



On superiority and weak stability of population states in evolutionary games

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Abstract

We revisit some concepts of superiority and weak stability of population states in evolutionary games with continuous strategy space. We prove a general stability result for replicator trajectories by introducing the concept of superiority with respect to a given closed set. Some important results in the literature regarding weak stability turn out to be special cases of our main result.

Keywords Evolutionary game theory · Evolutionarily stable strategy · Replicator dynamics · Games with continuous strategy space

1 Introduction

Evolutionary game theory models different situations in biology as well as economics. One such situation is conflict amongst individuals of a species for resources or mates or territory. The individuals participating in an evolutionary game can have finitely or infinitely many pure strategies. Evolutionary games with finite pure strategy sets have been widely studied in the literature (Maynard Smith 1982; Weibull 1995; Hofbauer and Sigmund 1988; Cressman 2003; Sandholm 2010). There are numerous results available regarding the evolutionary stability and dynamic stability of population states through various dynamics, for such games (Weibull 1995; Cressman 2003; Sandholm 2010).

Many competing situations in real life, such as harvest preemption, war of attrition, bargaining, etc., have continuous strategy spaces. An approach to study continuous evolutionary games is approximation by finite games. These approximations are successful in many appli-

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cations. However, there are situations where such approximations fail to give desired results (see Alós-Ferrer 2006).

It is only recently that the literature on evolutionary games with continuous strategy spaces started growing (Bomze and Pötscher 1989; Oechssler and Riedel 2001, 2002; Cressman 2005; Cressman and Hofbauer 2005; Cressman et al. 2006; Shaiju and Bernhard 2009; Hingu et al. 2016a). Here, a population state is a probability measure on its continuous strategy space. Certain static stability concepts such as evolutionary stable strategy (ESS) and uninvadability are studied. Replicator dynamics for evolution of population states is defined along with certain dynamic stability concepts such as Lyapunov stability and attracting property. There are a few more static stability and dynamic stability concepts defined for such games but they depend on the notion of “closeness” of population states.

In evolutionary games with continuous strategy spaces, the closeness of population states is mainly studied using strong and weak topologies. In strong topology, two population states are ‘close’ to one another, when only a small proportion of a population perturbs from their strategies resulting in the other population. Suppose in a population, instead of a small proportion being perturbed, the entire population changes its strategies to new strategies which are near to the old strategies, then the resulting new population is also ‘close’ to the old population in some sense. This type of perturbations are not captured by the distance used in the strong topology, thus leading us naturally to consider games in the weak topology.

Using the strong topology, the static stability concept of strong uninvadability (Bomze 1990) is defined which is stronger than the concept of uninvadability. Dynamic stability concepts of strong Lyapunov stability and strongly attracting property are also studied. With the help of the weak topology, the static stability concept of evolutionary robust strategy (\mathcal{ERS}) (Oechssler and Riedel 2002) and the dynamic stability concepts of weak Lyapunov stability and weakly attracting property are also discussed. Most of the results discussed in Oechssler and Riedel (2001), Oechssler and Riedel (2002) regarding the dynamic stability and evolutionary stability of a population state require that the entire population plays a single strategy. Such a population state is called a monomorphic population and is denoted by the Dirac measure δ_x .

Oechssler and Riedel studied evolutionary games with continuous strategy spaces with the strong topology in Oechssler and Riedel (2001) and proved that an uninvadable monomorphic population state is strongly Lyapunov stable. Moreover, if the initial population state is strongly close to this monomorphism and the payoff function is continuous, then it is weakly attracting. Strong stability of polymorphic population states has been studied in Hingu et al. (2016a). It is established that an uninvadable polymorphism is strongly attracting. A polymorphism is a population state with finite support.

Games with the weak topology have been studied by Oechssler and Riedel (2002) as well as by Cressman (2005) (see also Cressman and Hofbauer 2005; Cressman et al. 2006). Oechssler and Riedel prove that an \mathcal{ERS} is weakly Lyapunov stable in case of doubly symmetric games with compact strategy spaces. They also prove asymptotic stability of monomorphisms. Cressman however, discusses stability of polymorphic population states in the weak topology using the concepts of neighbourhood attracting strategy and neighbourhood superiority.

