Annals of Operations Research (2020) 287:233–255
https://doi.org/10.1007/s10479-019-03409-6

ORIGINAL RESEARCH

New dualities for mathematical programs with vanishing constraints

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Published online: 25 October 2019

Abstract
Recently, Mishra et al. (Ann Oper Res 243(1):249–272, 2016) formulate and study the Wolfe and the Mond–Weir type dual models for the mathematical programs with vanishing constraints. They establish the weak, strong, converse, restricted converse and strict converse duality results between the primal mathematical programs with vanishing constraints and the corresponding dual model under some assumptions. However, their models contain the calculation of the index sets, this makes it difficult to solve them from algorithm point of view. In this paper, we propose the new Wolfe and Mond–Weir type dual models for the mathematical programs with vanishing constraints, which do not involve the calculation of the index set. We show that the weak, strong, converse and restricted converse duality results hold between the primal mathematical programs with vanishing constraints and the corresponding new dual models under the same assumptions as the ones of Mishra et al.

Keywords Mathematical programs with vanishing constraints · Wolfe dual · Mond–Weir dual

Mathematics Subject Classification 49J52 · 90C33

This work is supported by National Natural Science Foundation of China (No. 11461015, 11961011, 11761014) of China, Guangxi Natural Science Foundation (No. 2015GXNSFAA139010, 2016GXNSFFA380009, 2017GXNSFAA198243) and Guangxi Key Laboratory of Automatic Detecting Technology (No.YQ17117).

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1 Introduction

Mathematical program with vanishing constraints (MPVC) was introduced by Achtziger and Kanzow (2008), and its general form is as follows:

$$
\begin{array}{lll}
min & f(x) & \\
s.t. & g_i(x) \leq 0, & i = 1, 2, \ldots, m, \\
 & h_j(x) = 0, & j = 1, 2, \ldots, p, \\
 & H_i(x) \geq 0, & i = 1, 2, \ldots, l, \\
 & G_i(x)H_i(x) \leq 0, & i = 1, 2, \ldots, l,
\end{array}
\tag{1.1}
$$

where $f : R^n \to R$ is Lipschitz continuous, $g : R^n \to R^m, h : R^n \to R^p,\ G, H : R^n \to R^l$ are all continuously differentiable functions. Throughout this paper, X denotes the feasible region of MPVC (1.1).

MPVC problem originates from the optimization topology design problems in mechanical structures (Achtziger and Kanzow 2008). At present, the corresponding research shows that the robot motion planning problem (Kirches et al. 2013) can be transformed into the MPVC problem. In addition, it is also widely used in the economic dispatch problem (Jabr 2012) and the nonlinear integer optimal control (Michael et al. 2013). As described in Achtziger and Kanzow (2008), the major difficulty in solving problem (1.1) is that it does not satisfy most of the standard constraint qualifications such as linearly independent constraint qualification (LICQ) and Mangasarian–Fromovitz constraint qualification (MFCQ) at any interesting feasible point, so that the standard optimization methods are likely to fail for this problem. The MPVC has attracted much attentions in recent years. Several theoretical properties and different numerical approaches for MPVC can be found in Achtziger and Kanzow (2008), Kirches et al. (2013), Hoheisel et al. (2010), Hu et al. (2014), Hu et al. (2017), Benko and Gfrerer (2017), Achtziger et al. (2013), Achtziger et al. (2012), Dorsch et al. (2012), Hoheisel and Kanzow (2009), Hoheisel and Kanzow (2008), Hoheisel and Kanzow (2007), Hoheisel et al. (2012), Izmailov and Pogosyan (2009) and Izmailov and Solodov (2009).

Duality is very important in optimization as the weak duality provides a lower bound to the objective function value of the primal problem. The classical Wolfe duality was introduced by Wolfe (1961), while the Mond–Weir duality was introduced by Mond and Weir (1981) for differentiable scalar functions. Later these duality models were extended to nondifferentiable functions by utilizing different generalizations of the notion of convexity for both scalar and vector cases (See Mishra et al. 2016; Antczak 2010; Chinchuluun et al. 2007; Askar and Tiwari 2009; Gulati and Mehndiratta 2010; Bot and Heinrich 2014; Lai and Huang 2012; Jefferson and Scott 2001; Lee and Lai 2005; Peterson 2001; Rockafellar 1999; Mishra and Shukla 2010; Mishra et al. 2012, 2015; Mishra and Jaiswal 2015; Pandey and Mishra 2016, 2017, 2018). Recently, Mishra et al. (2016) formulate and study Wolfe and Mond–Weir type dual models for the mathematical programs with vanishing constraints. They establish the weak, strong, converse, restricted converse and strict converse duality results between the primal mathematical programs with vanishing constraints and the corresponding dual models under some assumptions. Since their models involve the calculation of index set, it is not conducive for the numerical solutions of dual problems.

In this paper, the new Wolfe and Mond–Weir type dual models for the mathematical programs with vanishing constraints are proposed which do not involve the calculations of index sets. Under the same assumptions as the ones of Mishra et al, the weak, strong, converse and restricted converse duality results between the primal mathematical programs with vanishing constraints and the corresponding new dual models are established. We also verify the validity of these results through an example.

The outline of this paper is as follows: in Sect. 2, we give some preliminaries about the MPVC. In Sect. 3, we give the new Wolfe and Mond–Weir type dual models for MPVC and some duality results. We close with some final remarks in Sect. 4.

2 Preliminaries

Let $x^* \in X$ be any feasible point of the MPVC (1.1). The following index sets will be used in the sequel.

$$\begin{aligned} I_g(x^*) &= \{i | g_i(x^*) = 0\} \\ I_h(x^*) &= \{1, 2, \ldots, p\} \\ I_+(x^*) &= \{i | H_i(x^*) > 0\} \\ I_0(x^*) &= \{i | H_i(x^*) = 0\} \end{aligned} \tag{2.1}$$

$$\begin{aligned} I_{+0}(x^*) &= \{i | H_i(x^*) > 0, G_i(x^*) = 0\} \\ I_{+-}(x^*) &= \{i | H_i(x^*) > 0, G_i(x^*) < 0\} \\ I_{0+}(x^*) &= \{i | H_i(x^*) = 0, G_i(x^*) > 0\} \\ I_{00}(x^*) &= \{i | H_i(x^*) = 0, G_i(x^*) = 0\} \\ I_{0-}(x^*) &= \{i | H_i(x^*) = 0, G_i(x^*) < 0\} \end{aligned} \tag{2.2}$$

We also use the following Lagrangian function and its gradient:

$$L(y, \lambda, \mu, \eta^H, \eta^G) = f(y) + \sum_{i=1}^{m} \lambda_i g_i(y) + \sum_{j=1}^{p} \beta_j h_j(y) - \sum_{i=1}^{l} \eta_i^H H_i(y) + \sum_{i=1}^{l} \eta_i^G G_i(y)$$

and

$$\nabla L(y, \lambda, \mu, \eta^H, \eta^G) = \nabla f(y) + \sum_{i=1}^{m} \lambda_i \nabla g_i(y) + \sum_{j=1}^{p} \beta_j \nabla h_j(y) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(y) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(y).$$

We define the following index sets for $x \in X$:

$$\begin{aligned} I_g^+(x) &= \{i \in \{1, 2, \ldots, m\} | \lambda_i > 0\} \\ I_h^+(x) &= \{j \in I_h(x) | \mu_j > 0\} \\ I_h^-(x) &= \{j \in I_h(x) | \mu_j < 0\} \\ I_{0+}^+(x) &= \{i \in I_{0+}(x) | \eta_i^H > 0\} \\ I_{0+}^-(x) &= \{i \in I_{0+}(x) | \eta_i^H < 0\} \\ I_{00}^+(x) &= \{i \in I_{00}(x) | \eta_i^H > 0\} \\ I_{+0}^+(x) &= \{i \in I_{+0}(x) | \eta_i^H > 0\} \\ I_{+-}^+(x) &= \{i \in I_{+-}(x) | \eta_i^H > 0\} \\ I_{0-}^+(x) &= \{i \in I_{0-}(x) | \eta_i^H > 0\} \end{aligned}$$

$$I_{+0}^{++}(x) = \{i \in I_{+0}(x) | \eta_i^G > 0\}$$
$$I_{+-}^{++}(x) = \{i \in I_{+-}(x) | \eta_i^G > 0\} \tag{2.3}$$

In order to establish the corresponding duality results, we give the following definitions and theorem which can be found in Achtziger and Kanzow (2008).

Definition 2.1 Let $x^* \in X$ be a feasible point of the MPVC (1.1). The Abadie constraint qualification, denoted by ACQ, is said to hold at x^*, iff $T(x^*) = L(x^*)$, where

$$T(x^*) = \left\{ d \in R^n : \exists\{x^k\} \subseteq X, \exists\{t_k\} \downarrow 0, x^k \to x^* \ and \ \frac{x^k - x^*}{t_k} \to d \right\}$$

is the standard tangent cone of the MPVC (1.1) at x^*, and

$$\begin{aligned} L(x^*) = \{d \in R^n : & \nabla g_i(x^*)^T d \leq 0, \ i \in I_g(x^*), \ \nabla h_j(x^*)^T d = 0, \ j = 1, 2, \ldots, p, \\ & \nabla H_i(x^*)^T d = 0, \ i \in I_{0+}(x^*), \ \nabla H_i(x^*)^T d \geq 0, \ i \in I_{00}(x^*) \cup I_{0-}(x^*), \\ & \nabla G_i(x^*)^T d \leq 0, \ i \in I_{+0}(x^*)\} \end{aligned}$$

denotes the corresponding linearized cone of the MPVC (1.1) at x^*.

Definition 2.2 Let $x^* \in X$ be a feasible point of the MPVC (1.1). The VC-ACQ is said to hold at x^*, iff $L^{VC}(x^*) \subseteq T(x^*)$, where

$$\begin{aligned} L^{VC}(x^*) = \{d \in R^n : & \nabla g_i(x^*)^T d \leq 0, \ i \in I_g(x^*), \ \nabla h_j(x^*)^T d = 0, \ j = 1, 2, \ldots, p, \\ & \nabla H_i(x^*)^T d = 0, \ i \in I_{0+}(x^*), \ \nabla H_i(x^*)^T d \geq 0, \ i \in I_{00}(x^*) \cup I_{0-}(x^*), \\ & \nabla G_i(x^*)^T d \leq 0, \ i \in I_{00}(x^*) \cup I_{+0}(x^*)\} \end{aligned}$$

denotes the corresponding VC-linearized cone of the MPVC (1.1) at x^*.

Theorem 2.1 *Let $x^* \in X$ be a local minimum of the MPVC (1.1) such that VC-ACQ holds at x^*. Then, there exist Lagrange multipliers $\lambda_i \in R$ $(i = 1, 2, \ldots, m)$, $\mu_j \in R$ $(j \in I_h)$, $\eta_i^H, \eta_i^G \in R$ $(i = 1, 2, \ldots, l)$, such that*

$$\nabla L(x^*, \lambda, \mu, \eta^H, \eta^G) = 0 \tag{2.4}$$

and

$$\begin{aligned} & h_j(x^*) = 0 \ (j \in I_h(x^*)), \\ & \lambda_i \geq 0, \ g_i(x^*) \leq 0, \ \lambda_i g_i(x^*) = 0 \ (i = 1, 2, \ldots, m), \\ & \eta_i^H = 0 \ (i \in I_+(x^*)), \ \eta_i^H \geq 0 \ (i \in I_{00}(x^*) \cup I_{0-}(x^*)), \ \eta_i^H \ is \ free \ (i \in I_{0+}(x^*)), \\ & \eta_i^G = 0 \ (i \in I_{0+}(x^*) \cup I_{0-}(x^*) \cup I_{+-}(x^*)), \ \eta_i^G \geq 0 \ (i \in I_{00}(x^*) \cup I_{+0}(x^*)). \end{aligned} \tag{2.5}$$

The following concepts of convexity and generalized convexity play a vital role during the establishment of some duality theorems.

Definition 2.3 Let $S \subseteq R^n$ be any nonempty set and let $f : R^n \to R$ be continuously differentiable. Then, f is said to be convex at $x^* \in S$ on S, iff for any $x \in S$, one has

$$f(x) - f(x^*) \geq \langle \nabla f(x^*), x - x^* \rangle.$$

Definition 2.4 Let $S \subseteq R^n$ be any nonempty set and let $f : R^n \to R$ be continuously differentiable. Then, f is said to be quasiconvex at $x^* \in S$ on S, iff for any $x \in S$, one has

$$f(x) \leq f(x^*) \Rightarrow \langle \nabla f(x^*), x - x^* \rangle \leq 0.$$

Definition 2.5 Let $S \subseteq R^n$ be any nonempty set and let $f : R^n \to R$ be continuously differentiable. Then, f is said to be pseudoconvex at $x^* \in S$ on S, iff for any $x \in S$, one has

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \Rightarrow f(x) \geq f(x^*).$$

3 Two new duals for mathematical programms with vanishing constraints

Recently, Mishra et al. (2016) gives the Wolfe type dual and Mond–Weir type dual for the MPVC (1.1). It is noted that their models involve the calculation of the index set, which make it is complicated in the practical applications. In this section, we will propose the new Wolfe type and Mond–Weir type dual models which does not contain the calculation of index set. Also we will establish the weak and strong, converse, restricted converse (strictly converse) dual theorems under mild conditions. Moreover, we will utilize the example to explain their validity.

3.1 Wolfe and Mond–Weir type dual models proposed by Mishra et al

Firstly, we give the Wolfe type and Mond–Weir type duals of the MPVC (1.1) which were proposed by Mishra et al. (2016). For $x \in X$, the Wolfe type and Mond–Weir type duals of the MPVC (1.1) are as follows:

Wolfe type dual model:

$$\begin{aligned}
& max\ L(y, \lambda, \mu, \eta^H, \eta^G) \\
& s.t.\ \ \nabla L(y, \lambda, \mu, \eta^H, \eta^G) = 0 \\
& \qquad \lambda_i \geq 0, \forall i \notin I_g(x), \\
& \qquad \eta_i^H \geq 0, \forall i \in I_+(x), \\
& \qquad \eta_i^G \leq 0, \forall i \in I_{0+}(x), \eta_i^G \geq 0, \forall i \in I_{0-}(x) \cup I_{+-}(x).
\end{aligned}$$

Mond–Weir type dual models:

$$\begin{aligned}
& max\ f(y) \\
& s.t.\ \ \nabla L(y, \lambda, \mu, \eta^H, \eta^G) = 0 \\
& \qquad \lambda_i \in R^+, \lambda_i g_i(y) \geq 0, \forall i = 1, 2, \ldots, m, \\
& \qquad \mu_j \in R, \mu_j h_j(y) \geq 0, \forall j = 1, 2, \ldots, p, \\
& \qquad \eta_i^H \geq 0, \forall i \in I_+(x), \eta_i^H \in R, \forall i \in I_0(x), \\
& \qquad - \eta_i^H H_i(y) \geq 0, \forall i = 1, 2, \ldots, l, \\
& \qquad \eta_i^G \leq 0, \forall i \in I_{0+}(x), \eta_i^G \geq 0, \forall i \in I_{0-}(x) \cup I_{+-}(x), \\
& \qquad\qquad\qquad \eta_i^G \in R, \forall i \in I_{00}(x) \cup I_{+0}(x), \\
& \qquad \eta_i^G G_i(y) \geq 0, \forall i = 1, 2, \ldots, l.
\end{aligned}$$

Remark 3.1 For the above two models, we can directly see that they involve the calculation of the index set. So, it is difficult to deal with the dual problems from algorithm point of view, and this also limits the practicability of the models.

Subsequentially, we give the new dual models and establish the weak, strong, converse and restricted converse duality theorems.

3.2 New Wolfe type dual model

For $x \in X$, the new Wolfe type dual of the MPVC (1.1), VC-WD(x) for short, is as follows:

$$\begin{aligned} & max\, L(y, \lambda, \mu, \eta^H, \eta^G) \\ & s.t.\;\; \nabla L(y, \lambda, \mu, \eta^H, \eta^G) = 0 \\ & \qquad \lambda_i \geq 0, \forall i = 1, 2, \ldots, m, \\ & \qquad \eta_i^G = \upsilon_i H_i(x), \upsilon_i \geq 0, \forall i = 1, 2, \ldots, l \\ & \qquad \eta_i^H = \rho_i - \upsilon_i G_i(x), \rho_i \geq 0, \forall i = 1, 2, \ldots, l. \end{aligned} \tag{3.1}$$

Let $S_w(x) \subseteq R^n \times R^m \times R^p \times R^l \times R^l$ denote the feasible set, i.e.,

$$\begin{aligned} S_w(x) = \{ & (y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) : \nabla L(y, \lambda, \mu, \eta^H, \eta^G) = 0, \\ & \lambda_i \geq 0, \quad i = 1, 2, \ldots, m, \\ & \eta_i^G = \upsilon_i H_i(x), \upsilon_i \geq 0, \quad i = 1, 2, \ldots, l, \\ & \eta_i^H = \rho_i - \upsilon_i G_i(x), \rho_i \geq 0, \quad i = 1, 2, \ldots, l.\} \end{aligned} \tag{3.2}$$

We denote by

$$pr S_w(x) = \{ y \in R^n : (y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) \in S_w(x) \}$$

the projection of the set $S_w(x)$ on R^n.

To be independent of the MPVC (1.1), we consider another dual problem which is denoted by VC-WD as follows:

$$max\;\; L(y, \lambda, \mu, \eta^H, \eta^G)$$

subject to

$$(y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) \in \cap_{x \in X} S_w(x) \tag{3.3}$$

The set of all feasible points of the VC-WD is denoted by $S_w = \cap_{x \in X} S_w(x)$ and the projection of the set S_w on R^n is denoted by $pr S_w$.

Remark 3.2 Compared with the new Wolfe dual model and the Wolfe dual model which was proposed by Mishra et al. (2016), we can obviously see that they do not involve the calculation of the index set for the above new Wolfe model. Moreover, it is not difficult to find that some signs of multiplier are different between the new Wolfe dual model and the Mishra et al's Wolfe dual model. Firstly, $\lambda_i \geq 0, \forall i = 1, 2, \ldots, m$ is required in the new Wolfe dual model, but it only requires that $\lambda_i \geq 0, \forall i \notin I_g(x)$ in the Mishra et al's Wolfe dual model. Hence, our model can better explain the nonnegative of the multipliers corresponding to the inequality constraints $g_i(x) \leq 0,\; i = 1, 2, \ldots, m$. Secondly, $\eta_i^G = 0, \forall i \in I_{0+}(x)$ is required in the new Wolfe dual model, but it requires that $\eta_i^G \leq 0, \forall i \in I_{0+}(x)$ in the Mishra et al's Wolfe dual model. This implies that our model can better explain the complementary slackness of the multipliers corresponding to the constraint functions $G_i(x),\; i \in I_{0+}(x)$.

Remark 3.3 In the new Wolfe dual model, the significance of ρ_i and ν_i is the same as the one in Theorem 1 of Achtziger and Kanzow (2008). How to select them to evaluate the optimal solution can be found in Remark 1 of Achtziger and Kanzow (2008).

Firstly, we give the weak duality theorem. The theorem shows the relationship between a feasible point of the MPVC (1.1) and a feasible point of the new Wolfe type dual.

Theorem 3.1 *Let $x \in X$, $(y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) \in S_w$ be feasible points for the MPVC (1.1) and the VC-WD, respectively. If one of the following conditions holds:*

(1) $L(\cdot, \lambda, \mu, \eta^H, \eta^G)$ is convex at $y \in X \cup pr S_w$;
(2) $f, g_i(i \in I_g^+(x)), h_j(j \in I_h^+(x)), -h_j(j \in I_h^-(x)), -H_i(i \in I_{+0}(x) \cup I_{+-}(x) \cup I_{00}(x) \cup I_{0-}(x) \cup I_{0+}^+(x)), -H_i(i \in I_{0+}^-(x)), -G_i(i \in I_{0+}(x)), G_i(i \in I_{00}(x) \cup I_{+0}(x) \cup I_{0-}(x) \cup I_{+-}(x))$ are convex at $y \in X \cup pr S_w$;

Then $f(x) \geq L(y, \lambda, \mu, \eta^H, \eta^G)$.

Proof (1) Suppose $f(x) < L(y, \lambda, \mu, \eta^H, \eta^G)$,i.e.,

$$f(x) < f(y) + \sum_{i=1}^{m} \lambda_i g_i(y) + \sum_{j=1}^{p} \mu_j h_j(y) - \sum_{i=1}^{l} \eta_i^H H_i(y) + \sum_{i=1}^{l} \eta_i^G G_i(y). \tag{3.4}$$

Since $x \in X$ and (3.1), it follows that

$$\begin{aligned}
&g_i(x) < 0 \quad \lambda_i \geq 0 \quad i \notin I_g(x),\\
&g_i(x) = 0 \quad \lambda_i \geq 0 \quad i \in I_g(x),\\
&h_j(x) = 0 \quad \mu_j \in R \quad j \in I_h,\\
&-H_i(x) < 0 \quad \eta_i^H \geq 0 \quad i \in I_+(x),\\
&-H_i(x) = 0 \quad \eta_i^H \in R \quad i \in I_0(x),\\
&G_i(x) > 0 \quad \eta_i^G = 0 \quad i \in I_{0+}(x),\\
&G_i(x) = 0 \quad \eta_i^G \geq 0 \quad i \in I_{00}(x) \cup I_{+0}(x),\\
&G_i(x) < 0 \quad \eta_i^G \geq 0 \quad i \in I_{0-}(x) \cup I_{+-}(x),
\end{aligned}$$

that is,

$$\sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) - \sum_{i=1}^{l} \eta_i^H H_i(x) + \sum_{i=1}^{l} \eta_i^G G_i(x) \leq 0 \tag{3.5}$$

Adding (3.4) and (3.5), one has

$$\begin{aligned}
&f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) - \sum_{i=1}^{l} \eta_i^H H_i(x) + \sum_{i=1}^{l} \eta_i^G G_i(x)\\
&\quad < f(y) + \sum_{i=1}^{m} \lambda_i g_i(y) + \sum_{j=1}^{p} \mu_j h_j(y) - \sum_{i=1}^{l} \eta_i^H H_i(y) + \sum_{i=1}^{l} \eta_i^G G_i(y)
\end{aligned}$$

i.e.,

$$L(x, \lambda, \mu, \eta^H, \eta^G) < L(y, \lambda, \mu, \eta^H, \eta^G) \tag{3.6}$$

By the convexity of $L(\cdot, \lambda, \mu, \eta^H, \eta^G)$, it follows that

$$L(y, \lambda, \mu, \eta^H, \eta^G) + \langle \nabla L(y, \lambda, \mu, \eta^H, \eta^G), x - y\rangle \quad \leq L(x, \lambda, \mu, \eta^H, \eta^G)$$

In view of the first equation in (3.1), one has $L(x, \lambda, \mu, \eta^H, \eta^G) \geq L(y, \lambda, \mu, \eta^H, \eta^G)$. That is a contradiction to (3.6) and hence the result is proved.

(2) By the convexity of $g_i(i \in I_g^+(x)), h_j(j \in I_h^+(x)), -h_j(j \in I_h^-(x)), -H_i(i \in I_{+0}(x) \cup I_{+-}(x) \cup I_{00}(x) \cup I_{0-}(x) \cup I_{0+}^+(x)), -H_i(i \in I_{0+}^-(x)), -G_i(i \in I_{0+}(x)), G_i(i \in I_{00}(x) \cup I_{+0}(x) \cup I_{0-}(x) \cup I_{+-}(x)$ at $y \in X \cup prS_w$, $x \in X$, $(y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_w$, one has

$$\begin{aligned}
g_i(y) + \langle \nabla g_i(y), x - y\rangle &\leq g_i(x) \leq 0 \quad \lambda_i > 0, i \in I_g^+(x),\\
h_j(y) + \langle \nabla h_j(y), x - y\rangle &\leq h_j(x) = 0 \quad \mu_j > 0, j \in I_h^+(x),\\
h_j(y) + \langle \nabla h_j(y), x - y\rangle &\geq h_j(x) = 0 \quad \mu_j < 0, j \in I_h^-(x),\\
-H_i(y) - \langle \nabla H_i(y), x - y\rangle &\leq -H_i(x) \leq 0, \eta_i^H \geq 0, i \in I_{+0}(x)\\
&\qquad \cup I_{+-}(x) \cup I_{00}(x) \cup I_{0-}(x) \cup I_{0+}^+(x),\\
-H_i(y) - \langle \nabla H_i(y), x - y\rangle &\leq -H_i(x) = 0, \eta_i^H < 0, i \in I_{0+}^-(x),\\
G_i(y) + \langle \nabla G_i(y), x - y\rangle &\geq G_i(x) > 0, \eta_i^G = 0, i \in I_{0+}(x),\\
G_i(y) + \langle \nabla G_i(y), x - y\rangle &\leq G_i(x) = 0, \eta_i^G \geq 0, i \in I_{+0}(x) \cup I_{00}(x),\\
G_i(y) + \langle \nabla G_i(y), x - y\rangle &\leq G_i(x) < 0, \eta_i^G \geq 0, i \in I_{0-}(x) \cup I_{+-}(x),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^m \lambda_i g_i(y) + \sum_{j=1}^p \mu_j h_j(y) - \sum_{i=1}^l \eta_i^H H_i(y) + \sum_{i=1}^l \eta_i^G G_i(y)\\
&\quad + \left\langle \sum_{i=1}^m \lambda_i \nabla g_i(y) + \sum_{j=1}^p \mu_j \nabla h_j(y) - \sum_{i=1}^l \eta_i^H \nabla H_i(y) + \sum_{i=1}^l \eta_i^G \nabla G_i(y), x - y \right\rangle \leq 0
\end{aligned} \tag{3.7}$$

Also, by the convexity of f at $y \in X \cup prS_w$, one has

$$f(y) + \langle \nabla f(y), x - y\rangle \leq f(x) \tag{3.8}$$

Adding (3.7) and (3.8), one has

$$L(y, \lambda, \mu, \eta^H, \eta^G) + \langle \nabla L(y, \lambda, \mu, \eta^H, \eta^G), x - y\rangle \leq f(x)$$

In view of the first equation in (3.1), one has

$$L(y, \lambda, \mu, \eta^H, \eta^G) \leq f(x).$$

and hence the result is proved. □

The following strong duality theorem gives the condition under which the new Wolfe dual is solvable and the global maximum can be obtained.

Theorem 3.2 *Let $x^* \in X$ be a local minimum of the MPVC (1.1), such that the VC-ACQ holds at x^*. Then, there exist Lagrange multipliers $\bar{\lambda} \in R^m, \bar{\mu} \in R^p, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v} \in R^l$, such that $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ is a feasible point of the VC-WD(x^*) and*

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(x^*) + \sum_{j=1}^{p} \bar{\mu}_j h_j(x^*) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(x^*) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(x^*) = 0 \tag{3.9}$$

Moreover, if one of the following conditions holds:

(1) *$L(\cdot, \lambda, \mu, \eta^H, \eta^G)$ is convex at $y \in X \cup pr S_w(x^*)$;*
(2) *$f, g_i(i \in I_g^+(x^*)), h_j(j \in I_h^+(x^*)), -h_j(j \in I_h^-(x^*)), -H_i(i \in I_{+0}(x^*) \cup I_{+-}(x^*) \cup I_{00}(x^*) \cup I_{0-}(x^*) \cup I_{0+}^+(x^*)), H_i(i \in I_{0+}^-(x^*)), -G_i(i \in I_{0+}(x^*)), G_i(i \in I_{00}(x^*) \cup I_{+0}(x^*) \cup I_{0-}(x^*) \cup I_{+-}(x^*))$ are convex at $y \in X \cup pr S_w(x^*)$;*

Then, $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ is a global maximum of the VC-WD(x^*), that is,

$$L(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G) \geq L(y, \lambda, \mu, \eta^H, \eta^G), \forall (y, \lambda, \mu, \eta^H, \eta^G) \in S_w(x^*)$$

and

$$f(x^*) = L(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G).$$

Proof Since x^* is local minimum of the MPVC (1.1) and the VC-ACQ condition is satisfied at x^*, by Theorem 2.1, it follows that, there exist Lagrange multipliers $\bar{\lambda} \in R^m, \bar{\mu} \in R^p, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v} \in R^l$, such that the conditions (2.4) and (2.5) hold and hence $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ is a feasible point of the VC-WD(x^*). By Theorem 3.1, one has

$$f(x^*) \geq L(y, \lambda, \mu, \eta^H, \eta^G), \forall (y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_w(x^*) \tag{3.10}$$

Adding (3.9) and (3.10), one has

$$L(x^*, \lambda, \mu, \eta^H, \eta^G) \geq L(y, \lambda, \mu, \eta^H, \eta^G), \forall (y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_w(x^*)$$

that is, $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ is a global maximum of the VC-WD(x^*). Also, the local minimum of the MPVC (1.1) and the global minimum of the VC-WD(x^*) are equal. □

The following theorem is a converse duality theorem. It gives the condition under which a feasible point of the new Wolfe dual generates a global minimum of the MPVC (1.1).

Theorem 3.3 *Let $x \in X$ be any feasible solution of the MPVC (1.1) and let $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ be a feasible point of the VC-WD such that*

$$\begin{aligned} &\bar{\lambda}_i g_i(y^*) \geq 0 \quad i = 1, 2, \ldots, m, \\ &\bar{\mu}_j h_j(y^*) = 0 \quad j = 1, 2, \ldots, p, \\ &-\bar{\eta}_i^H H_j(y^*) \geq 0 \quad i = 1, 2, \ldots, l, \\ &\bar{\lambda}_i^G G_i(y^*) \geq 0 \quad i = 1, 2, \ldots, l. \end{aligned}$$

Moreover, if one of the following conditions holds:

(1) $L(\cdot, \lambda, \mu, \eta^H, \eta^G)$ is convex at $y^* \in X \cup pr S_w$;
(2) $f, g_i(i \in I_g^+(x)), h_j(j \in I_h^+(x)), -h_j(j \in I_h^-(x)), -H_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x)), -H_i(i \in I_{0+}^-(x)), G_i(i \in I_{+0}^{++}(x) \cup \in I_{+-}^{++}(x))$ are convex at $y^* \in X \cup pr S_w$;

Then, y^* is a global minimum of the MPVC (1.1).

Proof Suppose to the contrary that y^* is not a global minimum of the MPVC (1.1), i.e., there exists $\tilde{x} \in X$ such that

$$f(\tilde{x}) < f(y^*) \tag{3.11}$$

(1) Since $\tilde{x}$ and $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ be the feasible point for the MPVC (1.1) and the VC-WD, respectively. Combining the hypothesis in the theorem, one has

$$\begin{aligned} &\sum_{i=1}^{m} \bar{\lambda}_i g_i(\tilde{x}) + \sum_{j=1}^{p} \bar{\mu}_j h_j(\tilde{x}) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(\tilde{x}) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(\tilde{x}) \le 0 \\ &\quad \le \sum_{i=1}^{m} \bar{\lambda}_i g_i(y^*) + \sum_{j=1}^{p} \bar{\mu}_j h_j(y^*) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(y^*) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(y^*) \end{aligned} \tag{3.12}$$

Adding (3.11) and (3.12), one has

$$L(\tilde{x}, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G) < L(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G)$$

By the convexity of $L(\cdot, \lambda, \mu, \eta^H, \eta^G)$ at $y^* \in X \cup prS_w$, it follows that

$$\langle \nabla L(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G), \tilde{x} - y^* \rangle < 0,$$

this is a contradiction to the dual constraint (3.1) of the VC-WD (x) and hence the result is proved.

(2) Since $\tilde{x}$ and $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ be the feasible point for the MPVC (1.1) and the VC-WD, respectively. Combining the hypothesis in the theorem, one has

$$\begin{aligned} g_i(\tilde{x}) &\le g_i(y^*) \quad i \in I_g^+(\tilde{x}), \\ h_j(\tilde{x}) &= h_j(y^*) \quad j \in I_h^+(\tilde{x}) \cup I_h^-(\tilde{x}), \\ -H_i(\tilde{x}) &\le -H_i(y^*) \quad i \in I_{+0}^+(\tilde{x}) \cup I_{+-}^+(\tilde{x}) \cup I_{00}^+(\tilde{x}) \cup I_{0-}^+(\tilde{x}) \cup I_{0+}^+(\tilde{x}), \\ -H_i(\tilde{x}) &\ge -H_i(y^*) \quad i \in I_{0+}^-(\tilde{x}), \\ G_i(\tilde{x}) &\le G_i(y^*) \quad i \in I_{+0}^{++}(\tilde{x}) \cup I_{+-}^{++}(\tilde{x}), \end{aligned}$$

By the convexity of the fuction in the theorem, it follows that

$$\begin{aligned} &\langle \nabla g_i(y^*), \tilde{x} - y^* \rangle \le 0, \quad \bar{\lambda}_i > 0, \quad i \in I_g^+(\tilde{x}), \\ &\langle \nabla h_j(y^*), \tilde{x} - y^* \rangle \le 0, \quad \bar{\mu}_j > 0, \quad j \in I_h^+(\tilde{x}), \\ &\langle \nabla h_j(y^*), \tilde{x} - y^* \rangle \ge 0, \quad \bar{\mu}_j < 0, \quad j \in I_h^-(\tilde{x}), \\ &-\langle \nabla H_i(y^*), \tilde{x} - y^* \rangle \le 0, \quad \bar{\eta}_i^H \ge 0, \quad i \in I_{+0}^+(\tilde{x}) \cup I_{+-}^+(\tilde{x}) \cup I_{00}^+(\tilde{x}) \cup I_{0-}^+(\tilde{x}) \cup I_{0+}^+(\tilde{x}), \\ &-\langle \nabla H_i(y^*), \tilde{x} - y^* \rangle \ge 0, \quad \bar{\eta}_i^H \le 0, \quad i \in I_{0+}^-(\tilde{x}), \\ &\langle \nabla G_i(y^*), \tilde{x} - y^* \rangle \le 0, \quad \bar{\eta}_i^G \ge 0, \quad i \in I_{+0}^{++}(\tilde{x}) \cup I_{+-}^{++}(\tilde{x}), \end{aligned}$$

which implies that

$$\left\langle \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(y^*) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(y^*) - \sum_{i=1}^{l} \bar{\eta}_i^H \nabla H_i(y^*) + \sum_{i=1}^{l} \bar{\eta}_i^G \nabla G_i(y^*), \tilde{x} - y^* \right\rangle \le 0.$$

Using the above inequality and (3.1), one has

$$\langle \nabla f(y^*), \tilde{x} - y^* \rangle \geq 0.$$

By the convexity of f, it follows that

$$f(\tilde{x}) \geq f(y^*),$$

this is a contradiction to our hypothesis and hence the result is proved. □

The following theorem is restricted converse duality theorem which gives a sufficient condition for a feasible point of the MPVC (1.1) to be a global minimum by using the new Wolfe dual.

Theorem 3.4 *Let $x^* \in X$ be a feasible point of the MPVC (1.1) and let $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon})$ be a feasible point of the VC-WD such that $f(x^*) = L(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G)$. Moreover, if one of the following conditions holds:*

(1) $L(\cdot, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G)$ is convex at $y^* \in X \cup pr S_w$;
(2) $f, g_i(i \in I_g^+(x^*)), h_j(j \in I_h^+(x^*)), -h_j(j \in I_h^-(x^*)), -H_i(i \in I_{+0}^+(x^*) \cup I_{+-}^+(x^*) \cup I_{00}^+(x^*) \cup I_{0-}^+(x^*) \cup I_{0+}^+(x^*)), -H_i(i \in I_{0+}^-(x^*)), G_i(i \in I_{+0}^{++}(x^*) \cup \in I_{+-}^{++}(x^*))$ are convex at $y^* \in X \cup pr S_w$;

Then, x^* is a global minimum of the MPVC (1.1).

Proof Suppose to the contrary that $x^* \in X$ is not a global minimum of the MPVC (1.1), then there exists $\tilde{x} \in X$ such that

$$f(\tilde{x}) < f(x^*).$$

Combining the assumption in the theorem, it follows that

$$f(\tilde{x}) < L(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G),$$

a contradiction to the Theorem 3.1 and hence the result is proved. □

The following strict converse duality theorem gives a sufficient condition about the uniqueness of a local minimum of the MPVC (1.1) and a global maximum of the new Wolfe dual model.

Theorem 3.5 *Let $x^* \in X$ be a local minimum for the MPVC (1.1) such that the VC-ACQ at x^*. Assume the conditions of Theorem 3.2 hold and $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{\upsilon})$ be a global maximum of the VC-WD(x^*). If one of the following conditions holds:*

(1) $L(\cdot, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G)$ is strictly convex at $y \in X \cup pr S_w(x^*)$;
(2) f is strictly convex and $g_i(i \in I_g^+(x^*)), h_j(j \in I_h^+(x^*)), -h_j(j \in I_h^-(x^*)), -H_i(i \in I_{+0}(x^*) \cup I_{+-}(x^*) \cup I_{00}(x^*) \cup I_{0-}(x^*) \cup I_{0+}^+(x^*)), -H_i(i \in I_{0+}^-(x^*)), -G_i(i \in I_{0+}(x^*)), G_i(i \in I_{00}(x^*) \cup I_{+0}(x^*) \cup I_{0-}(x^*) \cup I_{+-}(x^*))$ are convex at $y \in X \cup pr S_w(x^*)$.

Then $x^* = y^*$.

Proof (1) Suppose that $x^* \neq y^*$. By Theorem 3.2, there exist Lagrange multipliers $\bar{\lambda} \in R^m$, $\bar{\mu} \in R^p$, $\bar{\eta}^G \in R^l$, $\bar{\eta}^H \in R^l$, $\bar{\rho} \in R^l$, $\bar{\upsilon} \in R^l$ such that $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon})$ be a global maximum of the VC-WD(x^*). Hence,

$$f(x^*) = L(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G) = L(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G). \tag{3.13}$$

In view of the feasibility of x^* for the MPVC (1.1) and the feasibility of $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{\upsilon})$ for the VC-WD(x^*), it follows that

$$\begin{aligned}
&g_i(x^*) < 0 \quad \tilde{\lambda}_i \geq 0 \quad i \notin I_g(x^*),\\
&g_i(x^*) = 0 \quad \tilde{\lambda}_i \geq 0 \quad i \in I_g(x^*),\\
&h_j(x^*) = 0 \quad \tilde{\mu}_j \in R \quad j \in I_h(x^*),\\
&-H_i(x^*) < 0 \quad \tilde{\eta}_i^H \geq 0 \quad i \in I_+(x^*),\\
&-H_i(x^*) = 0 \quad \tilde{\eta}_i^H \in R \quad i \in I_0(x^*),\\
&G_i(x^*) > 0 \quad \tilde{\eta}_i^G = 0 \quad i \in I_{0+}(x^*),\\
&G_i(x^*) = 0 \quad \tilde{\eta}_i^G \geq 0 \quad i \in I_{00}(x^*) \cup I_{+0}(x^*),\\
&G_i(x^*) < 0 \quad \tilde{\eta}_i^G \geq 0 \quad i \in I_{0-}(x^*) \cup I_{+-}(x^*),
\end{aligned}$$

that is,

$$\sum_{i=1}^{m} \tilde{\lambda}_i g_i(x^*) + \sum_{j=1}^{p} \tilde{\mu}_j h_j(x^*) - \sum_{i=1}^{l} \tilde{\eta}_i^H H_i(x^*) + \sum_{i=1}^{l} \tilde{\eta}_i^G G_i(x^*) \leq 0 \tag{3.14}$$

Adding (3.13) and (3.14), one has

$$L(x^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G) \leq L(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G) \tag{3.15}$$

By the strict convexity of $L(\cdot, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G)$, it follows that

$$\langle \nabla L(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G), x^* - y^* \rangle < 0$$

That is a contradiction to the first equation in (3.1) and hence the result is proved.

(2) By the strict convexity of f at y^*, one has

$$f(x^*) - f(y^*) >< \nabla f(y^*), x^* - y^* \rangle. \tag{3.16}$$

In view of the convexity of $g_i (i \in I_g^+(x^*))$, $h_j (j \in I_h^+(x^*))$, $-h_j (j \in I_h^-(x^*))$, $-H_i (i \in I_{+0}(x^*) \cup I_{+-}(x^*) \cup I_{00}(x^*) \cup I_{0-}(x^*) \cup I_{0+}^+(x^*))$, $-H_i (i \in I_{0+}^-(x^*))$, $-G_i (i \in I_{0+}(x^*))$, $G_i (i \in I_{00}(x^*) \cup I_{+0}(x^*) \cup I_{0-}(x^*) \cup I_{+-}(x^*)$ at $y^* \in X \cup prS_w(x^*)$, $x^* \in X$ and $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{\upsilon}) \in S_w(x^*)$, one has

$$\begin{aligned}
&g_i(y^*) + \langle \nabla g_i(y^*), x^* - y^* \rangle \leq g_i(x^*) \leq 0 \quad \tilde{\lambda}_i > 0, i \in I_g^+(x^*),\\
&h_j(y^*) + \langle \nabla h_j(y^*), x^* - y^* \rangle \leq h_j(x^*) = 0 \quad \tilde{\mu}_j > 0, j \in I_h^+(x^*),\\
&h_j(y^*) + \langle \nabla h_j(y^*), x^* - y^* \rangle \geq h_j(x^*) = 0 \quad \tilde{\mu}_j < 0, j \in I_h^-(x^*),\\
&-H_i(y^*) - \langle \nabla H_i(y^*), x^* - y^* \rangle \leq -H_i(x^*) \leq 0, \tilde{\eta}_i^H \geq 0, i \in I_{+0}(x^*)\\
&\qquad \cup I_{+-}(x^*) \cup I_{00}(x^*) \cup I_{0-}(x^*) \cup I_{0+}^+(x^*),\\
&-H_i(y^*) - \langle \nabla H_i(y^*), x^* - y^* \rangle \leq -H_i(x^*) = 0, \tilde{\eta}_i^H < 0, i \in I_{0+}^-(x^*),\\
&G_i(y^*) + \langle \nabla G_i(y^*), x^* - y^* \rangle \geq G_i(x^*) > 0, \tilde{\eta}_i^G = 0, i \in I_{0+}(x^*),\\
&G_i(y^*) + \langle \nabla G_i(y^*), x^* - y^* \rangle \leq G_i(x^*) = 0, \tilde{\eta}_i^G \geq 0, i \in I_{+0}(x^*) \cup I_{00}(x^*),\\
&G_i(y^*) + \langle \nabla G_i(y^*), x^* - y^* \rangle \leq G_i(x^*) < 0, \tilde{\eta}_i^G \geq 0, i \in I_{0-}(x^*) \cup I_{+-}(x^*),
\end{aligned}$$

which implies that

$$\sum_{i=1}^{m}\tilde{\lambda}_i g_i(y^*)+\sum_{j=1}^{p}\tilde{\mu}_j h_j(y^*)-\sum_{i=1}^{l}\tilde{\eta}_i^H H_i(y^*)+\sum_{i=1}^{l}\tilde{\eta}_i^G G_i(y^*) \\ +\left\langle\sum_{i=1}^{m}\tilde{\lambda}_i\nabla g_i(y^*)+\sum_{j=1}^{p}\tilde{\mu}_j\nabla h_j(y^*)-\sum_{i=1}^{l}\tilde{\eta}_i^H\nabla H_i(y^*)+\sum_{i=1}^{l}\tilde{\eta}_i^G\nabla G_i(y^*),x^*-y^*\right\rangle\leq 0 \tag{3.17}$$

Adding (3.16) and (3.17), one has

$$L(y^*,\tilde{\lambda},\tilde{\mu},\tilde{\eta}^H,\tilde{\eta}^G)<f(x^*).$$

This is a contradiction to (3.13) and hence the result is proved. □

In order to verify the validity of the new Wolfe dual model and the related theorems, we give the following example.

Example 1 Consider the following MPVC

$$\begin{aligned} &min\ f(x)=x_1^2+x_2^2\\ &s.t.\ \ H_1(x)=x_2\geq 0,\\ &\qquad G_1(x)H_1(x)=x_1x_2\leq 0, \end{aligned} \tag{3.18}$$

with $n=2, m=p=0, l=1$. The new Wolfe dual model to (3.18) is given by

$$\begin{aligned} &max\ L(y,\eta_1^H,\eta_1^G)=y_1^2+y_2^2-\eta_1^H y_2+\eta_1^G y_1\\ &s.t.\ \ \nabla L(y,\eta_1^H,\eta_1^G)=(2y_1+\eta_1^G,2y_2-\eta_1^H)^T=0,\\ &\qquad \eta_1^G=v_1x_2,\ v_1\geq 0\\ &\qquad \eta_1^H=\rho_1-v_1x_1,\ \rho_1\geq 0. \end{aligned} \tag{3.19}$$

(1) Let $x^*=(0,0)^T\in X$, $(y,\eta_1^H,\eta_1^G,\rho_1,v_1)=(0,0,0,0,0)\in S_W(x^*)$, one has

$$f(x^*)=0=L(0,0,0)$$

It can be verified that the hypothesis of Theorem 3.4 holds, since the positive definiteness of $\nabla^2 L(y,\eta_1^H,\eta_1^G)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$. Taking account (3.18), x^* is a global minimum of (3.18). So, Theorem 3.4 is verified.

(2) We can get $y_1=-\frac{1}{2}\eta_1^G$, $y_2=\frac{1}{2}\eta_1^H$ by (3.19). One has also

$$L(y,\eta_1^H,\eta_1^G)=-\frac{1}{4}\eta_1^{G\,2}-\frac{1}{4}\eta_1^{H\,2}\leq 0.$$

Since $f(x)=x_1^2+x_2^2\geq 0$, we can get $f(x)\geq L(y,\eta_1^H,\eta_1^G)$, Hence, Theorem 3.1 is verified.

(3) We can obtain that (3.18) satisfy VC-LICQ, since $\nabla H_1=(0,1)^T$, $\nabla G_1=(1,0)^T$. So we obtain that (3.18) satisfies VC-ACQ. By Theorem 2.1, there exist Lagrange multipliers $\eta_1^H,\eta_1^G,\rho_1,v_1\in R$ such that $(0,\eta_1^H,\eta_1^G,\rho_1,v_1)$ is a feasible point of the VC-WD(0) and

$$-\eta_1^H H_1(0)+\eta_1^G G_1(0)=0.$$

So, $(0,\eta_1^H,\eta_1^G,\rho_1,v_1)$ is a global maximum of the VC-WD(0) and $f(0)=0=L(0,\eta_1^H,\eta_1^G)$. Theorem 3.2 is verified.

3.3 New Mond–Weir type dual model

In this section, we discuss the new Mond–Weir type dual for MPVC (1.1). For $x \in X$, the new Mond–Weir type dual of the MPVC (1.1), VC-MWD(x) for short, is as follows:

$$\begin{aligned} & max\ f(y) \\ & s.t.\ \ \nabla L(y, \lambda, \mu, \eta^H, \eta^G) = 0, \\ & \quad \lambda_i \geq 0, \lambda_i g_i(y) \geq 0, \qquad i = 1, 2, \ldots, m, \\ & \quad \mu_j h_j(y) = 0, \qquad j = 1, 2, \ldots, p, \\ & \quad \eta_i^G G_i(y) \geq 0, \qquad i = 1, 2, \ldots, l, \\ & \quad \eta_i^G = v_i H_i(x), v_i \geq 0, \qquad i = 1, 2, \ldots, l, \\ & \quad -\eta_i^H H_i(y) \geq 0, \qquad i = 1, 2, \ldots, l, \\ & \quad \eta_i^H = \rho_i - v_i G_i(x), \rho_i \geq 0,\ i = 1, 2, \ldots, l. \end{aligned} \tag{3.20}$$

Let $S_{MW}(x) \subseteq R^n \times R^m \times R^p \times R^l \times R^l$ denote feasible point set, i.e.

$$\begin{aligned} S_{MW}(x) = \{ & (y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) : \nabla\psi(y, \lambda, \mu, \eta^H, \eta^G) = 0, \\ & \lambda_i \geq 0, \lambda_i g_i(y) \geq 0, \qquad i = 1, 2, \ldots, m, \\ & \mu_j h_j(y) = 0, \qquad j = 1, 2, \ldots, p, \\ & \eta_i^G G_i(y) \geq 0, \qquad i = 1, 2, \ldots, l, \\ & \eta_i^G = v_i H_i(x), v_i \geq 0, \qquad i = 1, 2, \ldots, l, \\ & -\eta_i^H H_i(y) \geq 0, \qquad i = 1, 2, \ldots, l, \\ & \eta_i^H = \rho_i - v_i G_i(x), \rho_i \geq 0,\ i = 1, 2, \ldots, l\}. \end{aligned} \tag{3.21}$$

We denote by

$$pr S_{MW}(x) = \{y \in R^n : (y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_{MW}(x)\}$$

the projection of the set $S_{MW}(x)$ on R^n.

Similar to the new Wolfe dual, we also consider another dual problem which is denoted by VC-MWD as follows:

$$\begin{aligned} & max\ \ f(y) \\ & s.t.\ \ \ (y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in \cap_{x \in X} S_{MW}(x) \end{aligned}$$

The set of all feasible points of the VC-MWD is denoted by $S_{MW} = \cap_{x \in X} S_{MW}(x)$ and the projection of the set S_{MW} on R^n is denoted by $pr S_{MW}$.

Remark 3.4 From the new Mond–Weir type dual model and the Mond–Weir type dual model which was proposed by Mishra et al. (2016), we can also see that they do not involve the calculation of the index set for the above new Mond–Weir type model. Moreover, it is not difficult to find that some signs of multiplier are different between the new Mond–Weir type dual model and the Mishra et al's Mond–Weir type dual model. Firstly, $\eta_i^G \geq 0, \forall i \in I_{00}(x) \cup I_{+0}(x),\ \eta_i^G = 0,\ i \in I_{0+}(x)$ is required in the new Mond–Weir dual model, but it requires that $\eta_i^G \in R, \forall i \in I_{00}(x) \cup I_{+0}(x),\ \eta_i^G \leq 0,\ i \in I_{0+}(x)$ in the Mishra et al's

Mond–Weir dual model, i.e., the signs of $\eta_i^G,\ i \in I_{00}(x) \cup I_{+0}(x) \cup I_{0+}(x)$ in our model are the same as the ones in Theorem 1 of Achtziger and Kanzow (2008). This shows that our model can better explain the complementary slackness and the nonnegative of the multipliers corresponding to the constraint function $G_i(x),\ i \in I_{00}(x) \cup I_{+0}(x) \cup I_{0+}(x)$. Secondly, $\eta_i^H \geq 0, \forall i \in I_{00}(x) \cup I_{0-}(x),\ \eta_i^H \in R,\ i \in I_{0+}(x)$ is required in the new Mond–Weir dual model, but it requires that $\eta_i^H \in R, \forall i \in I_{00}(x) \cup I_{0-}(x) \cup I_{0+}(x)$ in the Mishra et al's Mond–Weir dual model, i.e., the signs of $\eta_i^H,\ i \in I_{00}(x) \cup I_{0-}(x) \cup I_{0+}(x)$ in our model are the same as the ones in Theorem 1 of Achtziger and Kanzow (2008). This implies that our model can better explain the nonnegative of the multipliers corresponding to the constraint functions $H_i(x),\ i \in I_{00}(x) \cup I_{0-}(x) \cup I_{0+}(x)$.

Remark 3.5 Similar to the new Wolfe dual model, the significance of ρ_i and ν_i is also the same as the one in Theorem 1 of Achtziger and Kanzow (2008). The method about selecting these parameters to evaluate the optimal solution can also be found in Remark 1 of Achtziger and Kanzow (2008).

The following weak duality theorem shows the relationship between a feasible point of the MPVC (1.1) and a feasible point of the new Mond–Weir type dual.

Theorem 3.6 *Let $x \in X$ and $(y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_{MW}$ be feasible points for the MPVC (1.1) and the VC-MWD, respectively. Moreover, if one of the following conditions holds:*

(1) *$f(\cdot)$ is pseudoconvex and $\sum_{i=1}^{m} \lambda_i g_i(\cdot) + \sum_{j=1}^{p} \mu_j h_j(\cdot) - \sum_{i=1}^{l} \eta_i^H H_i(\cdot) + \sum_{i=1}^{l} \eta_i^G G_i(\cdot)$ is quasiconvex at $y \in X \cup pr S_{MW}$, respectively;*

(2) *$f(\cdot)$ is pseudoconvex and $g_i(i \in I_g^+(x)), h_j(j \in I_h^+(x)), -h_j(j \in I_h^-(x)), -H_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x)), -H_i(i \in I_{0+}^-(x)), G_i(i \in I_{+0}^{++}(x) \cup \in I_{+-}^{++}(x))$ are quasiconvex at $y \in X \cup pr S_{MW}$,respectively;*

Then, $f(x) \geq f(y)$.

Proof (1) Since $x \in X$ and $(y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_{MW}$, it follows that

$$\begin{aligned}
&g_i(x) \leq 0 \quad \lambda_i \geq 0 \quad i = 1, 2, \ldots, m,\\
&h_j(x) = 0 \quad \mu_j \in R \quad j = 1, 2, \ldots, p,\\
&-H_i(x) < 0 \quad \eta_i^H \geq 0 \quad i \in I_+(x),\\
&-H_i(x) = 0 \quad \eta_i^H \in R \quad i \in I_0(x),\\
&G_i(x) > 0 \quad \eta_i^G = 0 \quad i \in I_{0+}(x),\\
&G_i(x) = 0 \quad \eta_i^G \geq 0 \quad i \in I_{00}(x) \cup I_{+0}(x),\\
&G_i(x) < 0 \quad \eta_i^G \geq 0 \quad i \in I_{0-}(x) \cup I_{+-}(x).
\end{aligned}$$

By (3.20), it implies that

$$\begin{aligned}
&\sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) - \sum_{i=1}^{l} \eta_i^H H_i(x) + \sum_{i=1}^{l} \eta_i^G G_i(x)\\
&\quad \leq \sum_{i=1}^{m} \lambda_i g_i(y) + \sum_{j=1}^{p} \mu_j h_j(y) - \sum_{i=1}^{l} \eta_i^H H_i(y) + \sum_{i=1}^{l} \eta_i^G G_i(y)
\end{aligned}$$

Combining the quasiconvexity of $\sum_{i=1}^{m} \lambda_i g_i(\cdot) + \sum_{j=1}^{p} \mu_j h_j(\cdot) - \sum_{i=1}^{l} \eta_i^H H_i(\cdot) + \sum_{i=1}^{l} \eta_i^G G_i(\cdot)$, one has

$$\left\langle \sum_{i=1}^{m} \lambda_i \nabla g_i(y) + \sum_{j=1}^{p} \mu_j \nabla h_j(y) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(y) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(y), x - y \right\rangle \leq 0.$$

Using the above inequality and the first equation in (3.20), one has

$$\langle \nabla f(y), x - y \rangle \geq 0.$$

By the pseudoconvexity of f, it implies that

$$f(x) \geq f(y),$$

and hence the result is proved.

(2) By $x \in X$, $(y, \lambda, \mu, \eta^H, \eta^G, \rho, v) \in S_{MW}$, it follows that

$$\begin{aligned}
g_i(x) &\leq g_i(y) \quad i \in I_g^+(x), \\
h_j(x) &= h_j(y) \quad j \in I_h^+(x) \cup I_h^-(x), \\
-H_i(x) &\leq -H_i(y) \quad i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x), \\
-H_i(x) &\geq -H_i(y) \quad i \in I_{0+}^-(x), \\
G_i(x) &\leq G_i(y) \quad i \in I_{+0}^{++}(x) \cup I_{+-}^{++}(x),
\end{aligned}$$

By the quasiconvexity of f, $g_i(i \in I_g^+(x))$, $h_j(j \in I_h^+(x))$, $-h_j(j \in I_h^-(x))$, $-H_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, $H_i(i \in I_{0+}^-(x))$, $G_i(i \in I_{+0}^{++}(x) \cup \in I_{+-}^{++}(x))$, it implies that

$$\begin{aligned}
&\langle \nabla g_i(y), x - y \rangle \leq 0, \quad i \in I_g^+(x), \\
&\langle \nabla h_j(y), x - y \rangle \leq 0, \quad j \in I_h^+(x), \\
&\langle \nabla h_j(y), x - y \rangle \geq 0, \quad j \in I_h^-(x), \\
&-\langle \nabla H_i(y), x - y \rangle \leq 0, \quad i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x), \\
&-\langle \nabla H_i(y), x - y \rangle \geq 0, \quad i \in I_{0+}^-(x), \\
&\langle \nabla G_i(y), x - y \rangle \leq 0, \quad i \in I_{+0}^{++}(x) \cup I_{+-}^{++}(x).
\end{aligned}$$

From the above inequalities and (2.3), it follows that

$$\langle \sum_{i=1}^{m} \lambda_i \nabla g_i(y) + \sum_{j=1}^{p} \mu_j \nabla h_j(y) - \sum_{i=1}^{l} \eta_i^H \nabla H_i(y) + \sum_{i=1}^{l} \eta_i^G \nabla G_i(y), x - y \rangle \leq 0.$$

Combining the above inequality and (3.20), one has

$$\langle \nabla f(y), x - y \rangle \geq 0.$$

By the pseudoconvexity of f, it implies that

$$f(x) \geq f(y)$$

and hence the result is proved. □

The following strong duality theorem gives the condition under which the new Mond–Weir dual is solvable and the global maximum can be obtained.

Theorem 3.7 *Let $x^* \in X$ be a local minimum of the MPVC (1.1) such that the VC-ACQ holds at x^*. Then, there exist Lagrange multipliers $\bar{\lambda} \in R^m, \bar{\mu} \in R^p, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon} \in R^l$, such that $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon})$ is a feasible point of the VC-MWD(x^*), that is, $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon}) \in S_{MW}(x^*)$. Moreover, Theorem 3.6 holds, then $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G)$ is a global maximum of the VC-MWD(x^*).*

Proof Since $x^* \in X$ is a local minimum of the MPVC (1.1) and the VC-ACQ condition is satisfied at x^*. By theorem 2.1, it follows that, there exist Lagrange multipliers $\bar{\lambda} \in R^m, \bar{\mu} \in R^p, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon} \in R^l$ such that the conditions (2.4) and (2.5) hold and hence $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon})$ is a feasible point of the VC-MWD(x^*). By Theorem 3.6, it follows that

$$f(x^*) \geq f(y), \ \forall (y, \lambda, \mu, \eta^H, \eta^G, \rho, \upsilon) \in S_{MW}(x^*),$$

and hence $(x^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon}) \in S_{MW}(x)$ is a global maximum of the VC-MWD (x^*). □

The following converse duality theorem gives the condition under which a feasible point of the new Mond–Weir dual generates a global minimum of the MPVC (1.1).

Theorem 3.8 *Let $x \in X$ and $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon}) \in S_{MW}$ be feasible points for the MPVC (1.1) and the VC-MWD, respectively. Moreover, if one of the following conditions holds:*

(1) *$f(\cdot)$ is pseudoconvex and $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot) + \sum_{j=1}^{p} \bar{\mu}_j h_j(\cdot) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(\cdot) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(\cdot)$ is quasiconvex at $y^* \in X \cup pr S_{MW}$, respectively;*
(2) *$f(\cdot)$ is pseudoconvex and $g_i(i \in I_g^+(x)), h_j(j \in I_h^+(x)), -h_j(j \in I_h^-(x)), -H_i(i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x)), -H_i(i \in I_{0+}^-(x)), G_i(i \in I_{+0}^{++}(x) \cup \in I_{+-}^{++}(x))$ are quasiconvex at $y^* \in X \cup pr S_{MW}$;*
Then y^* is a global minimum of the MPVC (1.1).

Proof Suppose to the contrary that y^* is not a global minimum of the MPVC (1.1), that is, there exists $\tilde{x} \in X$, such that $f(\tilde{x}) < f(y^*)$.

(1) By the pseudoconvexity of $f(\cdot)$, one has

$$\langle \nabla f(y^*), \tilde{x} - y^* \rangle < 0. \tag{3.22}$$

Since $\tilde{x} \in X$, $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{\upsilon}) \in S_{MW}$, one has

$$\begin{aligned}
&\bar{\lambda}_i g_i(\tilde{x}) \leq \bar{\lambda}_i g_i(y^*) \qquad i = 1, 2, \ldots, m, \\
&\bar{\mu}_j h_j(\tilde{x}) = \bar{\mu}_j h_j(y^*) \qquad j = 1, 2, \ldots, p, \\
&-\bar{\eta}_i^H H_i(\tilde{x}) \leq -\bar{\eta}_i^H H_i(y^*) \quad i = 1, 2, \ldots, l, \\
&\bar{\eta}_i^G G_i(\tilde{x}) \leq \bar{\eta}_i^G G_i(y^*) \qquad i = 1, 2, \ldots, l.
\end{aligned}$$

which implies that

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(\tilde{x}) + \sum_{j=1}^{p} \bar{\mu}_j h_j(\tilde{x}) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(\tilde{x}) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(\tilde{x})$$

$$\leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(y^*) + \sum_{j=1}^{p} \bar{\mu}_j h_j(y^*) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(y^*) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(y^*)$$

By the quasiconvexity of $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot) + \sum_{j=1}^{p} \bar{\mu}_j h_j(\cdot) - \sum_{i=1}^{l} \bar{\eta}_i^H H_i(\cdot) + \sum_{i=1}^{l} \bar{\eta}_i^G G_i(\cdot)$, it follows that

$$\left\langle \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(y^*) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(y^*) - \sum_{i=1}^{l} \bar{\eta}_i^H \nabla H_i(y^*) + \sum_{i=1}^{l} \bar{\eta}_i^G \nabla G_i(y^*), \tilde{x} - y^* \right\rangle \leq 0. \tag{3.23}$$

Adding the inequalities (3.22) and (3.23), one has

$$\langle \nabla L(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G), \tilde{x} - y^* \rangle < 0,$$

this is a contradiction to (3.20) and hence the result is proved.

(2) Since $\tilde{x} \in X$, $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v}) \in S_{MW}$, one has

$$\begin{aligned}
\bar{\lambda}_i g_i(\tilde{x}) &\leq \bar{\lambda}_i g_i(y^*) \quad & i = 1, 2, \ldots, m, \\
\bar{\mu}_j h_j(\tilde{x}) &= \bar{\mu}_j h_j(y^*) \quad & j = 1, 2, \ldots, p, \\
-\bar{\eta}_i^H H_i(\tilde{x}) &\leq -\bar{\eta}_i^H H_i(y^*) \quad & i = 1, 2, \ldots, l, \\
\bar{\eta}_i^G G_i(\tilde{x}) &\leq \bar{\eta}_i^G G_i(y^*) \quad & i = 1, 2, \ldots, l.
\end{aligned}$$

Using the above inequalities and (2.3), it follows that

$$\begin{aligned}
g_i(\tilde{x}) &\leq g_i(y^*) \quad & i \in I_g^+(\tilde{x}), \\
h_j(\tilde{x}) &= h_j(y^*) \quad & j \in I_h^+(\tilde{x}) \cup I_h^-(\tilde{x}), \\
-H_i(\tilde{x}) &\leq -H_i(y^*) \quad & i \in I_{+0}^+(\tilde{x}) \cup I_{+-}^+(\tilde{x}) \cup I_{00}^+(\tilde{x}) \cup I_{0-}^+(\tilde{x}) \cup I_{0+}^+(\tilde{x}), \\
-H_i(\tilde{x}) &\geq -H_i(y^*) \quad & i \in I_{0+}^-(\tilde{x}), \\
G_i(\tilde{x}) &\leq G_i(y^*) \quad & i \in I_{+0}^{++}(\tilde{x}) \cup I_{+-}^{++}(\tilde{x}),
\end{aligned}$$

By the quasiconvexity of $g_i (i \in I_g^+(x))$, $h_j (j \in I_h^+(x))$, $-h_j (j \in I_h^-(x))$, $-H_i (i \in I_{+0}^+(x) \cup I_{+-}^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, $H_i (i \in I_{0+}^-(x))$, $G_i (i \in I_{+0}^{++}(x) \cup \in I_{+-}^{++}(x))$, it implies that

$$\begin{aligned}
&\langle \nabla g_i(y^*), \tilde{x} - y^* \rangle \leq 0, \quad i \in I_g^+(\tilde{x}), \\
&\langle \nabla h_j(y^*), \tilde{x} - y^* \rangle \leq 0, \quad j \in I_h^+(\tilde{x}), \\
&\langle \nabla h_j(y^*), \tilde{x} - y^* \rangle \geq 0, \quad j \in I_h^-(\tilde{x}), \\
&-\langle \nabla H_i(y^*), \tilde{x} - y^* \rangle \leq 0, \quad i \in I_{+0}^+(\tilde{x}) \cup I_{+-}^+(\tilde{x}) \cup I_{00}^+(\tilde{x}) \cup I_{0-}^+(\tilde{x}) \cup I_{0+}^+(\tilde{x}), \\
&-\langle \nabla H_i(y^*), \tilde{x} - y^* \rangle \geq 0, \quad i \in I_{0+}^-(\tilde{x}), \\
&\langle \nabla G_i(y^*), \tilde{x} - y^* \rangle \leq 0, \quad i \in I_{+0}^{++}(\tilde{x}) \cup I_{+-}^{++}(\tilde{x}),
\end{aligned}$$

Then, it follows that

$$\left\langle \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(y^*) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(y^*) - \sum_{i=1}^{l} \bar{\eta}_i^H \nabla H_i(y^*) + \sum_{i=1}^{l} \bar{\eta}_i^G \nabla G_i(y^*), \tilde{x} - y^* \right\rangle \leq 0.$$

Combining the above inequality and (3.20), one has

$$\langle \nabla f(y^*), \tilde{x} - y^* \rangle \geq 0$$

By the pseudoconvexity of $f(\cdot)$, it implies that

$$f(\tilde{x}) \geq f(y^*),$$

this is a contradiction to our hypothesis and hence the result is proved. □

The following restricted converse duality theorem gives a sufficient condition for a feasible point of the MPVC (1.1) to be a global minimum by utilizing the new Mond–Weir dual.

Theorem 3.9 *Let $x^* \in X$ and $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v}) \in S_{MW}$ be feasible points for the MPVC (1.1) and the VC-MWD, respectively, such that $f(x^*) = f(y^*)$. If the hypothesis of theorem 3.6 holds at $y^* \in X \cup prS_{MW}$, then x^* is a global minimum of the MPVC (1.1).*

Proof Suppose to the contrary that $x^* \in X$ is not a global minimum of the MPVC (1.1), then there exists $\tilde{x} \in X$ such that

$$f(\tilde{x}) \leq f(x^*).$$

From the assumptions in the theorem, it follows that

$$f(\tilde{x}) \leq f(y^*),$$

this is a contradiction to the Theorem 3.6 and hence the result is proved. □

The following strict converse duality theorem gives a sufficient condition about the uniqueness of a local minimum of the MPVC (1.1) and a global maximum of the new Wolfe dual model.

Theorem 3.10 *Let $x^* \in X$ be a local minimum for the MPVC (1.1) such that the VC-ACQ at x^*. Assume the conditions of Theorem 3.7 hold and $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{v})$ be a global maximum of the VC-WD(x^*). If one of the following conditions holds:*

(1) $f(\cdot)$ is strictly pseudoconvex and $\sum_{i=1}^{m} \tilde{\lambda}_i g_i(\cdot) + \sum_{j=1}^{p} \tilde{\mu}_j h_j(\cdot) - \sum_{i=1}^{l} \tilde{\eta}_i^H H_i(\cdot) + \sum_{i=1}^{l} \tilde{\eta}_i^G G_i(\cdot)$ is quasiconvex at $y^* \in X \cup prS_{MW}(x^*)$, respectively;

(2) $f(\cdot)$ is strictly pseudoconvex and $g_i(i \in I_g^+(x^*)), h_j(j \in I_h^+(x^*)), -h_j(j \in I_h^-(x^*)), -H_i(i \in I_{+0}^+(x^*) \cup I_{+-}^+(x^*) \cup I_{00}^+(x^*) \cup I_{0-}^+(x^*) \cup I_{0+}^+(x^*)), -H_i(i \in I_{0+}^-(x^*)), G_i(i \in I_{+0}^{++}(x^*) \cup \in I_{+-}^{++}(x^*))$ are quasiconvex at $y^* \in X \cup prS_{MW}(x^*)$, respectively;

Then, $x^* \neq y^*$.

Proof (1) Suppose that $x^* \neq y^*$. By Theorem 3.7, there exist Lagrange multipliers $\bar{\lambda} \in R^m$, $\bar{\mu} \in R^p$, $\bar{\eta}^G \in R^l$, $\bar{\eta}^H \in R^l$, $\bar{\rho} \in R^l$, $\bar{v} \in R^l$ such that $(y^*, \bar{\lambda}, \bar{\mu}, \bar{\eta}^H, \bar{\eta}^G, \bar{\rho}, \bar{v})$ be a global maximum of the VC-MWD(x^*). Hence,

$$f(x^*) = f(y^*). \tag{3.24}$$

Since $x^* \in X$ and $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{v}) \in S_{MW}$, it follows that

$$g_i(x^*) \leq 0 \quad \tilde{\lambda}_i \geq 0 \quad i = 1, 2, \ldots, m,$$
$$h_j(x^*) = 0 \quad \tilde{\mu}_j \in R \quad j = 1, 2, \ldots, p,$$

$$
\begin{aligned}
&-H_i(x^*)<0 \quad \tilde{\eta}_i^H \geq 0 \quad i \in I_+(x^*),\\
&-H_i(x^*)=0 \quad \tilde{\eta}_i^H \in R \quad i \in I_0(x^*),\\
&G_i(x^*)>0 \quad \tilde{\eta}_i^G=0 \quad i \in I_{0+}(x^*),\\
&G_i(x^*)=0 \quad \tilde{\eta}_i^G \geq 0 \quad i \in I_{00}(x^*) \cup I_{+0}(x^*),\\
&G_i(x^*)<0 \quad \tilde{\eta}_i^G \geq 0 \quad i \in I_{0-}(x^*) \cup I_{+-}(x^*).
\end{aligned}
$$

By (3.20), it implies that

$$
\begin{aligned}
&\sum_{i=1}^m \tilde{\lambda}_i g_i(x^*)+\sum_{j=1}^p \tilde{\mu}_j h_j(x^*)-\sum_{i=1}^l \tilde{\eta}_i^H H_i(x^*)+\sum_{i=1}^l \tilde{\eta}_i^G G_i(x^*)\\
&\quad \leq \sum_{i=1}^m \tilde{\lambda}_i g_i(y^*)+\sum_{j=1}^p \tilde{\mu}_j h_j(y^*)-\sum_{i=1}^l \tilde{\eta}_i^H H_i(y^*)+\sum_{i=1}^l \tilde{\eta}_i^G G_i(y^*)
\end{aligned}
$$

Combining the quasiconvexity of $\sum_{i=1}^m \tilde{\lambda}_i g_i(\cdot)+\sum_{j=1}^p \tilde{\mu}_j h_j(\cdot)-\sum_{i=1}^l \tilde{\eta}_i^H H_i(\cdot)+\sum_{i=1}^l \tilde{\eta}_i^G G_i(\cdot)$, one has

$$
\langle\sum_{i=1}^m \tilde{\lambda}_i \nabla g_i(y^*)+\sum_{j=1}^p \tilde{\mu}_j \nabla h_j(y^*)-\sum_{i=1}^l \tilde{\eta}_i^H \nabla H_i(y^*)+\sum_{i=1}^l \tilde{\eta}_i^G \nabla G_i(y^*), x^*-y^*\rangle \leq 0.
$$

Using the above inequality and the first equation in (3.20), one has

$$
\langle\nabla f(y^*), x^*-y^*\rangle \geq 0.
$$

By the strictly pseudoconvexity of f, it implies that

$$
f(x^*)>f(y^*),
$$

This is a contradiction to (3.24) and hence the result is proved.
(2) By using $x^* \in X$, $(y^*, \tilde{\lambda}, \tilde{\mu}, \tilde{\eta}^H, \tilde{\eta}^G, \tilde{\rho}, \tilde{v}) \in S_{MW}(x^*)$, it follows that

$$
\begin{aligned}
&g_i(x^*) \leq g_i(y^*) \quad i \in I_g^+(x^*),\\
&h_j(x^*)=h_j(y^*) \quad j \in I_h^+(x^*) \cup I_h^-(x^*),\\
&-H_i(x^*) \leq -H_i(y^*) \quad i \in I_{+0}^+(x^*) \cup I_{+-}^+(x^*) \cup I_{00}^+(x^*) \cup I_{0-}^+(x^*) \cup I_{0+}^+(x^*),\\
&-H_i(x^*) \geq -H_i(y^*) \quad i \in I_{0+}^-(x^*),\\
&G_i(x^*) \leq G_i(y^*) \quad i \in I_{+0}^{++}(x^*) \cup I_{+-}^{++}(x^*),
\end{aligned}
$$

In view of the quasiconvexity of $g_i(i \in I_g^+(x^*)), h_j(j \in I_h^+(x^*)), -h_j(j \in I_h^-(x^*)), -H_i(i \in I_{+0}^+(x^*) \cup I_{+-}^+(x^*) \cup I_{00}^+(x^*) \cup I_{0-}^+(x^*) \cup I_{0+}^+(x^*)), H_i(i \in I_{0+}^-(x^*)), G_i(i \in I_{+0}^{++}(x^*) \cup \in I_{+-}^{++}(x^*))$, it implies that

$$
\begin{aligned}
&\langle\nabla g_i(y^*), x^*-y^*\rangle \leq 0, \quad i \in I_g^+(x^*),\\
&\langle\nabla h_j(y^*), x^*-y^*\rangle \leq 0, \quad j \in I_h^+(x^*),\\
&\langle\nabla h_j(y^*), x^*-y^*\rangle \geq 0, \quad j \in I_h^-(x^*),
\end{aligned}
$$

$$-\langle \nabla H_i(y^*), x^* - y^* \rangle \le 0, \quad i \in I_{+0}^{+}(x^*) \cup I_{+-}^{+}(x^*) \cup I_{00}^{+}(x^*) \cup I_{0-}^{+}(x^*) \cup I_{0+}^{+}(x^*),$$
$$-\langle \nabla H_i(y^*), x^* - y^* \rangle \ge 0, \quad i \in I_{0+}^{-}(x^*),$$
$$\langle \nabla G_i(y^*), x^* - y^* \rangle \le 0, \quad i \in I_{+0}^{++}(x^*) \cup I_{+-}^{++}(x^*).$$

From the above inequalities and (2.3), it follows that

$$\left\langle \sum_{i=1}^{m} \tilde{\lambda}_i \nabla g_i(y^*) + \sum_{j=1}^{p} \tilde{\mu}_j \nabla h_j(y^*) - \sum_{i=1}^{l} \tilde{\eta}_i^H \nabla H_i(y^*) + \sum_{i=1}^{l} \tilde{\eta}_i^G \nabla G_i(y^*), x^* - y^* \right\rangle \le 0.$$

Combining the above inequality and (3.20), one has

$$\langle \nabla f(y^*), x^* - y^* \rangle \ge 0.$$

By the strictly pseudoconvexity of f, it implies that

$$f(x^*) > f(y^*)$$

This is a contradiction to (3.24) and hence the result is proved.

In order to verify the validity of the new Mond–Weir type dual, we continue to consider (3.18).

Example 2 For (3.18), its new Mond–Weir dual model is given by

$$\begin{aligned} & max\ f(y) = y_1^2 + y_2^2 \\ & s.t.\ \nabla L(y, \eta_1^H, \eta_1^G) = (2y_1 + \eta_1^G, 2y_2 - \eta_1^H)^T = 0, \\ & \quad \eta_1^G G_1(y) = \eta_1^G y_1 \ge 0, \\ & \quad \eta_1^G = v_1 x_2, v_1 \ge 0, \\ & \quad -\eta_1^H H_1(y) = -\eta_1^H y_2 \ge 0, \\ & \quad \eta_1^H = \rho_1 - v_1 x_1, \rho_1 \ge 0. \end{aligned} \tag{3.25}$$

(1) Let $x^* = (0, 0)^T \in X$, $(y^*, \eta_1^H, \eta_1^G, \rho_1, v_1) = (0, 0, 0, 0, 0) \in S_{MW}$ one has

$$f(x^*) = 0 = f(y^*)$$

It can be verified that the hypothesis of Theorem 3.8 hold, since the positive definiteness of $\nabla^2 L(y, \eta_1^H, \eta_1^G) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Taking into account (3.18), x^* is a global minimum of (3.18). So, Theorem 3.8 is verified.

(2) We can get $y_1 = -\frac{1}{2}\eta_1^G$, $y_2 = \frac{1}{2}\eta_1^H$ by (3.25), one has

$$L(y, \eta_1^H, \eta_1^G) = f(y) - \eta_1^H H_1(y) + \eta_1^G G_1(y) = -\frac{1}{4}{\eta_1^G}^2 - \frac{1}{4}{\eta_1^H}^2 \le 0,$$

which implies that

$$f(y) \le \eta_1^H H_1(y) - \eta_1^G G_1(y),$$

combining (3.25), one has $f(y) \le 0$. Since $f(x) = x_1^2 + x_2^2 \ge 0$, we can get $f(x) \ge f(y)$. Theorem 3.6 is verified.

(3) We can obtain that (3.18) satisfy VC-LICQ, since $\nabla H_1 = (0, 1)^T$, $\nabla G_1 = (1, 0)^T$. So we obtain that (3.18) satisfy VC-ACQ. By Theorem 2.1, there exist Lagrange multipliers $\eta_1^H, \eta_1^G, \rho_1, v_1 \in R$ such that $(0, \eta_1^H, \eta_1^G, \rho_1, v_1)$ is a feasible point of the VC-MWD(0).

Taking into account $f(y) \leq 0$, $(0, \eta_1^H, \eta_1^G, \rho_1, v_1)$ is a global maximum of the VC-MWD(0) and Theorem 3.7 is verified.

4 Conclusions

In this paper, we have formulated new Wolfe and Mond–Weir type dual models to the MPVC and have established the weak, strong, converse and restricted converse duality results under the assumptions of convexity, strict convexity, pseudoconvexity, strict pseudoconvexity and quasiconvexity. Also, the validity of the results is verified by an example. As the future work, some other dual models for the MPVC, like the mixed type dual, may be investigated by relaxing the convexity and generalized convexity assumptions.

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