\lambda_1^2 = -\lambda_1$ into (3.16), we arrive at $(\alpha_1 + 3\alpha_2)^2 \leq 0$, which is absurd due to the fact that $(\alpha_1, \alpha_2) \in \mathbb{R}_+^2 \setminus \{0\}$. Consequently, the conclusion of (i) in Theorem 3.1 does not hold for this setting. The reason is that the characteristic cone

$$\begin{aligned} K &:= \text{coneco}\{(0_2, 1) \cup \text{epi } g_1^*(\cdot, v), v \in \Theta\} \\ &= \text{coneco}\{(0, 0, 1), (v_1, v_2 + 1, 0) \mid v_1^2 + v_2^2 \leq 1\} \end{aligned}$$

is not closed for this problem. To see this, we select a sequence $c_m := (1, \frac{1}{m}, 0) = m(\frac{1}{m}, \frac{1}{m^2}, 0) \in K$ for all $m \in \mathbb{N}$. Clearly, it holds that $c_m \rightarrow c_0 := (1, 0, 0)$ as $m \rightarrow \infty$ but $c_0 \notin K$.

In the next proposition, we show how the optimality conditions obtained in (3.1) and (3.2) can be alternatively checked via verifiable criteria including a *robust Karush-Kuhn-Tucker* (KKT) condition. To this end, let $\bar{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP). One says

(see e.g., Chuong, 2020 for a more general nonsmooth setting) that the robust (KKT) condition of problem (UP) holds at \bar{x} if there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\bar{u}^r \in \Omega_r, r = 1, \dots, p$ and $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k, \bar{v}^l \in \Theta_l, l = 1, \dots, k$ such that

$$\sum_{r=1}^p \alpha_r \nabla_1 f_r(\bar{x}, \bar{u}^r) + \sum_{l=1}^k \lambda_l \nabla_1 g_l(\bar{x}, \bar{v}^l) = 0, \lambda_l g_l(\bar{x}, \bar{v}^l) = 0, l = 1, \dots, k, \tag{3.19}$$

$$\sum_{r=1}^p \alpha_r f_r(\bar{x}, \bar{u}^r) = \sum_{r=1}^p \alpha_r F_r(\bar{x}), \tag{3.20}$$

where $\nabla_1 f$ denotes the derivative of $f(\cdot, \cdot)$ with respect to the first variable. Note that in the particular case, where there are *no uncertain parameters* in the objective functions of problem (UP), the condition (3.20) is redundant and in this case, its corresponding robust (KKT) condition was examined in Chuong (2018) for an SOS-convex polynomial setting. It is worth noticing further that the *robust* (KKT) condition of problem (UP) holds at \bar{x} is equivalent to saying that the (classical) (KKT) condition of *robust* problem (RP) holds at \bar{x} .

Proposition 3.4 (Optimality characterizations via verifiable criteria) *Let $\bar{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP). The following statements are equivalent:*

- (i) *The optimality conditions in (3.1) and (3.2) are valid.*
- (ii) *The robust (KKT) condition of problem (UP) holds at \bar{x} .*
- (iii) *There exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k, \mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s$ such that*

$$\sum_{r=1}^p \alpha_r (2Q^r \bar{x} + \xi^r) + \sum_{r=1}^p \sum_{i=1}^m \alpha_r^i \xi_i^r + \sum_{l=1}^k \lambda_l (2M^l \bar{x} + \theta^l) + \sum_{l=1}^k \sum_{j=1}^s \lambda_l^j \theta_j^l = 0, \tag{3.21}$$

$$\lambda_l (\bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l) + \sum_{j=1}^s \lambda_l^j ((\theta_j^l)^T \bar{x} + \gamma_j^l) = 0, l = 1, \dots, k, \tag{3.22}$$

$$\sum_{r=1}^p \alpha_r (\bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r) + \sum_{r=1}^p \sum_{i=1}^m \alpha_r^i ((\xi_i^r)^T \bar{x} + \beta_i^r) = \sum_{r=1}^p \alpha_r F_r(\bar{x}), \tag{3.23}$$

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0, r = 1, \dots, p, \lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l \geq 0, l = 1, \dots, k, \tag{3.24}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega^r} f_r(\bar{x}, u^r)$ for $r = 1, \dots, p$.

Proof [(i) \implies (ii)] Assume that there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.1) and (3.2) hold. Following the proof of Theorem 3.1(ii), we first assert that there exist $\bar{u}^r \in \Omega_r, r = 1, \dots, p$ and $\bar{v}^l \in \Theta_l, l = 1, \dots, k$ such that

$$h(x) := \sum_{r=1}^p \alpha_r f_r(x, \bar{u}^r) + \sum_{l=1}^k \lambda_l g_l(x, \bar{v}^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^n, \tag{3.25}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r)$ for $r = 1, \dots, p$. Substituting $x := \bar{x}$ into (3.25), we obtain that

$$\sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) + \sum_{l=1}^k \lambda_l g_l(\bar{x}, \tilde{v}^l) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}). \tag{3.26}$$

As \bar{x} is a robust feasible point of problem (UP), it holds that $\lambda_l g_l(\bar{x}, \tilde{v}^l) \leq 0, l = 1, \dots, k$ and so,

$$\sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) + \sum_{l=1}^k \lambda_l g_l(\bar{x}, \tilde{v}^l) \leq \sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) \leq \sum_{r=1}^p \alpha_r F_r(\bar{x}), \tag{3.27}$$

where the last inequality holds due to the fact that $\alpha_r \geq 0$ and $F_r(\bar{x}) \geq f_r(\bar{x}, \tilde{u}^r)$ for $r = 1, \dots, p$. Combining (3.26) and (3.27) gives us

$$\sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) = \sum_{r=1}^p \alpha_r F_r(\bar{x}), \quad \sum_{l=1}^k \lambda_l g_l(\bar{x}, \tilde{v}^l) = 0. \tag{3.28}$$

Note that the last equality in (3.28) means that $\lambda_l g_l(\bar{x}, \tilde{v}^l) = 0$ for $l = 1, \dots, k$.

Observe now by (3.28) and (3.25) that $h(x) \geq 0 = h(\bar{x})$ for all $x \in \mathbb{R}^n$; i.e., \bar{x} is a minimizer of h on \mathbb{R}^n . This entails that $\nabla h(\bar{x}) = 0$, and so we arrive at

$$\begin{aligned} \sum_{r=1}^p \alpha_r \nabla_1 f_r(\bar{x}, \tilde{u}^r) + \sum_{l=1}^k \lambda_l \nabla_1 g_l(\bar{x}, \tilde{v}^l) &= 0, \quad \lambda_l g_l(\bar{x}, \tilde{v}^l) = 0, \quad l = 1, \dots, k, \\ \sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) &= \sum_{r=1}^p \alpha_r F_r(\bar{x}), \end{aligned}$$

which shows that the robust (KKT) condition of problem (UP) holds at \bar{x} .

[(ii) \implies (iii)] Assume that the robust (KKT) condition of problem (UP) holds at \bar{x} . This means that there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \bar{u}^r := (\bar{u}_1^r, \dots, \bar{u}_m^r) \in \Omega_r, r = 1, \dots, p$ and $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k, \bar{v}^l := (\bar{v}_1^l, \dots, \bar{v}_s^l) \in \Theta_l, l = 1, \dots, k$ such that (3.19) and (3.20) are valid. The relation $\bar{u}^r := (\bar{u}_1^r, \dots, \bar{u}_m^r) \in \Omega_r$ means that $A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \geq 0$ for $r = 1, \dots, p$.

By letting $\alpha_r^i := \alpha_r \bar{u}_i^r, r = 1, \dots, p, i = 1, \dots, m$, we obtain that $\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r = \alpha_r \left(A^r + \sum_{i=1}^m \bar{u}_i^r A_i^r \right) \geq 0, r = 1, \dots, p$ and that

$$\alpha_r f_r(x, \bar{u}^r) = \alpha_r (x^T Q^r x + (\xi^r)^T x + \beta^r) + \sum_{i=1}^m \alpha_r^i ((\xi_i^r)^T x + \beta_i^r), \quad r = 1, \dots, p. \tag{3.29}$$

Similarly, by letting $\lambda_l^j := \lambda_l \bar{v}_j^l, l = 1, \dots, k, j = 1, \dots, s$, we get by $\bar{v}^l := (\bar{v}_1^l, \dots, \bar{v}_s^l) \in \Theta_l, l = 1, \dots, k$ that $\lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l \geq 0, l = 1, \dots, k$ and that

$$\lambda_l g_l(x, \bar{v}^l) = \lambda_l (x^T M^l x + (\theta^l)^T x + \gamma^l) + \sum_{j=1}^s \lambda_l^j ((\theta_j^l)^T x + \gamma_j^l), \quad l = 1, \dots, k. \tag{3.30}$$

Substituting now (3.29) and (3.30) into (3.19) and (3.20), we arrive at the conclusion of (iii).

[(iii) \implies (i)] Assume that there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.21), (3.22), (3.23) and (3.24) hold. Proceeding similarly as in the proof of Theorem 3.1(ii), we assert from (3.24) that there exist $\tilde{u}^r \in \Omega_r, r = 1, \dots, p$ and $\tilde{v}^l \in \Theta_l, l = 1, \dots, k$ such that

$$\alpha_r f_r(x, \tilde{u}^r) = \alpha_r (x^T Q^r x + (\xi^r)^T x + \beta^r) + \sum_{i=1}^m \alpha_r^i ((\xi_i^r)^T x + \beta_i^r), \quad r = 1, \dots, p, \quad \forall x \in \mathbb{R}^n, \tag{3.31}$$

$$\lambda_l g_l(x, \tilde{v}^l) = \lambda_l (x^T M^l x + (\theta^l)^T x + \gamma^l) + \sum_{j=1}^s \lambda_l^j ((\theta_j^l)^T x + \gamma_j^l), \quad l = 1, \dots, k, \quad \forall x \in \mathbb{R}^n. \tag{3.32}$$

Consider the function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ given as above, i.e., $h(x) := \sum_{r=1}^p \alpha_r f_r(x, \tilde{u}^r) + \sum_{l=1}^k \lambda_l g_l(x, \tilde{v}^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}), x \in \mathbb{R}^n$. We conclude by (3.21), (3.22) and (3.32) that $\nabla h(\bar{x}) = 0$ and that $\lambda_l g_l(\bar{x}, \tilde{v}^l) = 0, l = 1, \dots, k$. This, together with the convexity of h , implies that

$$h(x) \geq h(\bar{x}) = \sum_{r=1}^p \alpha_r f_r(\bar{x}, \tilde{u}^r) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) = 0, \quad \forall x \in \mathbb{R}^n,$$

where the last equality holds by virtue of (3.23) and (3.31). Therefore, taking (3.31) and (3.32) into account, we deduce from $h(x) \geq 0$ for all $x \in \mathbb{R}^n$ that

$$\begin{aligned} & x^T \left(\sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l \right) x + \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T x \\ & + \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^n. \end{aligned}$$

This amounts to the matrix inequality in (3.1) (cf. Ben-Tal and Nemirovski, 2001, Simple lemma, p. 163). Consequently, the assertion of (i) is valid, which completes the proof. \square

Robust linear multiobjective optimization. Let us now consider a *robust linear* multiobjective optimization problem as

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u^1 \in \Omega_1} L_1(x, u^1), \dots, \max_{u^p \in \Omega_p} L_p(x, u^p) \right) \mid h_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k \right\}, \tag{RL}$$

where $L_r, r = 1, \dots, p$ and $h_l, l = 1, \dots, k$ are affine functions given by

$$\begin{aligned} L_r(x, u^r) & := (\xi^r)^T x + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T x + \beta_i^r), \quad x \in \mathbb{R}^n, u^r := (u_1^r, \dots, u_m^r) \in \Omega_r, \\ h_l(x, v^l) & := (\theta^l)^T x + \gamma^l + \sum_{j=1}^s v_j^l ((\theta_j^l)^T x + \gamma_j^l), \quad x \in \mathbb{R}^n, v^l := (v_1^l, \dots, v_s^l) \in \Theta_l \end{aligned}$$

with $\xi^r \in \mathbb{R}^n$, $\xi_i^r \in \mathbb{R}^n$, $\beta^r \in \mathbb{R}$, $\beta_i^r \in \mathbb{R}$, $i = 1, \dots, m$ and $\theta^l \in \mathbb{R}^n$, $\theta_j^l \in \mathbb{R}^n$, $\gamma^l \in \mathbb{R}$, $\gamma_j^l \in \mathbb{R}$, $j = 1, \dots, s$ fixed. Here, the uncertainty sets Ω_r , $r = 1, \dots, p$ and Θ_l , $l = 1, \dots, k$ are given as in (2.2).

The following corollary presents necessary/sufficient optimality conditions for (weak) Pareto solutions of problem (RL).

Corollary 3.5 (Optimality for robust linear optimization) *Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of problem (RL), i.e., $\bar{x} \in \{x \in \mathbb{R}^n \mid h_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k\}$.*

(i) *Assume that the characteristic cone $K := \text{coneco}\{(0_n, 1), (\theta^l + \sum_{j=1}^s v_j^l \theta_j^l, -\gamma^l - \sum_{j=1}^s v_j^l \gamma_j^l), v^l := (v_1^l, \dots, v_s^l) \in \Theta_l, l = 1, \dots, k\}$ is closed. If \bar{x} is a weak Pareto solution of problem (RL), then there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}$, $r = 1, \dots, p$, $i = 1, \dots, m$ and $\lambda_l \geq 0$, $\lambda_j^l \in \mathbb{R}$, $j = 1, \dots, s$, $l = 1, \dots, k$ such that*

$$\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_j^l \theta_j^l) = 0, \tag{3.33}$$

$$\sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_j^l \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \geq 0, \tag{3.34}$$

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0, r = 1, \dots, p, \lambda_l B^l + \sum_{j=1}^s \lambda_j^l B_j^l \geq 0, l = 1, \dots, k, \tag{3.35}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r)$ for $r = 1, \dots, p$.

(ii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}$, $r = 1, \dots, p$, $i = 1, \dots, m$ and $\lambda_l \geq 0$, $\lambda_j^l \in \mathbb{R}$, $j = 1, \dots, s$, $l = 1, \dots, k$ such that (3.33), (3.34) and (3.35) hold, then \bar{x} is a weak Pareto solution of problem (RL).*

(iii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$, $\alpha_r^i \in \mathbb{R}$, $r = 1, \dots, p$, $i = 1, \dots, m$ and $\lambda_l \geq 0$, $\lambda_j^l \in \mathbb{R}$, $j = 1, \dots, s$, $l = 1, \dots, k$ such that (3.33), (3.34) and (3.35) hold, then \bar{x} is a Pareto solution of problem (RL).*

Proof Observe first that the problem (RL) is a particular case of problem (RP) with $Q^r := M^l := 0_{n \times n}$, $r = 1, \dots, p$, $l = 1, \dots, k$. In this setting, we see that $\text{epi } h_l^*(\cdot, v^l) = (\theta^l + \sum_{j=1}^s v_j^l \theta_j^l, -\gamma^l - \sum_{j=1}^s v_j^l \gamma_j^l) + \{0_n\} \times \mathbb{R}_+$ for $v^l := (v_1^l, \dots, v_s^l) \in \Theta_l$, $l = 1, \dots, k$ and so,

$$K := \text{coneco}\left\{(0_n, 1), \left(\theta^l + \sum_{j=1}^s v_j^l \theta_j^l, -\gamma^l - \sum_{j=1}^s v_j^l \gamma_j^l\right), v^l := (v_1^l, \dots, v_s^l) \in \Theta_l, l = 1, \dots, k\right\}$$

$$= \text{coneco}\{\text{epi } h_l^*(\cdot, v^l) \mid v^l \in \Theta_l, l = 1, \dots, k\}.$$

Now, our desired conclusions are followed by invoking Theorem 3.1. □

Remark that the assertions (i) and (ii) in Corollary 3.5 develop optimality characterizations for robust weak Pareto solutions given in Chuong (2017, Theorem 3.1) by using a robust alternative approach and in Chuong (2018, Corollary 2.4) by using an SOS-convex

polynomial approach, where there were *no uncertain parameters* on the objective functions $L_r, r = 1, \dots, p$. The interested reader is referred to Chuong and Jeyakumar (2021) for related optimality conditions in terms of two-stage linear multiobjective approach.

We now consider the uncertain multiobjective optimization problem (UP), where $\Omega_r, r = 1, \dots, p$ and $\Theta_l, l = 1, \dots, k$ in (2.2) are replaced by the following *ball uncertainty sets*:

$$\Omega_r := \{u^r \in \mathbb{R}^m \mid \|u^r\| \leq 1\}, \quad \Theta_l := \{v^l \in \mathbb{R}^s \mid \|v^l\| \leq 1\}. \tag{3.36}$$

In this case, we obtain necessary/sufficient optimality conditions for robust (weak) Pareto solutions of problem (UP) under *ball uncertainty*.

Corollary 3.6 (Robust optimality with ball uncertainty) *Consider the problem (UP) with Ω_r and Θ_l given by (3.36), and let $\bar{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP).*

(i) *Assume that the characteristic cone $K := \text{coneco}\{(0_n, 1) \cup \text{epi } g_l^*(\cdot, v^l), v^l \in \Theta_l, l = 1, \dots, k\}$ is closed. If \bar{x} is a robust weak Pareto solution of problem (UP), then there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that*

$$\left(\begin{array}{c} \sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l \qquad \qquad \qquad \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right) \\ \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \end{array} \right) \geq 0, \tag{3.37}$$

$$\left(\begin{array}{c} \alpha_r I_m \quad \alpha_r^r \\ (\alpha_r^r)^T \quad \alpha_r \end{array} \right) \geq 0, r = 1, \dots, p, \quad \left(\begin{array}{c} \lambda_l I_s \quad \lambda_l^l \\ (\lambda_l^l)^T \quad \lambda_l \end{array} \right) \geq 0, l = 1, \dots, k, \tag{3.38}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r), \alpha^r := (\alpha_1^r, \dots, \alpha_m^r)$ and $\lambda^l := (\lambda_1^l, \dots, \lambda_s^l)$.

(ii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.37) and (3.38) hold, then \bar{x} is a robust weak Pareto solution of problem (UP).*

(iii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_l^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.37) and (3.38) hold, then \bar{x} is a robust Pareto solution of problem (UP).*

Proof Let A^r and $A_i^r, r = 1, \dots, p, i = 1, \dots, m$ be $(m + 1) \times (m + 1)$ matrices given by

$$A^r := I_{m+1}, \quad A_i^r := \begin{pmatrix} 0 & e_i \\ e_i^T & 0 \end{pmatrix},$$

where $e_i \in \mathbb{R}^m$ is a vector whose i th component is one and all others are zero. For each $r \in \{1, \dots, p\}$, we have

$$\begin{aligned} \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid A^r + \sum_{i=1}^m u_i^r A_i^r \geq 0\} \\ = \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid \begin{pmatrix} I_m & u^r \\ (u^r)^T & 1 \end{pmatrix} \geq 0\} \\ = \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid 1 - (u^r)^T I_m u^r \geq 0\} = \Omega_r, \end{aligned}$$

which means that Ω_r in (3.36) lands in the form of the spectrahedral sets in (2.2). In this way, we can verify that, for each $r \in \{1, \dots, p\}$,

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0 \Leftrightarrow \begin{pmatrix} \alpha_r I_m & \alpha_r^r \\ (\alpha_r^r)^T & \alpha_r \end{pmatrix} \geq 0,$$

where $\alpha_r \geq 0$ and $\alpha^r := (\alpha_r^1, \dots, \alpha_r^m)$ with $\alpha_r^i \in \mathbb{R}, i = 1, \dots, m$. Similarly, let B^l and $B_j^l, l = 1, \dots, k, j = 1, \dots, s$ be $(s + 1) \times (s + 1)$ matrices given by

$$B^l := I_{s+1}, \quad B_j^l := \begin{pmatrix} 0 & e_j \\ e_j^T & 0 \end{pmatrix},$$

where $e_j \in \mathbb{R}^s$ is a vector whose j th component is one and all others are zero. We can show that Θ_l in (3.36) lands in the form of the spectrahedral sets in (2.2) and that

$$\lambda_l B^l + \sum_{j=1}^s \lambda_j^l B_j^l \geq 0 \Leftrightarrow \begin{pmatrix} \lambda_l I_s & \lambda^l \\ (\lambda^l)^T & \lambda_l \end{pmatrix} \geq 0,$$

where $\lambda_l \geq 0, l = 1, \dots, k$ and $\lambda^l := (\lambda_j^1, \dots, \lambda_j^s)$ with $\lambda_j^j \in \mathbb{R}, j = 1, \dots, s$. Now, invoking Theorem 3.1, we arrive at the desired result. \square

Let us now consider the uncertain multiobjective optimization problem (UP), where $\Omega_r, r = 1, \dots, p$ and $\Theta_l, l = 1, \dots, k$ in (2.2) are replaced by the following *box* uncertainty sets:

$$\begin{aligned} \Omega_r &:= \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid |u_i^r| \leq 1, i = 1, \dots, m\}, \\ \Theta_l &:= \{v^l := (v_1^l, \dots, v_s^l) \in \mathbb{R}^s \mid |v_j^l| \leq 1, j = 1, \dots, s\}. \end{aligned} \tag{3.39}$$

In this case, we obtain necessary/sufficient optimality conditions for robust (weak) Pareto solutions of problem (UP) under *box uncertainty*.

Corollary 3.7 (Robust optimality with box uncertainty) *Consider the problem (UP) with Ω_r and Θ_l given by (3.39), and let $\bar{x} \in \mathbb{R}^n$ be a robust feasible point of problem (UP).*

(i) *Assume that the characteristic cone $K := \text{coneco}\{(0_n, 1) \cup \text{epi } g_l^*(\cdot, v^l), v^l \in \Theta_l, l = 1, \dots, k\}$ is closed. If \bar{x} is a robust weak Pareto solution of problem (UP), then there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_j^l \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that*

$$\begin{pmatrix} \sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l & \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_j^l \theta_j^l) \right) \\ \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_j^l \theta_j^l) \right)^T & \sum_{r=1}^p (\alpha_r B^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_j^l \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \end{pmatrix} \geq 0, \tag{3.40}$$

$$\alpha_r \geq |\alpha_r^i|, i = 1, \dots, m, r = 1, \dots, p, \lambda_l \geq |\lambda_j^l|, j = 1, \dots, s, l = 1, \dots, k, \tag{3.41}$$

where $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r)$.

(ii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_j^l \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.40) and (3.41) hold, then \bar{x} is a robust weak Pareto solution of problem (UP).*

(iii) *If there exist $(\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p, \alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_l \geq 0, \lambda_j^l \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that (3.40) and (3.41) hold, then \bar{x} is a robust Pareto solution of problem (UP).*

Proof Let A^r and $A_i^r, r = 1, \dots, p, i = 1, \dots, m$ be $(2m \times 2m)$ matrices given by

$$A^r := I_{2m}, \quad A_i^r := \begin{pmatrix} E_i & 0 \\ 0 & -E_i \end{pmatrix},$$

where E_i is an $(m \times m)$ diagonal matrix with one in the (i, i) th entry and zeros elsewhere. Then, we have

$$\begin{aligned} \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid A^r + \sum_{i=1}^m u_i^r A_i^r \geq 0\} \\ = \{u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m \mid 1 + u_i^r \geq 0, 1 - u_i^r \geq 0, i = 1, \dots, m\} = \Omega_r, \end{aligned}$$

which means that Ω in (3.39) lands in the form of the spectrahedral sets in (2.2). In this way, we can verify that, for each $r \in \{1, \dots, p\}$,

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0 \Leftrightarrow \alpha_r \geq |\alpha_r^i|, \quad i = 1, \dots, m,$$

where $\alpha_r \geq 0$ and $\alpha_r^i \in \mathbb{R}, i = 1, \dots, m$. Similarly, let B^l and $B_j^l, l = 1, \dots, k, j = 1, \dots, s$ be $(2s \times 2s)$ matrices given by

$$B^l := I_{2s}, \quad B_j^l := \begin{pmatrix} E_j & 0 \\ 0 & -E_j \end{pmatrix},$$

where E_j is an $(s \times s)$ diagonal matrix with one in the (j, j) th entry and zeros elsewhere. We can show that Θ_l in (3.36) lands in the form of the spectrahedral sets in (2.2) and that

$$\lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l \geq 0 \Leftrightarrow \lambda_l \geq |\lambda_l^j|, \quad j = 1, \dots, s,$$

where $\lambda_l \geq 0, l = 1, \dots, k$ and $\lambda_l^j \in \mathbb{R}, j = 1, \dots, s$. Now, invoking Theorem 3.1, we arrive at the desired result. □

4 Finding robust Pareto solutions via semidefinite programming

In this section, we provide a way to find robust (weak) Pareto solutions for the uncertain multiobjective optimization problem (UP) by solving a related semidefinite programming (SDP) problem. Since the underlying problems are convex, our approach is based on commonly used weighted-sum scalarization methods that allow one to find both robust weakly Pareto solutions and robust Pareto solutions. This is done by defining robust weighted-sum scalarization optimization problems of (RP) as follows.

For each $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, we consider a robust weighted-sum optimization problem of the form:

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{r=1}^p \alpha_r \max_{u^r \in \Omega_r} f_r(x, u^r) \mid g_l(x, v^l) \leq 0, \forall v^l \in \Theta_l, l = 1, \dots, k \right\}, \quad (RP_\alpha)$$

where $f_r, r = 1, \dots, p$ and $g_l, l = 1, \dots, k$ are bi-functions given as in (2.3), and $\Omega_r, r = 1, \dots, p$ and $\Theta_l, l = 1, \dots, k$ are uncertainty sets given as in (2.2).

We propose a semidefinite programming (SDP) reformulation problem for the problem (RP_α) as follows:

$$\begin{aligned}
 \inf_{(w, W, \lambda^r, t^l, \tilde{Z}^r, Z^l)} & \left\{ \sum_{r=1}^p \alpha_r (\lambda^r + \text{Tr}(\tilde{Z}^r A^r)) \mid \begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \succeq 0, w \in \mathbb{R}^n, W \in S^n, \right. & (RP_\alpha^*) \\
 & \beta^r + (\xi^r)^T w + \text{Tr}(W Q^r) - \lambda^r = 0, r = 1, \dots, p, \\
 & \beta_i^r + (\xi_i^r)^T w + \text{Tr}(\tilde{Z}^r A_i^r) = 0, i = 1, \dots, m, r = 1, \dots, p, \\
 & t^l + \text{Tr}(Z^l B^l) \leq 0, l = 1, \dots, k, \\
 & \gamma^l + (\theta^l)^T w + \text{Tr}(W M^l) - t^l = 0, l = 1, \dots, k, \\
 & \gamma_j^l + (\theta_j^l)^T w + \text{Tr}(Z^l B_j^l) = 0, j = 1, \dots, s, l = 1, \dots, k, \\
 & \lambda^r \in \mathbb{R}, t^l \in \mathbb{R}, \tilde{Z}^r \succeq 0, Z^l \succeq 0, r = 1, \dots, p, l = 1, \dots, k \left. \right\}.
 \end{aligned}$$

We are now ready to establish solution relationships between the robust multiobjective optimization problem (RP) and the (scalar) semidefinite programming (SDP) reformulation problem (RP_α^*) , which is a relaxation problem of the corresponding robust weighted-sum optimization problem (RP_α) . This result provides us a way to find robust (weak) Pareto solutions of the uncertain multiobjective optimization problem (UP) by solving the (scalar) SDP reformulation problem (RP_α^*) .

Theorem 4.1 (Robust (weak) Pareto solutions via SDP) *Let the characteristic cone $K := \text{coneco}\{(0_n, 1) \cup \text{epi } g_l^*(\cdot, v^l), v^l \in \Theta_l, l = 1, \dots, k\}$ be closed, and assume that there exist $\tilde{u}^r := (\tilde{u}_1^r, \dots, \tilde{u}_m^r) \in \mathbb{R}^m, r = 1, \dots, p$ and $\tilde{v}^l := (\tilde{v}_1^l, \dots, \tilde{v}_s^l) \in \mathbb{R}^s, l = 1, \dots, k$ such that*

$$A^r + \sum_{i=1}^m \tilde{u}_i^r A_i^r \succ 0, B^l + \sum_{j=1}^s \tilde{v}_j^l B_j^l \succ 0. \tag{4.1}$$

Then, the following assertions hold:

(i) *If \bar{x} is a robust weak Pareto solution of problem (UP) , then there exist $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ and $\tilde{Z}_0^r \succeq 0, Z_0^l \succeq 0, r = 1, \dots, p, l = 1, \dots, k$ such that $(\bar{x}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) , where $\bar{W} := \bar{x} \bar{x}^T, \bar{\lambda}^r := \bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r$ and $\bar{t}^l := \bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l$.*

(ii) **(Finding robust weak Pareto solutions)** *Let $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ be such that the problem (RP_α) admits an optimal solution. If $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) , then \bar{w} is a robust weak Pareto solution of problem (UP) .*

(iii) **(Finding robust Pareto solutions)** *Let $\alpha \in \text{int} \mathbb{R}_+^p$ be such that the problem (RP_α) admits an optimal solution. If $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) , then \bar{w} is a robust Pareto solution of problem (UP) .*

Proof (SDP reformulation with optimal solutions for (RP_α^*)) Let $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$ be such that the problem (RP_α) admits an optimal solution and assume that \bar{x} is an optimal solution of problem (RP_α) . Denote by $\text{val}(RP_\alpha)$ and $\text{val}(RP_\alpha^*)$ the optimal values of problems (RP_α) and (RP_α^*) , respectively.

We first show that there exist $\tilde{Z}_0^r \succeq 0, Z_0^l \succeq 0, r = 1, \dots, p, l = 1, \dots, k$ such that $(\bar{x}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$, where $\bar{\lambda}^r := \bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r, \bar{t}^l :=$

$\bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l$ and $\bar{W} := \bar{x} \bar{x}^T$, is an optimal solution of problem (RP_α^*) and

$$\text{val}(\text{RP}_\alpha) = \text{val}(\text{RP}_\alpha^*) = \sum_{r=1}^p \alpha_r (\bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r)). \tag{4.2}$$

Let $F_r(\bar{x}) := \max_{u^r \in \Omega_r} f_r(\bar{x}, u^r)$ for $r = 1, \dots, p$. As \bar{x} is an optimal solution of problem (RP_α) , we obtain that

$$\text{val}(\text{RP}_\alpha) = \sum_{r=1}^p \alpha_r F_r(\bar{x}), \tag{4.3}$$

and that

$$\max_{v^l := (v_1^l, \dots, v_s^l) \in \Theta_l} \{\bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l + \sum_{j=1}^s v_j^l ((\theta_j^l)^T \bar{x} + \gamma_j^l)\} \leq 0, \quad l = 1, \dots, k,$$

which can be equivalently expressed as

$$[v^l := (v_1^l, \dots, v_s^l) \in \mathbb{R}^s, B^l + \sum_{j=1}^s v_j^l B_j^l \geq 0] \Rightarrow \bar{t}^l + \sum_{j=1}^s v_j^l \bar{t}_j^l \leq 0, \quad l = 1, \dots, k, \tag{4.4}$$

where $\bar{t}^l := \bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l$ and $\bar{t}_j^l := (\theta_j^l)^T \bar{x} + \gamma_j^l$. The condition (4.1) shows that the strict feasibility condition holds for (4.4). Hence, invoking a strong duality in semidefinite programming (cf. Blekherman et al., 2012, Theorem 2.15), we find $Z_0^l \geq 0, l = 1, \dots, k$ such that

$$\begin{aligned} \bar{t}^l + \text{Tr}(Z_0^l B^l) &\leq 0, \quad l = 1, \dots, k, \\ \bar{t}_j^l + \text{Tr}(Z_0^l B_j^l) &= 0, \quad j = 1, \dots, s, l = 1, \dots, k. \end{aligned}$$

Similarly, by $\max_{u^r := (u_1^r, \dots, u_m^r) \in \Omega_r} \{\bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r + \sum_{i=1}^m u_i^r ((\xi_i^r)^T \bar{x} + \beta_i^r)\} = F_r(\bar{x}), r = 1, \dots, p$, we have

$$[u^r := (u_1^r, \dots, u_m^r) \in \mathbb{R}^m, A^r + \sum_{i=1}^m u_i^r A_i^r \geq 0] \Rightarrow \bar{\lambda}^r + \sum_{i=1}^m u_i^r \bar{\lambda}_i^r \leq F_r(\bar{x}), \quad r = 1, \dots, p,$$

where $\bar{\lambda}^r := \bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r$ and $\bar{\lambda}_i^r := (\xi_i^r)^T \bar{x} + \beta_i^r$. Therefore, we can find $\tilde{Z}_0^r \geq 0, r = 1, \dots, p$ such that

$$\begin{aligned} \bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r) &\leq F_r(\bar{x}), \quad r = 1, \dots, p, \\ \bar{\lambda}_i^r + \text{Tr}(\tilde{Z}_0^r A_i^r) &= 0, \quad i = 1, \dots, m, r = 1, \dots, p. \end{aligned} \tag{4.5}$$

Now, letting $\bar{W} := \bar{x} \bar{x}^T$, we see that $(\bar{x}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is a feasible point of problem (RP_α^*) , and it also entails that

$$\text{val}(\text{RP}_\alpha^*) \leq \sum_{r=1}^p \alpha_r (\bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r)) \leq \sum_{r=1}^p \alpha_r F_r(\bar{x}) = \text{val}(\text{RP}_\alpha), \tag{4.6}$$

where the second inequality holds by noting (4.5) and the fact that $\alpha_r \geq 0, r = 1, \dots, p$, while the equality holds by (4.3).

Next, we show that $\text{val}(\text{RP}_\alpha) \leq \text{val}(\text{RP}_\alpha^*)$. To do this, let $(w, W, \lambda^r, t^l, \tilde{Z}^r, Z^l, r = 1, \dots, p, l = 1, \dots, k)$ be a feasible point of problem (RP_α^*) . Then, $w \in \mathbb{R}^n, W \in S^n, \lambda^r \in \mathbb{R}, t^l \in \mathbb{R}, \tilde{Z}^r \geq 0, Z^l \geq 0, r = 1, \dots, p, l = 1, \dots, k$ and

$$\begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \succeq 0, \tag{4.7}$$

$$\begin{aligned} \beta^r + (\xi^r)^T w + \text{Tr}(W Q^r) - \lambda^r &= 0, r = 1, \dots, p, \\ \beta_i^r + (\xi_i^r)^T w + \text{Tr}(\tilde{Z}^r A_i^r) &= 0, i = 1, \dots, m, r = 1, \dots, p, \\ t^l + \text{Tr}(Z^l B^l) &\leq 0, l = 1, \dots, k, \end{aligned} \tag{4.8}$$

$$\gamma^l + (\theta^l)^T w + \text{Tr}(W M^l) - t^l = 0, l = 1, \dots, k, \tag{4.9}$$

$$\gamma_j^l + (\theta_j^l)^T w + \text{Tr}(Z^l B_j^l) = 0, j = 1, \dots, s, l = 1, \dots, k. \tag{4.10}$$

Note, for any $v^l := (v_1^l, \dots, v_s^l) \in \Theta_l, l = 1, \dots, k$, that $\text{Tr}[Z^l(B^l + \sum_{j=1}^s v_j^l B_j^l)] \geq 0, l = 1, \dots, k$ due to $Z^l \geq 0$ and $B^l + \sum_{j=1}^s v_j^l B_j^l \geq 0$. Then, we deduce from (4.8) and (4.9) that, for given $v^l := (v_1^l, \dots, v_s^l) \in \Theta_l, l = 1, \dots, k$,

$$\begin{aligned} \gamma^l + (\theta^l)^T w + \text{Tr}(w w^T M^l) - \sum_{j=1}^s v_j^l \text{Tr}(Z^l B_j^l) &\leq \gamma^l + (\theta^l)^T w + \text{Tr}(w w^T M^l) + \text{Tr}(Z^l B^l) \\ &= t^l + \text{Tr}(Z^l B^l) - \text{Tr}((W - w w^T) M^l) \leq 0, l = 1, \dots, k, \end{aligned}$$

where we should note that $\text{Tr}((W - w w^T) M^l) \geq 0$ due to $M^l \geq 0$ and, by (4.7), $W - w w^T \succeq 0$. Therefore, by (4.10), for given $v^l := (v_1^l, \dots, v_s^l) \in \Theta_l, l = 1, \dots, k$,

$$g_l(w, v^l) = \gamma^l + (\theta^l)^T w + \text{Tr}(w w^T M^l) - \sum_{j=1}^s v_j^l \text{Tr}(Z^l B_j^l) \leq 0, l = 1, \dots, k.$$

This shows that w is a feasible point of problem (RP_α) , and so

$$\text{val}(\text{RP}_\alpha) \leq \sum_{r=1}^p \alpha_r F_r(w), \tag{4.11}$$

where $F_r(w) := \max_{u^r \in \Omega_r} f_r(w, u^r), r = 1, \dots, p$.

Similarly, for given $u^r := (u_1^r, \dots, u_m^r) \in \Omega_r, r = 1, \dots, p$, we have

$$\beta^r + (\xi^r)^T w + \text{Tr}(w w^T Q^r) - \sum_{i=1}^m u_i^r \text{Tr}(\tilde{Z}^r A_i^r) \leq \lambda^r + \text{Tr}(\tilde{Z}^r A^r), r = 1, \dots, p,$$

which entails that

$$\begin{aligned} F_r(w) &= \max_{u^r := (u_1^r, \dots, u_m^r) \in \Omega_r} \{ \beta^r + (\xi^r)^T w + \text{Tr}(w w^T Q^r) - \sum_{i=1}^m u_i^r \text{Tr}(\tilde{Z}^r A_i^r) \} \\ &\leq \lambda^r + \text{Tr}(\tilde{Z}^r A^r), r = 1, \dots, p. \end{aligned}$$

This together with (4.11) shows that $\text{val}(\text{RP}_\alpha) \leq \sum_{r=1}^p \alpha_r (\lambda^r + \text{Tr}(\tilde{Z}^r A^r))$, and so $\text{val}(\text{RP}_\alpha) \leq \text{val}(\text{RP}_\alpha^*)$ inasmuch as $(w, W, \lambda^r, t^l, \tilde{Z}^r, Z^l, r = 1, \dots, p, l = 1, \dots, k)$ was arbitrarily taken.

Now, taking (4.6) into account, we arrive at the conclusion that

$$\text{val}(\text{RP}_\alpha) = \text{val}(\text{RP}_\alpha^*) = \sum_{r=1}^p \alpha_r (\bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r)),$$

which further ensures that $(\bar{x}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) . In other words, (4.2) has been established.

(i) Let \bar{x} be a robust weak Pareto solution of problem (UP). Under the closedness of the cone K , we invoke Theorem 3.1(i) to assert that there exist $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$, $\alpha_r^i \in \mathbb{R}, r = 1, \dots, p, i = 1, \dots, m$ and $\lambda_j \geq 0, \lambda_j^j \in \mathbb{R}, j = 1, \dots, s, l = 1, \dots, k$ such that

$$\left(\begin{array}{c} \sum_{r=1}^p \alpha_r Q^r + \sum_{l=1}^k \lambda_l M^l \qquad \qquad \qquad \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right) \\ \frac{1}{2} \left(\sum_{r=1}^p (\alpha_r \xi^r + \sum_{i=1}^m \alpha_r^i \xi_i^r) + \sum_{l=1}^k (\lambda_l \theta^l + \sum_{j=1}^s \lambda_l^j \theta_j^l) \right)^T \sum_{r=1}^p (\alpha_r \beta^r + \sum_{i=1}^m \alpha_r^i \beta_i^r) + \sum_{l=1}^k (\lambda_l \gamma^l + \sum_{j=1}^s \lambda_l^j \gamma_j^l) - \sum_{r=1}^p \alpha_r F_r(\bar{x}) \end{array} \right) \geq 0 \tag{4.12}$$

$$\alpha_r A^r + \sum_{i=1}^m \alpha_r^i A_i^r \geq 0, r = 1, \dots, p, \lambda_l B^l + \sum_{j=1}^s \lambda_l^j B_j^l \geq 0, l = 1, \dots, k. \tag{4.13}$$

Denoting by C the robust feasible set of problem (UP), it holds that C is also the feasible set of the corresponding robust weighted-sum optimization problem (RP_α) . Proceeding similarly as in the proof of Theorem 3.1(ii), we conclude by (4.12) and (4.13) that

$$\sum_{r=1}^p \alpha_r F_r(x) \geq \sum_{r=1}^p \alpha_r F_r(\bar{x}) \text{ for all } x \in C,$$

which shows that \bar{x} is an optimal solution of problem (RP_α) . Then, as shown above, there exist $\tilde{Z}_0^r \geq 0, Z_0^l \geq 0, r = 1, \dots, p, l = 1, \dots, k$ such that $(\bar{x}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) , where $\bar{\lambda}^r := \bar{x}^T Q^r \bar{x} + (\xi^r)^T \bar{x} + \beta^r, \bar{t}^l := \bar{x}^T M^l \bar{x} + (\theta^l)^T \bar{x} + \gamma^l$ and $\bar{W} := \bar{x} \bar{x}^T$.

(ii) **(Finding a robust weak Pareto solution of (UP) via (RP_α^*))** Let $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$ be such that the problem (RP_α) admits an optimal solution. Then, as shown by (4.2), it holds that

$$\text{val}(\text{RP}_\alpha) = \text{val}(\text{RP}_\alpha^*). \tag{4.14}$$

Now, let $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ be an optimal solution of problem (RP_α^*) . Then, $\bar{w} \in \mathbb{R}^n, \bar{W} \in S^n, \bar{\lambda}^r \in \mathbb{R}, \bar{t}^l \in \mathbb{R}, \tilde{Z}_0^r \geq 0, Z_0^l \geq 0, r = 1, \dots, p, l = 1, \dots, k$ and

$$\text{val}(\text{RP}_\alpha^*) = \sum_{r=1}^p \alpha_r (\bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r)), \tag{4.15}$$

$$\begin{pmatrix} 1 & \bar{w}^T \\ \bar{w} & \bar{W} \end{pmatrix} \succeq 0, \tag{4.16}$$

$$\beta^r + (\xi^r)^T \bar{w} + \text{Tr}(\bar{W} Q^r) - \bar{\lambda}^r = 0, r = 1, \dots, p, \tag{4.17}$$

$$\beta_i^r + (\xi_i^r)^T \bar{w} + \text{Tr}(\tilde{Z}_0^r A_i^r) = 0, i = 1, \dots, m, r = 1, \dots, p, \tag{4.18}$$

$$\bar{t}^l + \text{Tr}(Z_0^l B^l) \leq 0, l = 1, \dots, k, \tag{4.19}$$

$$\gamma^l + (\theta^l)^T \bar{w} + \text{Tr}(\bar{W} M^l) - \bar{t}^l = 0, l = 1, \dots, k, \tag{4.20}$$

$$\gamma_j^l + (\theta_j^l)^T \bar{w} + \text{Tr}(Z_0^l B_j^l) = 0, j = 1, \dots, s, l = 1, \dots, k. \tag{4.21}$$

Arguing as above, we can conclude from (4.16), (4.19), (4.20) and (4.21) that \bar{w} is a feasible point of problem (RP_α) , and so \bar{w} is also a robust feasible point of problem (UP) . Similarly, we get by (4.16), (4.17) and (4.18) that

$$F_r(\bar{w}) \leq \bar{\lambda}^r + \text{Tr}(\tilde{Z}_0^r A^r), r = 1, \dots, p. \tag{4.22}$$

We claim that \bar{w} is a robust weak Pareto solution of problem (UP) . Assume on the contrary that there exists another robust feasible point \hat{x} of problem (UP) such that

$$F_r(\hat{x}) < F_r(\bar{w}), r = 1, \dots, p,$$

where $F_r(w) := \max_{u^r \in \Omega_r} f_r(w, u^r), r = 1, \dots, p$. Note here that \hat{x} is also a feasible point of problem (RP_α) . Then, by (4.22) and (4.15), we have

$$\text{val}(\text{RP}_\alpha) \leq \sum_{r=1}^p \alpha_r F_r(\hat{x}) < \sum_{r=1}^p \alpha_r F_r(\bar{w}) \leq \text{val}(\text{RP}_\alpha^*),$$

which contradicts (4.14). Consequently, \bar{w} is a robust weak Pareto solution of problem (UP) .

(iii) **(Finding a robust Pareto solution of (UP) via (RP_α^*))** Let $\alpha := (\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$ be such that the problem (RP_α) admits an optimal solution and assume that $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$ is an optimal solution of problem (RP_α^*) . Clearly, the assertions in (4.14)–(4.22) are valid for this case. Arguing similarly as in the proof of (ii), we can show that there is no other robust feasible point \hat{x} of problem (UP) such that

$$F_r(\hat{x}) \leq F_r(\bar{w}), r = 1, \dots, p \text{ and } F_r(\hat{x}) < F_r(\bar{w}) \text{ for some } r \in \{1, \dots, p\},$$

where $F_r(w) := \max_{u^r \in \Omega_r} f_r(w, u^r), r = 1, \dots, p$. Therefore, \bar{w} is a robust Pareto solution of problem (UP) . □

Remark 4.2 If the uncertainty sets $\Omega_r, r = 1, \dots, p$ and $\Theta_l, l = 1, \dots, k$ in (2.2) have nonempty interiors, we can find $\tilde{u}^r := (\tilde{u}_1^r, \dots, \tilde{u}_m^r) \in \mathbb{R}^m, r = 1, \dots, p$ and $\tilde{v}^l := (\tilde{v}_1^l, \dots, \tilde{v}_s^l) \in \mathbb{R}^s, l = 1, \dots, k$ such that (4.1) holds.

Under the closedness of the cone K and the validation of (4.1), assume that the problem (RP_α) admits an optimal solution for each $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$. Then, Procedure 1 below presents the above SDP scheme for finding robust (weak) Pareto solutions of problem (UP) .

Procedure 1 SDP scheme to find robust (weak) Pareto solutions of (UP)

Input: An uncertain convex quadratic multiobjective problem of the form (UP) .

Step 1. Take $\alpha := (\alpha_1, \dots, \alpha_p) \in \text{int}\mathbb{R}_+^p$ (or $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$).

Step 2. Solve the SDP problem (RP_α^*) to find its optimal solution.

Step 3. If there is no optimal solution of (RP_α^*) , go back to Step 1;

otherwise, find an optimal solution of (RP_α^*) as $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \tilde{Z}_0^r, Z_0^l, r = 1, \dots, p, l = 1, \dots, k)$.

Step 4. Extract \bar{w} from the optimal solution of (RP_α^*) .

Output: \bar{w} is a robust (weak) Pareto solution of (UP) .

5 Conclusion and perspectives

In this paper, we have considered a convex quadratic multiobjective optimization problem, where both the objective and constraint functions involve data uncertainty. To handle the proposed uncertain multiobjective program, we have employed the robust deterministic approach to examine robust optimality conditions and find robust (weak) Pareto solutions of the underlying uncertain multiobjective problem. More specifically, necessary conditions and sufficient conditions in terms of linear matrix inequalities for robust (weak) Pareto optimality of the multiobjective optimization problem have been established. It has been shown that the obtained optimality conditions can be verified by way of checking alternatively other criteria including a robust Karush-Kuhn-Tucker condition. In addition, we have proved that a (scalar) relaxation problem of a robust weighted-sum optimization program of the multiobjective optimization problem can be solved by using a semidefinite programming (SDP) problem. This result has demonstrated that the obtained SDP scheme can be employed to numerically calculate a robust (weak) Pareto solution of the uncertain multiobjective problem that is implemented in MATLAB.

It would be interesting to know how we can employ another approach such as the *primal* reformulation/relaxation as in Ben-Tal et al. (2015) to find (weak) Pareto solutions of a robust convex quadratic multiobjective optimization problem and perform a comparison with the results obtained in this paper. Besides, a land combat vehicle system or other defence systems (cf. Nguyen et al., 2016; Nguyen and Cao, 2017, 2019) can be expressed in terms of multiobjective optimization problems that contain *uncertainty* data due to the incomplete, error estimation or noisy information. For example, unknown environment factors such as land surface, air and sea surroundings or weather would be shown up when a defence system is operating in a new terrain. The different types of operation scenarios and physical links between defence system components pose a challenge to the decision maker in providing the “best” combination (solution) of defence system configuration. For a force design problem, this would have to be scaled up to what is the “best” combination of defence systems to achieve multiple objectives under various threats and scenarios assumptions. It would be of great interest to see how we can employ our obtained SDP scheme to develop a suit of efficient algorithms, which allow end-users to easily elicit the decision maker’s preferences and perform a good visualization in large scale defence scenarios for uncertain force design problems.

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Appendix: Numerical examples

The following examples show how we can employ the SDP reformulation scheme obtained in Theorem 4.1 (Procedure 1) to find robust (weak) Pareto solutions of uncertain convex quadratic multiobjective optimization problems.

The first example is dealt with a family of uncertain convex quadratic multiobjective problems of (EU_c) by specifying the value c of the objectives, while the second example is concerned with a family of uncertain convex quadratic multiobjective problems of (EU_n) by specifying the dimension n of the decision variables.

Example 6.1 (Calculating Pareto solutions via SDP) Let us consider a family of uncertain convex quadratic multiobjective problems:

$$\min_{x \in \mathbb{R}^2} \{ (f_1(x, u), f_2(x, u), f_3(x, u), f_4(x, u)) \mid g_l(x, v) \leq 0, l = 1, 2, 3, 4, 5, 6 \}, \quad (\text{EU}_c)$$

where $c \in [0, +\infty)$ is a given parameter, $u := (u_1, u_2) \in \Omega$ and $v := (v_1, v_2) \in \Theta$ are uncertain parameters and $f_r, r = 1, 2, 3, 4$ and $g_l, l = 1, 2, 3, 4, 5, 6$ are bi-functions given by

$$\begin{aligned} f_1(x, u) &:= x_1^2 + x_2^2 - cx_1 + 1 + u_1 + u_2, & f_2(x, u) &:= x_2^2 + 1 - u_1 + u_2, \\ f_3(x, u) &:= x_2^2 + 2x_2 + u_1, & f_4(x, u) &:= x_1^2 + 4x_2^2 - cx_1 + x_2 - u_2, \\ g_1(x, v) &:= x_2^2 - x_1 + v_1x_2 + v_2x_1 - 2c - 4, & g_2(x, v) &:= -x_1 + v_1 - 3, \\ g_3(x, v) &:= -x_2 - v_2 - 3, & g_4(x, v) &:= x_1^2 + x_2^2 - v_1x_1 - v_2x_2 - c^2 - 2c - 5, \\ g_5(x, v) &:= x_2 - 4, & g_6(x, v) &:= x_1 - c - 4, \end{aligned} \quad x := (x_1, x_2) \in \mathbb{R}^2.$$

Here, the uncertainty sets Ω and Θ are given by

$$\Omega := \{ u := (u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \leq 1 \}, \quad \Theta := \{ v := (v_1, v_2) \in \mathbb{R}^2 \mid \frac{v_1^2}{4} + \frac{v_2^2}{9} \leq 1 \}. \quad (6.1)$$

Now, we consider the robust convex quadratic multiobjective problem of (EU_c) as follows:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \{ & (\max_{u \in \Omega} f_1(x, u), \max_{u \in \Omega} f_2(x, u), \max_{u \in \Omega} f_3(x, u), \\ & \max_{u \in \Omega} f_4(x, u)) \mid g_l(x, v) \leq 0, l = 1, 2, 3, 4, 5, 6, \forall v \in \Theta \}. \end{aligned} \quad (\text{ER}_c)$$

Note that the problem (ER_c) can be expressed in terms of problem (RP) , where $\Omega_r := \Omega, r = 1, 2, 3, 4, \Theta_l := \Theta, l = 1, 2, 3, 4, 5, 6$ are described respectively by

$$A^r := I_3, B^l := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1^r := B_1^l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2^r := B_2^l = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $f_r, r = 1, 2, 3, 4, g_l, l = 1, 2, 3, 4, 5, 6$ are given respectively by $Q^1 := I_2, Q^2 := Q^3 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q^4 := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \xi^1 := (-c, 0), \xi^2 := 0_2, \xi^3 := (0, 2), \xi^4 := (-c, 1), \xi_i^r := 0_2, r = 1, 2, 3, 4, i = 1, 2, \beta^1 := \beta^2 := 1, \beta^3 := \beta^4 := 0, \beta_1^1 := \beta_2^1 := \beta_2^2 := \beta_1^3 := 1, \beta_1^2 := \beta_2^4 := -1, \beta_2^3 := \beta_1^4 := 0$ and $M^1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, M^2 := M^3 := M^5 := M^6 := 0_{2 \times 2}, M^4 := I_2, \theta^1 := \theta^2 := (-1, 0), \theta^3 := (0, -1), \theta^4 := 0_2, \theta^5 := (0, 1), \theta^6 := (1, 0), \theta_1^1 := (0, 1), \theta_1^2 := (1, 0), \theta_2^1 := \theta_2^2 := \theta_1^3 := \theta_2^3 := \theta_1^5 := \theta_2^5 := \theta_1^6 := \theta_2^6 := 0_2, \theta_1^4 := (-1, 0), \theta_2^4 := (0, -1), \gamma^1 := -2c - 4, \gamma^2 := \gamma^3 := -3, \gamma^4 := -c^2 - 2c - 5, \gamma^5 := -4, \gamma^6 := -c - 4, \gamma_1^1 := \gamma_2^1 := \gamma_2^2 := \gamma_1^3 := \gamma_1^4 := \gamma_2^4 := \gamma_1^5 := \gamma_2^5 := \gamma_1^6 := \gamma_2^6 := 0, \gamma_1^2 := 1, \gamma_3^3 := -1.$

We now use the SDP reformulation, obtained in Theorem 4.1, to find a robust (weak) Pareto solution of problem (EU_c) . Taking $\tilde{x} := (c, 1)$, we see that $g_l(\tilde{x}, v) < 0$ for all $v \in \Theta$ and $l = 1, 2, 3, 4, 5, 6$. Thus, the characteristic cone $K := \text{coneco}\{(0_2, 1) \cup \text{epi } g_l^*(\cdot, v), v \in \Theta, l = 1, 2, 3, 4, 5, 6\}$ is closed by virtue of Remark 3.2. Moreover, by taking $\tilde{u} := \tilde{v} := (0, 0) \in \mathbb{R}^2$,

it holds that $A^r + \sum_{i=1}^2 \tilde{u}_i A_i^r > 0, r = 1, 2, 3, 4$ and $B^l + \sum_{j=1}^2 \tilde{v}_j B_j^l > 0, l = 1, 2, 3, 4, 5, 6$.

Namely, all the assumptions of Theorem 4.1 are fulfilled in this setting.

Let $\alpha := (1, 1, 1, 1) \in \text{int}\mathbb{R}_+^4$, and consider a corresponding robust weighted-sum optimization problem of (ER_c) as follows:

$$\inf_{x \in \mathbb{R}^2} \left\{ \sum_{r=1}^4 \max_{u \in \Omega} f_r(x, u) \mid g_l(x, v) \leq 0, l = 1, 2, 3, 4, 5, 6, \forall v \in \Theta \right\}. \tag{E_c}$$

Note that the (scalar) weighted-sum problem (E_c) has optimal solutions as its objective function is a continuous function and its feasible set is a compact set.

The SDP reformulation problem of (E_c) is given by

$$\begin{aligned} \inf_{(w, W, \lambda^r, t^l, \tilde{Z}^r, Z^l)} \left\{ \sum_{r=1}^4 (\lambda^r + \text{Tr}(\tilde{Z}^r A^r)) \mid \begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \succeq 0, w \in \mathbb{R}^2, W \in S^2, \right. \\ \beta^r + (\xi^r)^T w + \text{Tr}(W Q^r) - \lambda^r = 0, r = 1, 2, 3, 4, \\ \beta_i^r + (\xi_i^r)^T w + \text{Tr}(\tilde{Z}^r A_i^r) = 0, i = 1, 2, r = 1, 2, 3, 4, \\ t^l + \text{Tr}(Z^l B^l) \leq 0, l = 1, 2, 3, 4, 5, 6, \\ \gamma^l + (\theta^l)^T w + \text{Tr}(W M^l) - t^l = 0, l = 1, 2, 3, 4, 5, 6, \\ \gamma_j^l + (\theta_j^l)^T w + \text{Tr}(Z^l B_j^l) = 0, j = 1, 2, l = 1, 2, 3, 4, 5, 6, \\ \left. \lambda^r \in \mathbb{R}, t^l \in \mathbb{R}, \tilde{Z}^r \succeq 0, Z^l \succeq 0, r = 1, 2, 3, 4, l = 1, 2, 3, 4, 5, 6 \right\}. \tag{E_c^*} \end{aligned}$$

Using the Matlab toolbox CVX (see e.g., Grant and Boyd, 2014), we solve the problem (E_c^{*}) with (for instance) $c := 4$ and the solver returns the weighted-sum optimal value as -1.17157 and an optimal solution of problem (E_c^{*}) with $c := 4$ as $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \bar{Z}_0^r, Z_0^l, r = 1, 2, 3, 4, l = 1, 2, 3, 4, 5, 6)$, where $\bar{w} = (2.0000, -2.9400\text{e-}08) \approx (2, 0)$. By Theorem 4.1(ii), we conclude that $\bar{w} = (2, 0)$ is a robust (weak) Pareto solution of problem (EU_c) with $c := 4$. (In this setting, we can re-check directly that \bar{w} is a (weak) Pareto solution of problem (ER_c) with $c := 4$.)

Similarly, we test with some other values of c (see Table 1). These the numerical tests are conducted on a computer with a 1.90GHz Intel(R) Core(TM) i7-8650U and 16.0GB RAM, equipped with MATLAB R2018b. In Table 1, “Robust Pareto Solutions” are optimal decision variables $x := (x_1, x_2)$ of (EU_c) and “Weighted-sum Values” are optimal values of (E_c^{*}) with the corresponding values of c .

Example 6.2 (Calculating Pareto solutions with higher dimensional decision variables) Let us consider a family of uncertain convex quadratic multiobjective problems:

$$\min_{x \in \mathbb{R}^n} \left\{ (f_1(x, u), f_2(x, u), f_3(x, u), f_4(x, u), f_5(x, u)) \mid g_l(x, v) \leq 0, l = 1, 2, 3 \right\}, \tag{EU_n}$$

where $n \in \mathbb{N}, n \geq 3$, is a given parameter, $u := (u_1, u_2) \in \Omega$ and $v := (v_1, v_2, v_3) \in \Theta$ are uncertain parameters and $f_r, r = 1, 2, 3, 4, 5$ and $g_l, l = 1, 2, 3$ are bi-functions given by

$$\begin{aligned} f_1(x, u) &:= x_1^2 + x_2^2 - nx_n + 1 + u_1 - u_2, & f_2(x, u) &:= x_2^2 - x_1 - \dots - x_n + 1 - u_1 + u_2, \\ f_3(x, u) &:= x_2^2 + 2x_2 - x_n + u_1 + u_2, & f_4(x, u) &:= x_1^2 + 4x_2^2 - x_1 + nx_n - u_2, \end{aligned}$$

Table 1 Robust Pareto solutions for (EU_c) with other values of c

c	Robust Pareto solutions	Weighted-sum values	CPU time (s)
1	(0.5000, 0)	+6.32843	0.50
5	(2.5000, 0)	−5.67157	0.44
10	(5.0000, 0)	−43.1716	0.41
25	(12.5000, 0)	−305.672	0.56
32	(16.0000, 0)	−505.172	0.47
60	(30.0000, 0)	−1793.17	0.44
100	(50.0001, 0)	−4993.17	0.35
210	(105.0000, 0)	−22043.2	0.53
301	(150.5000, 0)	−45293.7	0.50

$$f_5(x, u) := x_1 + x_2 + \dots + x_n - u_1 + u_2,$$

$$g_1(x, v) := x_2^2 - x_1 + v_1x_2 + v_2x_1 - 2n, \quad g_2(x, v) := x_1^2 + x_2^2 + \dots + x_n^2 - n + v_1 - v_3,$$

$$g_3(x, v) := x_1^2 + x_2^2 - v_1x_1 - v_2x_2 - x_n - n + v_3, \quad x \in \mathbb{R}^n.$$

Here, the uncertainty sets Ω and Θ are given by

$$\Omega := \{u := (u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 + \frac{1}{2}u_2^2 \leq 1, u_1 \geq 0\},$$

$$\Theta := \{v := (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1^2 + v_2^2 + v_3^2 \leq 1\}.$$

Now, we consider the *robust* convex quadratic multiobjective problem of (EU_n) as follows:

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u \in \Omega} f_1(x, u), \max_{u \in \Omega} f_2(x, u), \max_{u \in \Omega} f_3(x, u), \max_{u \in \Omega} f_4(x, u), \max_{u \in \Omega} f_5(x, u) \right) \mid g_l(x, v) \leq 0, \right. \\ \left. (\text{ER}_n) \right.$$

$$l = 1, 2, 3, \forall v \in \Theta \}.$$

Note that the problem (ER_n) can be expressed in terms of problem (RP), where $\Omega_r := \Omega, r = 1, 2, 3, 4, 5, \Theta_l := \Theta, l = 1, 2, 3$ are described respectively by

$$A^r := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_1^r := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2^r := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ B^l := I_4, B_1^l := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, B_2^l := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, B_3^l := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $f_r, r = 1, 2, 3, 4, 5, g_l, l = 1, 2, 3$ are given respectively by

$$Q^1 := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q^2 := Q^3 := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q^4 := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$Q^5 := 0_{n \times n}, \xi^1 := (0, 0, \dots, 0, -n), \xi^2 := (-1, -1, \dots, -1), \xi^3 := (0, 2, 0, \dots, 0, -1), \xi^4 := (-1, 0, \dots, 0, n), \xi^5 := (1, 1, \dots, 1), \xi_i^r := 0_n, r = 1, 2, 3, 4, 5, i = 1, 2, \beta^1 := \beta^2 := 1, \beta^3 := \beta^4 := \beta^5 := 0, \beta_1^1 := \beta_2^2 := \beta_3^3 := \beta_2^5 := 1, \beta_1^2 := \beta_1^3 := \beta_2^4 := \beta_1^5 := -1, \beta_1^4 := 0$ and $M^1 := Q_3, M^2 := I_n, M^3 := Q_1, \theta^1 := (-1, 0, \dots, 0), \theta^2 := 0_n, \theta^3 := (0, \dots, 0, -1), \theta_1^1 := (0, 1, 0, \dots, 0), \theta_2^1 := (1, 0, \dots, 0), \theta_3^1 := 0_n, \theta_1^2 := \theta_2^2 := \theta_3^2 := 0_n, \theta_1^3 := (-1, 0, \dots, 0), \theta_2^3 := (0, -1, 0, \dots, 0), \theta_3^3 := 0_n, \gamma^1 := -2n, \gamma^2 := \gamma^3 := -n, \gamma_1^1 := \gamma_2^1 := \gamma_3^1 := \gamma_2^2 := \gamma_1^3 := \gamma_2^3 := 0, \gamma_1^2 := \gamma_3^3 := 1, \gamma_3^2 := -1.$

We now use the SDP reformulation, obtained in Theorem 4.1, to find a robust (weak) Pareto solution of problem (EU_n). Taking $\bar{x} := 0_n$, we see that $g_l(\bar{x}, v) < 0$ for all $v \in \Theta$ and $l = 1, 2, 3$. Thus, the characteristic cone $K := \text{coneco}\{(0_n, 1) \cup \text{epi } g_l^*(\cdot, v), v \in \Theta, l = 1, 2, 3\}$ is closed by virtue of Remark 3.2. Moreover, by taking $\bar{u} := (\frac{1}{2}, 0) \in \mathbb{R}^2, \bar{v} := 0_3 \in \mathbb{R}^3$, it holds that $A^r + \sum_{i=1}^2 \bar{u}_i A_i^r \succ 0, r = 1, 2, 3, 4, 5$ and $B^l + \sum_{j=1}^3 \bar{v}_j B_j^l \succ 0, l = 1, 2, 3$. Namely, all the assumptions of Theorem 4.1 are fulfilled in this setting.

Let $\alpha := (1, 1, 1, 1, 1) \in \text{int}\mathbb{R}_+^5$, and consider a corresponding robust weighted-sum optimization problem of (ER_n) as follows:

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{r=1}^5 \max_{u \in \Omega} f_r(x, u) \mid g_l(x, v) \leq 0, l = 1, 2, 3, \forall v \in \Theta \right\}. \tag{E_n}$$

Note that the (scalar) weighted-sum problem (E_n) has optimal solutions as its objective function is a continuous function and its feasible set is a compact set.

The SDP reformulation problem of (E_n) is given by

$$\inf_{(w, W, \lambda^r, t^l, \tilde{Z}^r, Z^l)} \left\{ \sum_{r=1}^5 (\lambda^r + \text{Tr}(\tilde{Z}^r A^r)) \mid \begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \geq 0, w \in \mathbb{R}^n, W \in S^n, \right. \tag{E_n^*}$$

$$\left. \begin{aligned} \beta^r + (\xi^r)^T w + \text{Tr}(W Q^r) - \lambda^r &= 0, r = 1, 2, 3, 4, 5, \\ \beta_i^r + (\xi_i^r)^T w + \text{Tr}(\tilde{Z}^r A_i^r) &= 0, i = 1, 2, r = 1, 2, 3, 4, 5, \\ t^l + \text{Tr}(Z^l B^l) &\leq 0, l = 1, 2, 3, \\ \gamma^l + (\theta^l)^T w + \text{Tr}(W M^l) - t^l &= 0, l = 1, 2, 3, \\ \gamma_j^l + (\theta_j^l)^T w + \text{Tr}(Z^l B_j^l) &= 0, j = 1, 2, 3, l = 1, 2, 3, \\ \lambda^r \in \mathbb{R}, t^l \in \mathbb{R}, \tilde{Z}^r \geq 0, Z^l \geq 0, r &= 1, 2, 3, 4, 5, l = 1, 2, 3 \end{aligned} \right\}.$$

Using the Matlab toolbox CVX (see e.g., Grant and Boyd, 2014), we solve the problem (E_n^{*}) with (for instance) $n := 5$ and the solver returns the weighted-sum optimal value as +7.56512 and an optimal solution of problem (E_n^{*}) with $n := 5$ as $(\bar{w}, \bar{W}, \bar{\lambda}^r, \bar{t}^l, \bar{Z}_0^r, Z_0^l, r =$

Table 2 Robust Pareto solutions for (EU_n) with other values of n

n	Robust Pareto solutions	Weighted-sum values	CPU time (s)
3	(0.2079, -0.1350, 1.2346)	+8.20822	0.55
8	(0.2277, -0.1390, 0, ..., 0, 2.5524)	+6.88761	0.73
20	(0.2363, -0.1405, 0, ..., 0, 4.3023)	+5.13695	0.74
40	(0.2403, -0.1412, 0, ..., 0, 6.2055)	+3.2336	0.75
100	(0.2439, -0.1418, 0, ..., 0, 9.9250)	-0.486061	0.78
201	(0.2457, -0.1421, 0, ..., 0, 14.1246)	-4.68571	1.02
500	(0.2472, -0.1424, 0, ..., 0, 22.3272)	-12.8883	1.58
850	(0.2479, -0.1425, 0, ..., 0, 29.1291)	-19.6902	3.33
1000	(0.9844, -0.5689, 0, ..., 0, 63.1906)	-54.5551	5.57

1, 2, 3, 4, 5, $l = 1, 2, 3$), where $\bar{w} = (0.2206, -0.1376, 0, 0, 1.8757)$. By Theorem 4.1(iii), we conclude that $\bar{w} = (0.2206, -0.1376, 0, 0, 1.8757)$ is a robust (weak) Pareto solution of problem (EU_n) with $n := 5$. (In this setting, we can re-check directly that \bar{w} is a (weak) Pareto solution of problem (ER_n) with $n := 5$.)

Similarly, we test with some other values of n (see Table 2). These the numerical tests are conducted on a computer with a 1.90GHz Intel(R) Core(TM) i7-8650U and 16.0GB RAM, equipped with MATLAB R2018b. In Table 2, “Robust Pareto Solutions” are optimal decision variables $x := (x_1, \dots, x_n)$ of (EU_n) and “Weighted-sum Values” are optimal values of (E_n^*) with the corresponding values of n .

References

- Ben-Tal, A., Dick-Den, H., & Jean-Philippe, V. (2015). Deriving robust counterparts of nonlinear uncertain inequalities. *Mathematical Programming*, 149, 265–299.
- Ben-Tal, A., El Ghaoui, L., & Nemirovski, A. (2009). *Robust optimization*. Princeton University Press.
- Ben-Tal, A., & Nemirovski, A. (2001). *Lectures on modern convex optimization: Analysis, algorithms, and engineering applications*. SIAM.
- Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM Review*, 53(3), 464–501.
- Blekherman, G., Parrilo, P. A., & Thomas, R. (2012). *Semidefinite optimization and convex algebraic geometry*. SIAM.
- Boj, R. I., Grad, S. M., & Wanka, G. (2009). *Duality in vector optimization*. Springer.
- Burachik, R. S., & Jeyakumar, V. (2005). A dual condition for the convex subdifferential sum formula with applications. *Journal of Convex Analysis*, 12, 229–233.
- Chinchuluun, A., & Pardalos, P. M. (2007). A survey of recent developments in multiobjective optimization. *Annals of Operations Research*, 154, 29–50.
- Chuong, T. D. (2017). Robust alternative theorem for linear inequalities with applications to robust multi-objective optimization. *Operations Research Letters*, 45(6), 575–580.
- Chuong, T. D. (2018). Linear matrix inequality conditions and duality for a class of robust multiobjective convex polynomial programs. *SIAM Journal on Optimization*, 28(3), 2466–2488.
- Chuong, T. D. (2020). Robust optimality and duality in multiobjective optimization problems under data uncertainty. *SIAM Journal on Optimization*, 30(2), 1501–1526.
- Chuong, T. D., & Jeyakumar, V. (2017). A generalized Farkas lemma with a numerical certificate and linear semi-infinite programs with SDP duals. *Linear Algebra and its Applications*, 515, 38–52.
- Chuong, T. D., & Jeyakumar, V. (2021). Adjustable robust multi-objective linear optimization: Pareto optimal solutions via conic programming. (Submitted for publication).

- Deb, K., Pratap, A., Agarwal, S., & Meyarivan, T. (2002). A fast and elitist multiobjective genetic algorithm: NSGA-II. *IEEE Transactions on Evolutionary Computation*, 6(2), 182–197.
- Doolittle, E. K., Kerivin, H. L. M., & Wiecek, M. M. (2018). Robust multiobjective optimization with application to Internet routing. *Annals of Operations Research*, 271, 487–525.
- Ehrgott, M. (2005). *Multicriteria optimization*. Springer.
- Ehrgott, M., Ide, J., & Schobel, A. (2014). Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, 239(1), 7–31.
- Eichfelder, G., Niebling, J., & Rocktaschel, S. (2020). An algorithmic approach to multiobjective optimization with decision uncertainty. *Journal of Global Optimization*, 77, 3–25.
- Engau, A., & Sigler, D. (2020). Pareto solutions in multicriteria optimization under uncertainty. *European Journal of Operational Research*, 281, 357–368.
- Georgiev, P. G., Luc, D. T., & Pardalos, P. M. (2013). Robust aspects of solutions in deterministic multiple objective linear programming. *European Journal of Operational Research*, 229, 29–36.
- Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Perez, J. (2014). Robust solutions of multiobjective linear semi-infinite programs under constraint data uncertainty. *SIAM Journal on Optimization*, 24(3), 1402–1419.
- Grant, M., & Boyd, S. (2014). *CVX: Matlab software for disciplined convex programming*, Version 2.1. <http://cvxr.com/cvx>
- Hadka, D. (2014). *MOEA framework user guide: A free and open source java framework for multiobjective optimization* (Version 2.4). Retrieved May, 2018, from <http://www.moeaframework.org/>
- Helton, J. W., & Nie, J. (2010). Semidefinite representation of convex sets. *Mathematical Programming*, 122(1), 121–64.
- Ide, J., & Schobel, A. (2016). Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts. *OR Spectrum*, 38(1), 235–271.
- Jahn, J. (2004). *Vector Optimization: Theory, applications, and extensions*. Springer.
- Jeyakumar, V. (2003). Characterizing set containments involving infinite convex constraints and reverse-convex constraints. *SIAM Journal on Optimization*, 13(4), 947–959.
- Jeyakumar, V., & Li, G. (2010). Strong duality in robust convex programming: Complete characterizations. *SIAM Journal on Optimization*, 20, 3384–3407.
- Kuroiwa, D., & Lee, G. M. (2012). On robust multiobjective optimization. *Vietnam Journal of Mathematics*, 40(2–3), 305–317.
- Lee, J. H., & Jiao, L. (2021). Finding efficient solutions in robust multiple objective optimization with SOS-convex polynomial data. *Annals of Operations Research*, 296, 803–820.
- Lee, J. H., & Lee, G. M. (2018). On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems. *Annals of Operations Research*, 269(1–2), 419–438.
- Luc, D. T. (1989). *Theory of vector optimization*. Lecture notes in economics and mathematical systems. Springer.
- Miettinen, K. (1999). *Nonlinear multiobjective optimization*. Kluwer.
- Nguyen, M.-T., & Cao, T. (2017). A hybrid decision making model for evaluating land combat vehicle system. In G. Syme, D. Hatton MacDonald, B. Fulton, & J. Piantadosi (Eds.), *22nd International congress on modelling and simulation (MODSIM2017)*, Hobart, Australia (pp. 1399–1405).
- Nguyen, M.-T., & Cao, T. (2019). A multi-method approach to evaluate land combat vehicle system. *International Journal of Applied Decision Sciences*, 12(4), 337–360.
- Nguyen, M.-T., Cao, T., & Chau, W. (2016). Bayesian network analysis tool for land combat vehicle system evaluation. In *24th National conference of the Australian Society for Operations Research (ASOR)*, Canberra, Australia.
- Niebling, J., & Eichfelder, G. (2019). A branch-and-bound-based algorithm for nonconvex multiobjective optimization. *SIAM Journal on Optimization*, 29(1), 794–821.
- Peacock, J., Blumson, D., Mangalasinghe, J., Hepworth, A., Coutts, A., & Lo, E. (2019). Baselineing the whole-of-force capability and capacity of the Australian Defence Force. In *Proceedings of the MODSIM conference*, Canberra, Australia.
- Rahimi, M., & Soleimani-damaneh, M. (2018a). Isolated efficiency in nonsmooth semi-infinite multi-objective programming. *Optimization*, 67(11), 1923–1947.
- Rahimi, M., & Soleimani-damaneh, M. (2018b). Robustness in deterministic vector optimization. *Journal of Optimization Theory and Applications*, 179(1), 137–162.
- Ramana, M., & Goldman, A. J. (1995). Some geometric results in semidefinite programming. *Journal of Global Optimization*, 7, 33–50.
- Rockafellar, R. T. (1970). *Convex analysis*. Princeton University Press.
- Sion, M. (1958). On general minimax theorems. *Pacific Journal of Mathematics*, 8, 171–176.

- Soyster, A. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, *1*, 1154–1157.
- Steuer, R. E. (1986). *Multiple criteria optimization: Theory, computation, and application*. Wiley.
- Vinzant, C. (2014). What is a spectrahedron? *Notices of the American Mathematical Society*, *61*(5), 492–494.
- Wiecek, M. M. (2007). Advances in cone-based preference modeling for decision making with multiple criteria. *Decision Making in Manufacturing and Services*, *1*, 153–173.
- Zalinescu, C. (2002). *Convex analysis in general vector spaces*. World Scientific.
- Zamani, M., Soleimani-damaneh, M., & Kabgani, A. (2015). Robustness in nonsmooth nonlinear multi-objective programming. *European Journal of Operational Research*, *247*, 370–378.
- Zhou, A., Qu, B.-Y., Li, H., Zhao, S.-Z., Suganthan, P. N., & Zhang, Q. (2011). Multiobjective evolutionary algorithms: A survey of the state of the art. *Swarm and Evolutionary Computation*, *1*, 32–49.
- Zhou-Kangas, Y., & Miettinen, K. (2019). Decision making in multiobjective optimization problems under uncertainty: Balancing between robustness and quality. *OR Spectrum*, *41*, 391–413.

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