Eshel and Sansone (2003) study the stability of the replicator dynamics trajectories under the maximal-shift topology. They consider the stability in continuous evolutionary games with the pure strategy space as a subset of the real line through the concept of continuously replicator stable strategies (CRSS). The authors also discuss similarities and differences between a population state when it is a CRSS, continuously stable strategy (CSS) and neighbourhood invader strategy (NIS). CSS was first defined by Eshel and Motro (1981) whereas, NIS was first defined by Apaloo (1997).

In order to study the stability of a population state P^* with respect to the replicator dynamics, the initial population state $Q(0)$ is chosen from a neighborhood (strong or weak) in such a way that the support of P^* is contained in the support of $Q(0)$. This is a necessary condition for studying the stability, as the replicator dynamics is derived based on the principal of relative change of frequencies of available strategies. Thus, in general, a population state P^* is not stable with respect to its complete neighbourhood. van Veelen and Spreij show that if a population state is asymptotically stable with respect to its complete neighbourhood then it is necessarily a polymorphic population state, irrespective of the underlying topology (Veelen and Spreij 2009, Propositions 13 and 14).

While discussing the weak stability of polymorphic population states, Cressman (2005) restricts initial population states to those which are not only near to it in the weak sense but also have their support near the support of the polymorphism. He illustrates this to be a necessary condition by giving an implicit example in which a monomorphic population is destabilized when a small mass is introduced on a point which is far away (in the strong sense) from the monomorphism but still in its weak neighbourhood.

Motivated by Cressman (2005), in this paper we give an explicit example (Example 1 in Sect. 2) to illustrate the above fact. We introduce a generalized notion of superiority of population states using which we prove the main result (Theorem 4 in Sect. 3) establishing weak convergence of replicator trajectories. Before discussing the main result, in the next section, we introduce some preliminary notions and definitions along with further motivations.

2 Preliminaries and problem description

This section is divided into two parts. The first subsection gives preliminary definitions and notations. In the second subsection, we discuss some existing results regarding the weak convergence of replicator trajectories.

2.1 Preliminaries

Let (S, d) be a compact metric space. Consider a two-player symmetric evolutionary game $G = (S, u)$, with the pure strategy space S and the payoff function $u : S \times S \rightarrow \mathbb{R}$. The game is symmetric in the sense that the role of the players is not important: that is, $u(z, w)$ is the payoff for the strategy z when played against the strategy w . Through out this paper, it is assumed that the payoff function u is measurable and bounded.

A population state of this evolutionary game G , is a probability measure Q on (S, \mathcal{B}) , where \mathcal{B} is the Borel σ -algebra of S . The set of all such population states is $\Delta = \Delta(S)$. The average payoff of a population state P against population state Q is given by,

$$E(P, Q) = \int_S \int_S u(z, w) Q(dw) P(dz).$$

We now recall the concepts of evolutionary stable strategy (ESS) and uninvadability.

Definition 1 (Maynard Smith 1974; Bomze and Pötscher 1989) A population state P is called *evolutionarily stable* if for every “mutation” $Q \neq P$, there is an invasion barrier $\epsilon(Q) > 0$, that is, for all $0 < \eta \leq \epsilon(Q)$,

$$E(P, (1 - \eta)P + \eta Q) > E(Q, (1 - \eta)P + \eta Q).$$

Definition 2 (Vickers and Cannings 1987) A population state P is called *uninvadable* if, in Definition 1, $\epsilon(Q)$ can be chosen independent of $Q \in \Delta$.

For η small enough the population state $(1 - \eta)P + \eta Q$ is *close* to P . However, all population states that are close to P need not be expressed in this form when S is infinite. The closeness of population states in the infinite case is usually studied using two topologies; strong topology and weak topology. In each case, we have a static stability definition stronger than uninvadability. We restrict ourselves to the weak topology in the sequel.

The weak topology on Δ can be induced by the Prohorov metric (Billingsley 1999) given by,

$$\pi(P, Q) = \inf\{\epsilon > 0 : P(B) \leq Q(B^\epsilon) + \epsilon \text{ and } Q(B) \leq P(B^\epsilon) + \epsilon, \text{ for all } B \in \mathcal{B}\}$$

where $B^\epsilon = \{z \in S : d(z, B) < \epsilon\}$. This metric can also be defined as

$$\pi(P, Q) = \inf\{\epsilon > 0 : P(F) \leq Q(F^\epsilon) + \epsilon \text{ for all closed sets } F \in \mathcal{B}\}, \tag{1}$$

because of the regularity of probability measures and the symmetry of the condition in the definition (Billingsley 1999).

Next we recall the definition of evolutionarily robust strategy (\mathcal{ERS}).

Definition 3 (Oechssler and Riedel 2002) A population state P is called *evolutionarily robust strategy* if, there exists an $\epsilon > 0$ such that for all $Q \neq P$, with $\pi(Q, P) < \epsilon$,

$$E(P, Q) > E(Q, Q).$$

We can verify that an \mathcal{ERS} is uninvadable. However, the converse is not true (Oechssler and Riedel 2002).

To study the evolutionary dynamics of population states more explicitly, the replicator dynamics is introduced (Oechssler and Riedel 2001, 2002) as described below. The success (or lack of success) of a strategy $z \in S$ against a strategy $w \in S$ is given by

$$\sigma(z, w) := u(z, w) - u(w, w).$$

The average success (or lack of success) of a strategy $z \in S$ against a population $Q \in \Delta$ is given by

$$\sigma(z, Q) := E(\delta_z, Q) - E(Q, Q).$$

The replicator dynamics is derived based on the idea that the relative increment in the frequency of strategies in a set $B \in \mathcal{B}$ is given by the average success of strategies in B . That is, for every $B \in \mathcal{B}$,

$$Q'(t)(B) = \frac{dQ(t)}{dt}(B) = \int_B \sigma(z, Q(t)) Q(t)(dz), \tag{2}$$

where $Q(t)$ denotes the population state at time t .

The replicator dynamics (2) can be also written as

$$Q'(t) = F(Q(t)), \tag{3}$$

where $F(Q(t))(B) = \int_B \sigma(z, Q(t)) Q(t)(dz)$; $B \in \mathcal{B}$ that is, $F(Q(t))$ is the signed measure whose Radon–Nikodym derivative $\frac{dF(Q(t))}{dQ(t)}$, w.r.t. $Q(t)$ is $\sigma(\cdot, Q(t))$.

Remark 1 We can show that, for every initial state $Q(0) \in \Delta$, there exists a unique solution $Q(t)$, lying in Δ , of the replicator dynamics (2) defined for all $t \geq 0$ (Oechssler and Riedel 2001).

Now, we recall a few dynamic stability definitions for population states.

Definition 4 Let P be a rest point of the replicator dynamics, i.e., $F(P) = 0$.

(a) P is called *weakly Lyapunov stable* if for all $\epsilon > 0$, there exists an $\eta > 0$ such that,

$$\pi(Q(0), P) < \eta \Rightarrow \pi(Q(t), P) < \epsilon \quad \text{for all } t > 0.$$

(b) P is called *weakly attracting* if there exists an $\eta > 0$ such that $Q(t)$ converges to P weakly, whenever $\pi(Q(0), P) < \eta$.

(c) P is called *weakly asymptotically stable* if it is weakly Lyapunov stable and weakly attracting.

In the next subsection we discuss the weak convergence of replicator dynamics trajectories.

2.2 Weak convergence

The weak stability of population states has been studied by Oechssler and Riedel (2002). They proved that in a doubly symmetric game, an $\mathcal{ER}\mathcal{S}$ population state is weakly Lyapunov stable. Moreover, they give sufficient conditions for a monomorphic $\mathcal{ER}\mathcal{S}$ to be asymptotically stable with respect to the weak topology in symmetric as well as doubly symmetric games. Cressman and Hofbauer (2005), established weak convergence of replicator trajectories to an $\mathcal{ER}\mathcal{S}$ which is weakly Lyapunov stable.

Cressman (2005) studied the stability of monomorphic and polymorphic population states with respect to the weak topology using neighbourhood superior and neighbourhood attracting strategies which are defined as follows.

Definition 5 (Cressman 2005)

(a) The population state $P^* \in \Delta$ with finite support (polymorphism) is *neighbourhood attracting* (with respect to (2)) if, for all initial $Q(0)$ with positive weight on each $x \in \text{supp}(P^*)$ and with support sufficiently close to the support of P^* , $Q(t)$ converges to P^* in the weak topology.

(b) The population state $P^* \in \Delta$ is *neighbourhood superior* if, for all other Q with support sufficiently close to support of P^* ,

$$E(P^*, Q) > E(Q, Q).$$

Cressman proved that a monomorphic population state δ_x is neighbourhood attracting when x strongly dominates every other strategy w in a two-strategy game between x and w for all w which are sufficiently close but not equal to x . Moreover, the dominance is determined by the second order Taylor expansion of the payoff function $u(z, w)$ about $z = w = x$. When the pure strategy space is a subset of the real line, the dominance is through the following conditions.

$$u_{11} < 0 \quad \text{and} \quad u_{11} + 2u_{12} < 0, \quad (4)$$

where u_{11} is the second order partial derivative of $u(z, w)$ evaluated at (x, x) with respect to the first variable and so on, assuming that u has continuous partial derivatives up to third order and x is in the interior of S . Both the conditions mentioned in (4) are essentially on the first partial derivative of u to ensure that the dominant strategy x is a rest point of the replicator dynamics. The following theorem summarizes this result.

Theorem 1 (Cressman 2005) *Suppose x is in the interior of $S \subset \mathbb{R}^n$ and the domination in the two-strategy game between x and w for all w sufficiently close but not equal to x is determined by the second order Taylor expansion of $u(z, w)$ about $z = w = x$. If x strongly dominates all such w , then δ_x is neighbourhood attracting. The converse is also true.*

The above theorem gives a necessary and sufficient condition for a monomorphism to be neighbourhood attracting, where as the following theorem gives a sufficient condition for a polymorphism (i.e., population state with finite support) to be neighbourhood attracting.

Theorem 2 (Cressman 2005) *If P^* is a neighbourhood superior polymorphism, then P^* is neighbourhood attracting.*

Cressman gives one more set of conditions for a polymorphic population state to be neighbourhood attracting based on the strategy domination of the points in the support of the polymorphism as stated in the following theorem.

Theorem 3 (Cressman 2005) *Suppose there are m strategies in the support of P^* and P^* is globally asymptotically stable equilibrium in the corresponding m -strategy (finite) game.*

- (a) *If, for all $x \in \text{supp}(P^*)$, x strongly dominates every other nearby w in the game whose strategy set S' is sufficiently close to support of P^* in Hausdroff distance, then P^* is neighbourhood attracting.*
- (b) *Suppose, for all $x \in \text{supp}(P^*)$ and nearby w , that x strongly dominates in the two-strategy game between x and w and that x weakly dominates w in the three-strategy game $\{x, w, z\}$ with z near $\text{supp}(P^*) \setminus \{x\}$. Then P^* is neighbourhood attracting.*

In Cressman et al. (2006), the authors introduce a similar concept of neighbourhood attraction for a set Q of probability measures whose support contains that of P^* . For a doubly symmetric game, it is proved that a neighbourhood superior monomorphism is attracting for the set Q .

In view of Example 3 in Cressman (2005), we discuss the following example which shows that a neighbourhood attracting polymorphism may not attract trajectories starting from population states having support away from that of the polymorphism.

Example 1 Consider the evolutionary game with $S = [-1, 1]$ and the payoff function defined by

$$u(z, w) = \begin{cases} -\frac{z^4}{2} + \frac{z^3}{3}w + z^2 - zw & \text{if } |z - w| \leq 1, \\ 0 & \text{if } |z - w| > 1. \end{cases}$$

We can easily verify that $u_{11} < 0$ and $u_{11} + 2u_{12} < 0$ at $z = w = \pm 1$. