



On the Condorcet efficiency of evaluative voting (and other voting rules) with trichotomous preferences

Abdelhalim El Ouafdi¹ · Dominique Lepelley¹ · Hatem Smaoui¹

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Abstract

We investigate the propensity of evaluative voting (2, 1, 0) to fulfill Condorcet majority conditions in a framework where preferences are supposed to be trichotomous and only three candidates are in contention. In this framework, we also compare evaluative voting to other voting rules, including Borda rule, plurality rule and approval voting.

Keywords Voting by evaluation · Three-valued scale · Approval voting · Scoring rules · Probability

1 Introduction and motivation

The *three-valued scale evaluative voting* is a new voting rule recently considered by voting theorists.¹ It proceeds as follows: each voter evaluates each candidate and gives her a score belonging to $\{0, 1, 2\}$. In other words, each voter is given the possibility to form three groups of candidates: those she appreciates and all these candidates receive 2 points, those she does not appreciate who receives 0 point and an intermediate group of candidates who obtain 1 point.² The voters' preferences are said to be *trichotomous*. Of course, one of these groups can be empty (we ignore in what follows the case where two groups are empty, i.e. the case where all the candidates belong to the same group). The winner is the candidate obtaining the highest number of points. Notice that what we call here Evaluative Voting (2, 1, 0) can be considered as a particular case of *Range Voting* (where the set of ratings that a voter can give is not limited to $\{0, 1, 2\}$) or an extension of *approval voting* (where the set of marks is $\{0, 1\}$, which implies that preferences are dichotomous). According to Hillinger (2005),

¹ This voting rule has been introduced by Felsenthal (1989) and Hillinger (2004, 2005).

² Condorcet (1793) was the first to propose a voting rule in which voters were required to partition the candidates into three groups according to their preferences.

✉ Dominique Lepelley
dominique.lepelley@univ-reunion.fr
Abdelhalim El Ouafdi
abdelhalim.el-ouafdi@univ-reunion.fr
Hatem Smaoui
hatem.smaoui@univ-reunion.fr

¹ CEMOI, Université de La Réunion, Saint-Denis, France

<i>A</i>	<i>A</i>	<i>B</i>	<i>B</i>	<i>C</i>	<i>C</i>	<i>AB</i>	<i>AB</i>	<i>AC</i>	<i>AC</i>	<i>BC</i>	<i>BC</i>
<i>B</i>	<i>C</i>	<i>A</i>	<i>C</i>	<i>A</i>	<i>B</i>	<i>C</i>	—	<i>B</i>	—	<i>A</i>	—
<i>C</i>	<i>B</i>	<i>C</i>	<i>A</i>	<i>B</i>	<i>A</i>	—	<i>C</i>	—	<i>B</i>	—	<i>A</i>
<i>R</i> ₁	<i>R</i> ₂	<i>R</i> ₃	<i>R</i> ₄	<i>R</i> ₅	<i>R</i> ₆	<i>R</i> ₇	<i>R</i> ₈	<i>R</i> ₉	<i>R</i> ₁₀	<i>R</i> ₁₁	<i>R</i> ₁₂
<i>A</i>	<i>A</i>	<i>B</i>	<i>B</i>	<i>C</i>	<i>C</i>	—	—	—	—	—	—
<i>BC</i>	—	<i>AC</i>	—	<i>AB</i>	—	<i>A</i>	<i>BC</i>	<i>B</i>	<i>AC</i>	<i>C</i>	<i>AB</i>
—	<i>BC</i>	—	<i>AC</i>	—	<i>AB</i>	<i>BC</i>	<i>A</i>	<i>AC</i>	<i>B</i>	<i>AB</i>	<i>C</i>
<i>R</i> ₁₃	<i>R</i> ₁₄	<i>R</i> ₁₅	<i>R</i> ₁₆	<i>R</i> ₁₇	<i>R</i> ₁₈	<i>R</i> ₁₉	<i>R</i> ₂₀	<i>R</i> ₂₁	<i>R</i> ₂₂	<i>R</i> ₂₃	<i>R</i> ₂₄

Fig. 1 The possible trichotomous preferences on three candidates

the motivation for advocating for a three-valued scale is twofold: (1) the two-valued range of *approval voting* is not discriminating enough (in addition to feeling positive or negative about candidates, one may also feel neutral); (2) the electorate is generally poorly informed on candidates and issues, and a finer division of the voting scale appears to be both unnecessary and possibly confusing to the voters. Three-values scale evaluative voting will be simply denoted by *EV* in the remainder of this paper.³

A number of studies have been recently conducted to analyze *EV*, both from an empirical point of view (Baujard et al. 2013, 2014; Baujard and Igersheim 2009; Igersheim et al. 2015; Lebon et al. 2015) and a theoretical perspective (Smaoui and Lepelley 2013; Alcantud and Laruelle 2014; see also Felsenthal 2012, Pivato 2014 and Macé 2015 who adopt the more general perspective of range voting). Most of these studies have demonstrated that *EV* has many good properties. However, Felsenthal (2012) and Smaoui and Lepelley (2013) emphasize some possible difficulties with *EV*: this voting rule does not always choose the Condorcet winner (*CW*) when such a candidate exists (a Condorcet winner beats each of the other candidates in pairwise majority comparisons) and is susceptible to select the Condorcet loser (a candidate who loses each of her majority comparisons, denoted by *CL*). In other words, *EV* violates both the *CW* condition (a *CW* should be selected when such a candidate exists) and the *CL* condition (a *CL* should not be elected when such a candidate exists). According to Smaoui and Lepelley (2013), these violations constitute the main flaw of *EV* and Felsenthal (2012) considers the possible *CL* election as “intolerable”. It is thus of interest to try to evaluate the likelihood of these violations. Also, we would like to know how *EV* performs when compared to usual voting rules, such as Plurality Rule (*PR*) or approval voting (*AV*). The current paper, where we limit our investigation to three candidate elections, is a first analytical step in this direction.⁴

We consider an appropriate framework where preferences are trichotomous and in which the *EV* winner can be easily identified and compared to the *CW* (or the *CL*), when such a candidate exists. With three candidates, the number of possible trichotomous preferences is $3^3 = 27$ (number of ways to put three objects in three cells); as we ignore the case where all the candidates are put in a same group, $27 - 3 = 24$ possible preferences are left and we can enumerate and number these preferences as indicated in Fig. 1.

³ The score vector (2, 1, 0) can be replaced with any positive linear transformation without changing the election winner. Some authors consider the score vector (1, 0, -1); Alcantud and Laruelle (2014) refer to this rule as *Dis&approval Voting*.

⁴ Smaoui and Lepelley (2013) have run simulations to obtain some information on this issue. We will go back to their findings in the discussion of our results.

We suppose that voters’ preferences are anonymous and we denote by n_i ($1 \leq i \leq 24$) the number of voters with preference R_i on the three candidates. Thus, n_7 denotes the number of voters who put A and B in the first group (with EV , each of them gives two points to A and to B), C in the intermediate group (with EV , C receives 1 point from each of these n_7 voters) and no candidate in the third group. A (trichotomous) voting situation reports the value of each n_i and can be represented by a 24-tuple $x = (n_1, n_2, \dots, n_{24})$ such that $n_i \geq 0$ ($1 \leq i \leq 24$) and $\sum_{i=1}^{24} n_i = n$, where n is the total number of voters. We denote by $V(n)$ the set of all voting situations with n voters.

To illustrate the violation of Condorcet conditions by the EV rule, consider the following simple example with 5 voters and two voting situations: x (3 voters with preference R_1 and 2 voters with preference R_{15}) and y (2 voters with preference R_{15} and 3 voters with preference R_{22}):

3	2	2	3
A	B	B	–
B	–	–	AC
C	AC	AC	B
x		y	

In x , candidate A is the CW . When EV is applied, the scores of A , B and C are respectively 6, 7 and 0 points. It is therefore B and not A who wins the election. In y , candidate B is the CL . It is, however, B who wins the election, the scores of A , B and C with EV being respectively 3, 4 and 3.

To calculate the likelihood of Condorcet condition violations, we will assume that all the possible (trichotomous) voting situations in $V(n)$ are equally likely to occur: it is an IAC-like assumption, where IAC stands for *Impartial Anonymous Culture*, a model very often used in this kind of investigation. We assume very large electorate (i.e. $n \rightarrow \infty$) and we define the CW efficiency of a voting rule as the probability of electing the CW , given that such a candidate exists. For a voting rule F , $CWE(F)$ will denote this CW efficiency. Similarly, $CLE(F)$ denotes the CL efficiency of F , i.e. the probability of electing a candidate different from the CL , given that a CL exists.

At this stage of our study, an important observation is worth mentioning. We will consider in this paper that voters vote sincerely, and this assumption distinguishes our analysis from some previous works. Felsenthal (1989) compared EV and AV and shows that, when voters use their dominant strategies, the selected candidate under these two rules must be the same in three-candidate elections. By contrast, in a framework where the preference revelation is sincere, it is clear that EV and AV do not always lead to the same winner; as a consequence, the CW efficiency and the CL efficiency under EV are different than that of AV .

In another work, Felsenthal et al. (1990) computed the CW efficiency of PR and AV . Their study, which does not consider EV , differs from ours in three respects. First, they assume sophisticated voting from the voters (who eliminate dominated strategies), whereas, as said above, we consider sincere voting. Second, they resort in their paper to simulation methods, while our computations are based on an analytical approach. Third, Felsenthal et al. (1990) suppose that voters have linear preferences over the candidates; a special feature of our contribution is to consider a framework where voters’ preferences are trichotomous.⁵

Our study is organized as follows. We start by extending the four rules we would like to compare with EV to our trichotomous preference framework (Sect. 2). Then we present in Sects. 3 and 4 the probabilistic results we have obtained for the CW efficiency and the CL

⁵ Note however that, in spite of these differences, both studies conclude that AV is more Condorcet efficient than PR .

efficiency of the voting rules under study, by considering the case of large electorates. Our results are summarized and discussed in Sect. 5.

2 Extending scoring rules and approval voting to the trichotomous framework

In addition to *EV*, four voting rules will be studied (and compared to *EV*) in this paper: plurality rule (*PR*), negative plurality rule (*NPR*), Borda rule (*BR*) and approval voting (*AV*). The first three rules belong to the class of (simple) scoring rules that are usually defined in a context where each voter expresses his (her) preferences by a strict order on the set of candidates. A scoring rule gives points to candidates according to their rank in voters' preference orders. Each candidate obtains a score equal to the total of points she received and the candidate with the highest score is selected. For elections with three candidates, a scoring rule can be represented by a scoring vector $(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_3 \leq \lambda_2 \leq \lambda_1$ and $\lambda_3 < \lambda_1$, indicating that every candidate receives λ_i points each time she is ranked in position i in an individual preference (strict) ranking. *PR*, *NPR* and *BR* use respectively the scoring vectors $(1, 0, 0)$, $(1, 1, 0)$ and $(2, 1, 0)$. Approval voting (*AV*) is defined in a different context where all individual preferences are dichotomous. Each individual preference consists of two groups, that of the approved candidates and that of the disapproved candidates. Each candidate receives 1 point each time she is approved and 0 point whenever she is disapproved; the winner is the candidate with the highest total number of points.

In our framework, the score of each candidate under *EV* for a voting situation x in $V(n)$ is easily calculated as follows:

$$\begin{aligned} S_{EV}(A, x) &= 2(n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14}) \\ &\quad + (n_3 + n_5 + n_{11} + n_{15} + n_{17} + n_{19} + n_{22} + n_{24}) \\ S_{EV}(B, x) &= 2(n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16}) \\ &\quad + (n_1 + n_6 + n_9 + n_{13} + n_{17} + n_{20} + n_{21} + n_{24}) \\ S_{EV}(C, x) &= 2(n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18}) \\ &\quad + (n_2 + n_4 + n_7 + n_{13} + n_{15} + n_{20} + n_{22} + n_{23}) \end{aligned}$$

As our aim in this note is not only to compute the Condorcet efficiency of *EV* but also to compare its performance to the one of other voting rules (*PR*, *NPR*, *BR* and *AV*), and since these rules are generally introduced in a context where voters' preferences are (strict) linear orders (with or without ties), we have to consider how these rules can be implemented in the trichotomous framework.

In order to adapt *PR*, *NPR* and *BR* to our framework, we make use of a general method that could be applied to any positional voting rule using a scoring vector $(\lambda_1, \lambda_2, \lambda_3)$. This method was first proposed by Black (1976) and has been recently used by Diss et al. (2010) and Gehrlein and Lepelley (2015) to extend positional voting rules to dichotomous preferences. We start by reducing each trichotomous individual preference to a weak order on the three candidates, ignoring the empty group when the trichotomous preference does not correspond to a strict order. This way of doing is justified by the observation that the scoring (positional) rules are based on the notion of an ordinal ranking and not on an evaluation (or rating) principle: the only relevant information for applying these voting rules is the ranking of each candidate in the voters' preferences; the strength of preferences is only a consequence of this ordinal ranking. After this reduction, we compute the number of points that each candidate

receives from each voter in the following way. If the order resulting from the reduction process of the trichotomous preference is a strict order, then the candidates receive, as explained above, α_i points for a i th position. If this order is a weak order like $X \succ Y \sim Z$ (X is preferred to Y and Z , Y and Z are ex aequo), as for example in R_{13} , then X receives λ_1 points and Y and Z receive an average $(\lambda_2 + \lambda_3)/2$ points each. If this order is a weak order like $X \sim Y \succ Z$ (X and Y are ex aequo and both are preferred to Z), as for example in R_7 , then X and Y receive an average $(\lambda_1 + \lambda_2)/2$ points each and Z receives λ_3 points. Therefore, our extended scoring rules treat all voters equally, since they consistently allocate the same $\lambda_1 + \lambda_2 + \lambda_3$ points to the candidates for each voter, and this extension allows to take into consideration the various types of preferences we consider here.

With these assumptions, the scores of A , B and C under PR are:

$$\begin{aligned}
 S_{PR}(A, x) &= (n_1 + n_2 + n_{13} + n_{14} + n_{19}) + \frac{1}{2}(n_7 + n_8 + n_9 + n_{10} + n_{22} + n_{24}) \\
 S_{PR}(B, x) &= (n_3 + n_4 + n_{15} + n_{16} + n_{21}) + \frac{1}{2}(n_7 + n_8 + n_{11} + n_{12} + n_{20} + n_{24}) \\
 S_{PR}(C, x) &= (n_5 + n_6 + n_{17} + n_{18} + n_{23}) + \frac{1}{2}(n_9 + n_{10} + n_{11} + n_{12} + n_{20} + n_{22})
 \end{aligned}$$

Under the NPR , we easily obtain:

$$\begin{aligned}
 S_{NPR}(A, x) &= n - [(n_4 + n_6 + n_{11} + n_{12} + n_{20}) + \frac{1}{2}(n_{15} + n_{16} + n_{17} + n_{18} + n_{21} + n_{23})] \\
 S_{NPR}(B, x) &= n - [(n_2 + n_5 + n_9 + n_{10} + n_{22}) + \frac{1}{2}(n_{13} + n_{14} + n_{17} + n_{18} + n_{19} + n_{23})] \\
 S_{NPR}(C, x) &= n - [(n_1 + n_3 + n_7 + n_8 + n_{24}) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21})]
 \end{aligned}$$

The scores under the BR are given as:

$$\begin{aligned}
 S_{BR}(A, x) &= 2(n_1 + n_2 + n_{13} + n_{14} + n_{19}) + \frac{3}{2}(n_7 + n_8 + n_9 + n_{10} + n_{22} + n_{24}) \\
 &\quad + (n_3 + n_5) + \frac{1}{2}(n_{15} + n_{16} + n_{17} + n_{18} + n_{21} + n_{23}) \\
 S_{BR}(B, x) &= 2(n_3 + n_4 + n_{15} + n_{16} + n_{21}) + \frac{3}{2}(n_7 + n_8 + n_{11} + n_{12} + n_{20} + n_{24}) \\
 &\quad + (n_2 + n_4) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21}) \\
 S_{BR}(C, x) &= 2(n_5 + n_6 + n_{17} + n_{18} + n_{23}) + \frac{3}{2}(n_9 + n_{10} + n_{11} + n_{12} + n_{20} + n_{22}) \\
 &\quad + (n_2 + n_4) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21})
 \end{aligned}$$

AV is easy to implement when the trichotomous preference is R_7 to R_{24} : in these cases, we ignore the empty group and the preferences are actually dichotomous. When the trichotomous preference corresponds to a strict order (R_1 to R_6), we assume that the voter will approve the candidate ranked in first position (with certainty) and that she will approve the candidate ranked in second position with probability $1/2$ (We shall come back to that assumption in the last Section). Thus, we obtain for approval voting (AV) the following scores:

$$S_{AV}(A, x) = (n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} + n_{24}) + \frac{1}{2}(n_3 + n_5)$$

$$S_{AV}(B, x) = (n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} + n_{24}) + \frac{1}{2}(n_1 + n_6)$$

$$S_{AV}(C, x) = (n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18} + n_{20} + n_{22} + n_{23}) + \frac{1}{2}(n_2 + n_4)$$

3 Results on the Condorcet winner efficiency with trichotomous preferences

Let $CWE(F, n)$ be the CW efficiency of the voting rule F , when all considered voting situations are in $V(n)$. According to the definition given in Sect. 2, we have: $CWE(F) = \lim_{n \rightarrow \infty} CWE(F, n)$. To compute this limiting probability for the different rules under consideration in this paper, we will use a method based on Ehrhart theory and on algorithms for counting integer points in rational polytopes. This method, which is partly original, will be explained and illustrated in the following subsection;⁶ it will then be applied throughout the rest of the paper. We first recall in the following paragraph some definitions related to the notions of polytope and quasi-polynomial and we give a brief overview of Ehrhart theory. These mathematical tools are now well known and widely used in probability calculations in voting theory.⁷

A rational polytope P in \mathbb{R}^d is a bounded subset of \mathbb{R}^d defined by a system of integer linear (in)equalities. Note that P could be not full-dimensional (this is the case when the linear system describing P contains some equalities) and could be semi-open (this is the case when some of the inequalities describing P are strict). A parametric polytope (with a single parameter n) of dimension d is a sequence of (possibly empty) d -dimensional rational polytopes P_n ($n \in \mathbb{N}$) of the form $P_n = \{x \in \mathbb{R}^d : Mx \leq bn + c\}$, where M is an $m \times d$ integer matrix and b and c are two integer vectors with m components. An important instance of parametric polytopes is obtained when the constant term c is equal to the zero vector. In this case, P_n is denoted nP and corresponds to the dilatation, by the positive integer factor n , of the rational polytope P defined by $P = \{x \in \mathbb{R}^d : Mx \leq b\}$. Calculating the number of integer solutions of a parametric linear system ($Mx \leq bn + c$) amounts to calculating the number of integer coordinate points belonging to the parametric polytope P_n defined by this system. By Ehrhart theory, we know that this number is a quasi-polynomial in n , of degree d , i.e. a polynomial expression $f(n)$ of the form $f(n) = \sum_{k=0}^d c_k(n)n^k$, where the coefficients $c_k(n)$ are rational periodic numbers in n . A rational periodic number of period q on the integer variable n is a function $u : \mathbb{Z} \rightarrow \mathbb{Q}$ such that $u(n) = u(n')$ whenever $n \equiv n' \pmod{q}$. Each coefficient $c_k(n)$ can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial $f(n)$ and defined as the least common multiple (*lcm*) of the periods of all coefficients.

⁶ The reader is referred to El Ouafdi et al. (2020) for further details on this method.

⁷ For a general background on Ehrhart theory and on the general problem of counting integer points in rational polytopes, see for example Beck and Robins (2015). We refer also to Lepelley et al. (2008) and Wilson and Pritchard (2007) for more details on the use of these tools in probability calculations under IAC hypothesis in voting theory..

3.1 Existence of a Condorcet winner

For a voting situation x in $V(n)$ and two different candidates w, w' , we denote by $P_x(w, w')$ the number of voters that prefer w to w' (i.e. the number of voters who put w in a group higher than the group in which they put w'). The numbers involved in the binary comparisons between A and B and between A and C are:

$$\begin{aligned}
 P_x(A, B) &= n_1 + n_2 + n_5 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} \\
 P_x(B, A) &= n_3 + n_4 + n_6 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} \\
 P_x(A, C) &= n_1 + n_2 + n_5 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} \\
 P_x(C, A) &= n_3 + n_4 + n_6 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21}
 \end{aligned}$$

With this notation, A is the CW if and only if $P_x(A, B) > P_x(B, A)$ and $P_x(A, C) > P_x(C, A)$. We denote by $CW(A, n)$ the event “ A is the CW ”, when the number of voters is equal to n . Let $\Pr(CW(A, n))$ and $\Pr(CW(A))$ be the probability and the limiting probability (when $n \rightarrow \infty$) of this event. We have:

$$\Pr(CW(A, n)) = \frac{|CW(A, n)|}{|V(n)|}$$

In this identity, $|V(n)|$ and $|CW(A, n)|$ denote the cardinalities of sets $V(n)$ and $CW(A, n)$ respectively. The number $|V(n)|$ is well known and is given by $|V(n)| = \binom{n+23}{23}$. Indeed, with n voters and 24 possible (trichotomous) individual preferences, the number of voting situations is equal to the number of ways n objects can be chosen from a set of 24 objects, where repetition is allowed. Note that $|V(n)|$ is a polynomial of degree 23 and that the coefficient of the leading term of this polynomial is equal to $1/23!$. The computation of $|CW(A, n)|$ as a function of n is more involved. By definition, $CW(A, n)$ is the set of all integer solutions, $x = (n_i)_{i=1}^{24}$, of the following system:

$$S_n : \begin{cases} n_i \geq 0, & i = 1, \dots, 24 \\ n_1 + n_2 + \dots + n_{24} = n \\ P_x(A, B) - P_x(B, A) > 0 \\ P_x(A, C) - P_x(C, A) > 0 \end{cases}$$

All (in)equalities in this system are linear and have integer coefficients on the variables n_i and on the integer parameter n . This defines, for each value of yn , a parametric (semi-open) rational polytope (of dimension 23), P_n , which is the dilatation, by the factor n , of the (semi-open) rational polytope P_1 . To obtain $|CW(A, n)|$, we have to compute the quasi-polynomial describing the number of integer points belonging to P_n . In voting theory, to perform this type of calculation, we usually resort to a computer program based on (parameterized) Barvinok algorithm (see [barvinok]). This program performs very well when the calculations concern voting events with three candidates and individual preferences are expressed by strict orders. In this case there are only 6 variables and the quasi-polynomials are generally of degree 5. With 24 variables and a degree 23, as in the case we are studying, the use of Barvinok algorithm does not make it possible to obtain the desired results. Other software such as LattE with its new version Latte integrale (see [latte]) and Normaliz ([normaliz]) allow, in some cases, to calculate quasi-polynomials corresponding to polytopes of dimension 23. However, as our goal is to calculate the limit value of $\Pr(CW(A, n))$, we do not need to know the exact expression of $|CW(A, n)|$. Indeed, $\Pr(CW(A, n))$ is the quotient of a quasi-polynomial ($|CW(A, n)|$) by a polynomial ($|V(n)|$). Since these two polynomial expressions are of the same

degree (23) and since we already know the expression of $|V(n)|$, we have only to calculate the coefficient of the leading term of $|CW(A, n)|$. We know, again by Ehrhart theory, that this coefficient is independent of n and is equal to the volume⁸ of P_1 . So we have:

$$\Pr(CW(A)) = \lim_{n \rightarrow +\infty} \frac{|CW(A, n)|}{|V(n)|} = \frac{\text{Vol}(P_1)}{1/23!} = 23! \text{Vol}(P_1) \tag{1}$$

Now, it only remains to calculate $\text{Vol}(P_1)$. In general, algorithms that compute the volume of polytopes are not very efficient when, as in all the cases studied in this paper, the number of variables is equal to 24. However, recent improvements in algorithms such as LattE and Normaliz have made it possible to obtain some results describing the probability of voting events with four candidates, requiring the calculation of the volumes of certain polytopes of dimension 24 (see Schürmann 2013; Bruns and Söger 2015; Brandt et al. 2016). To compute $\text{Vol}(P_1)$ and all the other volumes involved in the calculations developed in the remainder of this paper, we will not use any algorithm of direct volume computation. Instead, we will apply a new method based on the Ehrhart theory and on the combined use of two software, LattE integrale and lrs ([Lrs]).

The command (count –ehrhart-polynomial) in the first program makes it possible to calculate in a reasonable time (from a few seconds to a few hours) the quasi-polynomial describing the number of integer points belonging to the dilatation nP of a polytope P by an integer factor n . With LattE integrale, this computation is possible only when P is an integral polytope (i.e. when all its vertices have integer coordinates). In this case, the quasi-polynomial associated with nP has period equal to 1 and hence is simply a polynomial. The second program, lrs, allows to obtain (usually within seconds) the coordinates of all vertices of a rational polytope. To calculate $\text{Vol}(P_1)$, the idea is then the following. We start by dilating P_1 by a positive integer factor k such that the obtained polytope kP_1 is integral; for this, k must be a multiple of the period of P_1 . Now, we know by Ehrhart that the period of P_1 is a divisor of the lcm of the denominators of the vertices of P_1 . It suffices then to take k equal to this number that we can easily obtain by applying lrs. After this step, we apply LattE integrale to the integral polytope kP_1 and we obtain the polynomial associated with the dilated polytope nkP_1 . It is obvious that if δ is the coefficient of the leading term of this polynomial, then $\delta = \text{Vol}(kP_1) = k^{24}\text{Vol}(P_1)$. Finally, we have:

$$\text{Vol}(P_1) = \frac{\delta}{k^{24}}$$

Applying this method, we obtained:

$$\text{Vol}(P_1) = \frac{6914641}{550632520243029171744276480000}$$

Multiplying this number by $23!$, as indicated in (1), we obtain the probability of having candidate A as CW :

$$\Pr(CW(A)) = \frac{2233429043}{6879707136} = 0.3246401$$

Since each of the three candidates A, B and C can be a CW , multiplying $\Pr(CW(A))$ by 3 gives the probability $\Pr(CW)$ that a CW exists:

$$\Pr(CW) = \frac{2233429043}{2293235712} = 0.9739204$$

⁸ Here, by the volume of P_1 we mean its relative volume, i.e. the volume of P_1 relative to its affine span.

Remark Candidate A is the CL if and only if $P_x(A, B) < P_x(B, A)$ and $P_x(A, C) < P_x(C, A)$.

Given the symmetry of the notions of Condorcet winner and Condorcet loser, it can be noticed that

$$|CW(A, n)| = |CL(A, n)|$$

and this implies that:

$$\Pr(CL) = \Pr(CW).$$

3.2 Condorcet winner efficiency of voting rules

Let $CWE(F, n)$ be the CW efficiency of the voting rule F , when all considered voting situations are in $V(n)$ and recall that $CWE(F) = \lim_{n \rightarrow \infty} CWE(F, n)$. By definition, $CWE(F, n)$ is the conditional probability to have the CW elected under F , given that a CW exists. We denote by $CW(F, A, n)$ the event “ A is the CW and A is elected under F ” and, as in the previous subsection, we denote by $CW(A, n)$ the event “ A is a CW ”. As the three candidates A, B and C are symmetric in our probabilistic model, it is possible to assume without loss of generality that A is the CW , and we have:

$$CWE(F, n) = \frac{\Pr(CW(F, A, n))}{\Pr(CW(A, n))}$$

Since $\Pr(CW(F, A, n)) = |CW(F, A, n)|/|V(n)|$ and $\Pr(CW(A, n)) = |CW(A, n)|/|V(n)|$, we obtain:

$$CWE(F, n) = \frac{|CW(F, A, n)|}{|CW(A, n)|} \tag{2}$$

Voting situations belonging to $CW(F, A, n)$ are the integer solutions of the following linear system:

$$S_n^F \begin{cases} n_i \geq 0, & i = 1, \dots, 24 \\ n_1 + n_2 + \dots + n_{24} = n \\ P_x(A, B) - P_x(B, A) > 0 \\ P_x(A, C) - P_x(C, A) > 0 \\ S_x(F, A) - S_x(F, B) > 0 \\ S_x(F, A) - S_x(F, C) > 0 \end{cases}$$

Let P_1^F be the (semi-open) polytope defined by S_1^F (the system obtained when $n = 1$) and let P_1 be the polytope defined in the previous subsection. Taking the limit in equality (2) and reasoning as in the previous subsection, we obtain:

$$CWE(F) = \frac{\text{Vol}(P_1^F)}{\text{Vol}(P_1)} \tag{3}$$

We have already calculated $\text{Vol}(P_1)$. To calculate $\text{Vol}(P_1^F)$ for the five rules under consideration, we replace successively, in system S_n^F , the voting rule F by EV, PR, NPR, BR and AV (by referring to the scores defined in Sect. 2) and then we apply, as in the previous subsection, the calculation method based on the LattE and lrs algorithms. We then apply formula (3). The results we have obtained are summarized in the Table 1 which gives, for each F , the exact values of $CWE(F)$ and $\Pr(CW(F, A))$, the limiting probability of the event “ A is the CW and A is elected under F ”.

Table 1 Condorcet winner efficiencies of five voting rules

	$\Pr(CW(F, A) = 23! \text{Vol}(P_1^F))$	$CWE(F)$
<i>EV</i>	$\frac{14202256426752049}{50565847449600000}$	$\frac{747487180355371}{863984392950000} \approx 0.8651628$
<i>PR</i>	$\frac{6188893189678377041}{22463437455746924544}$	$\frac{6188893189678377041}{7292533334267676672} \approx 0.8486616$
<i>NPR</i>	$\frac{21756141802346747501}{85691213438976000000}$	$\frac{43512283604693495002}{55637613939420140625} \approx 0.7820660$
<i>BR</i>	$\frac{609912099396357654321671}{2011623085859143680000000}$	$\frac{49402880051104970000055351}{52897340557040627821250000} \approx 0.9339388$
<i>AV</i>	$\frac{721524879167199418097}{2428171994529792000000}$	$\frac{721524879167199418097}{788282080439721000000} \approx 0.9153131$

Table 2 Condorcet loser efficiencies of four voting rules

	$\Pr(CL(F, A))$	$CLE(F)$
<i>EV</i>	$\frac{161266847754571}{50565847449600000}$	$\frac{16254436618295429}{16415703466050000} \approx 0.99017606$
<i>PR</i>	$\frac{63431612518594165392461}{8774780256151142400000000}$	$\frac{2785214221179717034607539}{2848645833698311200000000} \approx 0.97773271$
<i>NPR</i>	$\frac{38536640041404497909}{5484237660094464000000}$	$\frac{16254436618295429}{1780403646061444500000} \approx 0.9783551$
<i>AV</i>	$\frac{371518918953857}{1011316948992000000}$	$\frac{327942550402046143}{328314069321000000} \approx 0.99886840$

4 Condorcet loser election and other results with trichotomous preferences

4.1 Condorcet loser efficiency

The computations are very similar to those we have conducted in Sect. 3: denoting the *CL* efficiency of rule *F* by $CLE(F, n)$, the event “*A* is the *CL* and *A* is elected under *F*” by $CL(F, A, n)$ and the event “*A* is a *CL*” by $CL(A, n)$, we have just to replace (2) with

$$CLE(F, n) = 1 - \frac{|CL(F, A, n)|}{|CL(A, n)|}$$

and

$$\begin{aligned} P_x(A, B) - P_x(B, A) &> 0 \\ P_x(A, C) - P_x(C, A) &> 0 \end{aligned}$$

with,

$$\begin{aligned} P_x(A, B) - P_x(B, A) &< 0 \\ P_x(A, C) - P_x(C, A) &< 0 \end{aligned}$$

in the definition of S_n^F .

As *BR* never selects the *CL* (Fishburn and Gehrlein 1976), we only consider *EV*, *PR*, *NPR* and *AV* in this subsection. The results are displayed in Table 2.

4.2 Other Results

A strengthening of the Condorcet winner condition sometimes used in the literature is based on the notion of an Absolute *CW* (*ACW*): a candidate is an *ACW* when more than one half

of the voters rank this candidate (and only this candidate) in first position. In our framework, A is a ACW iff:

$$n_1 + n_2 + n_{13} + n_{14} + n_{19} > n/2.$$

One can define in the same way the notion of Absolute CL (ACL): a candidate is an ACL when more than one half of the voters rank this candidate (and only this candidate) in last position. Thus, A is an ACL when:

$$n_4 + n_5 + n_{11} + n_{12} + n_{20} > n/2.$$

Of course, when such candidates exist, the ACW should be elected and the ACL should not.

Felsenthal (2012) and Smaoui and Lepelley (2013) observe that EV violates these two conditions⁹ and we are interested in this subsection in the computation of the likelihood of such violations. It is of interest to notice that AV also violates both the ACW and the ACL conditions; and among the scoring rules, PR is the only rule verifying the ACW condition (Lepelley 1992), whereas BR and NPR verifies the ACL condition.

Given the symmetry of our model, the probability of having an ACW is equal to the probability of having an ACL and it turns out that this probability is very low. Using the same technique as above, we obtain:

$$Pr(ACW) = Pr(ACL) = 32709/8388608 = 0.0038992.$$

Denoting by $ACWE(F)$ the probability of having the ACW elected under F , given that such a candidate exists, we obtain the following results:

$$ACWE(EV) = 20508589358593657/20510395637760000 = 0.9999119.$$

$$ACWE(NPR) = 13945016265602573/14598301564455552 = 0.9552492.$$

$$ACWE(BR) = 20450740328214773603501/20450812337541810000000 = 0.9999965.$$

$$ACWE(AV) = 12863582803932107/12864120144003072 = 0.9999582.$$

Regarding the election of the ACL , the probabilities we obtain are close to 0 and, consequently, the ACL efficiencies are close to 1:

$$Pr(ACL, EV) = 7414393657/20510395637760000 = 0.0000004$$

and

$$ACLE(EV) = 20510388223366343/20510395637760000 = 0.99999964.$$

$$Pr(ACL, PR) = 42835450539276625/119589286416019881894 = 0.0003582$$

and

$$ACLE(PR) = 119546450965480605359/119589286416019881984 = 0.99964181.$$

$$Pr(ACL, AV) = 292081/8576080096002048 = 3.405763 \dots \times 10^{-11}$$

and

$$ACLE(AV) = 8576080095709967/8576080096002048 = 0.999999999.$$

⁹ As the CL election, the ACL election and the ACW non election are qualified as "intolerable" by Felsenthal (2012).

Table 3 Condorcet efficiencies of five voting rules with trichotomous preferences

	<i>EV</i>	<i>PR</i>	<i>NPR</i>	<i>BR</i>	<i>AV</i>
<i>CW</i> eff.	0.8651628	0.8486616	0.7820660	0.9339388	0.9153131
<i>CL</i> eff.	0.9901761	0.9777327	0.9783551	1	0.9988684
<i>ACW</i> eff.	0.9999119	1	0.9552492	0.9999965	0.9999582
<i>ACL</i> eff.	0.9999996	0.9996418	1	1	0.9999999

5 Conclusions and final remark

All the results we have obtained in this study are summarized in Table 3.

The following conclusions emerge from the examination of these figures:

- The hierarchy of *PR*, *NPR* and *BR* regarding *CW* efficiency is consistent with what we could expect from previous studies (Gehrlein and Lepelley 2011); in other words, moving from linear orders to trichotomous preferences does not modify the ranking of the scoring rules: *BR* is better than *PR*, itself better than *NPR*. Moreover, our study confirms the **superiority of *BR*** over all the other (one-stage) voting rules in terms of Condorcet efficiency in three-candidate elections.
- The **Condorcet winner efficiency of *EV* holds a middle position** between the *CW* efficiencies of *PR* and *NPR* on the one hand, and the *CW* efficiencies of *AV* and *BR* on the other hand.¹⁰
- Compared to *PR* and *NPR*, ***EV* reduces the risk of electing the *CL*** and the performance of *AV* on this issue is even better. This observation, added to the preceding one, gives a strong argument for using *EV* or *AV* instead of *PR* in political elections.
- The probability of not electing the absolute Condorcet winner when such a candidate exists appears to be **very low**, except for *NPR*; and it turns out that, if the election of an absolute Condorcet loser can occur under *EV*, *PR* and *AV*, such an event is **highly unlikely** in our framework. Consequently, we should not worry about these possibilities as such.
- Generally speaking, the comparison of ***EV* and *AV* in terms of Condorcet efficiency is to the advantage of *AV***. Notice that our framework and our probabilistic assumption play an important role in this conclusion: among the 24 possible trichotomous preferences, 18 are actually dichotomous preferences; as we consider every possible preference as equally likely, this peculiarity is in favor of *AV* since we know that when all the preferences are dichotomous, *AV* always selects the *CW* (Brams and Fishburn, 1983). Let $\alpha = (n_1 + n_2 + \dots + n_6)/n$ be the proportion of voters having strict preference orders (or linear orders) in the electorate (our study assumes that α is on average equal to $6/24 = 1/4$). It can be expected that the *CW* efficiency of *EV* increases when α increases. To check this conjecture, we have computed the *CW* efficiency of *EV* for various values of α , assuming that for each specific value under consideration, the corresponding voting situations are equally likely to occur. For comparison, we have also computed the *CW* efficiency of *AV* under the same assumptions. The results are shown in Table 4 (exact fractions are here omitted).

We observe the expected increase for the Condorcet efficiency of *EV* with an upper limit at $\frac{41}{45} = 0.911111$, which corresponds to the *CW* efficiency of *BR* with strict preferences.

¹⁰ The *CW* Efficiency ranking of these five voting rules obtained from the simulations conducted by Smaoui and Lepelley (2013) is similar with however one exception: they obtain that *EV* stands before *AV*. Their simulations are based on a framework completely different from the one used in the current study.

Table 4 CW efficiency of EV and AV as function of the proportion of voters with linear orders

α	0	1/4	1/3	1/2	2/3	3/4	1
$CWE_{\alpha}(EV)$	0.854666	0.863551	0.873434	0.894253	0.905886	0.908669	0.911111
$CWE_{\alpha}(AV)$	1	0.910172	0.893512	0.890087	0.901906	0.906364	0.911111

For AV, the Condorcet efficiency decreases when α moves from 0 to 1/2 and then increases when α moves from 1/2 to 1. We note that for $1/2 \leq \alpha < 1$, EV performs slightly better than AV.

The good performance of AV could also be due to our assumption that voters with R_1 to R_6 (strict) preferences approve their candidate ranked in second position with probability 1/2. If, following Gehrlein and Lepelley (2015), we reject this assumption,¹¹ the AV scores become:

$$S'_{AV}(A, x) = (n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} + n_{24})$$

$$S'_{AV}(B, x) = (n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} + n_{24})$$

$$S'_{AV}(C, x) = (n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18} + n_{20} + n_{22} + n_{23})$$

and we obtain:

$$CWE'(AV) = \frac{21895717301068657}{25126076733750000} = 0.87143399.$$

The CW efficiency of AV is reduced but remains slightly higher than the CW efficiency of EV (0.8651628), leading to the conclusion that, in our framework, the superiority of AV on EV seems to be rather robust.

Finally, it is of interest to notice that one can easily compute from our results the CW efficiency of PER and NPER, where PER stands for Plurality Elimination Rule and NPER for negative plurality elimination rule. PER and NPER are two-stage voting rules where the candidate with the lowest score under PR and NPER (respectively) is eliminated at the first stage. While never electing the CL, PER and NPER are susceptible to elect a candidate different from the CW and NPER can even elect a candidate different from the ACW. To obtain $CWE(PER)$, it is easily observed that (1) starting from a voting situation in which a CL exists and is elected under NPR, if we inverse the preference orders of every voter, then we obtain a voting situation in which a CW exists and is ranked first by the smallest number of voters and hence is eliminated under PER; (2) similarly, if we inverse the voters' preferences in a voting situation where the CW gets the minimum number of first ranks (and hence is eliminated under PER), then we obtain a voting situation in which the CL is elected under NPR. Using these symmetry arguments, it can be concluded that:

$$CWE(PER) = CLE(NPR) = 0.9783551.$$

And from similar arguments, we obtain:

$$CWE(NPER) = CLE(PR) = 0.9777327$$

and

$$ACWE(NPER) = ACLE(PR) = 0.9996418.$$

¹¹ This assumption is tantamount to consider that AV works as BR in presence of linear orders, whereas Gehrlein and Lepelley (2015) consider that, in this context, AV works as PR.

We conclude that, in our trichotomous framework as well as in the usual framework where only linear orders are considered (see Gehrlein and Lepelley 2011), two-stage voting rules perform better than one-stage rules in electing the CW in three-alternative elections.

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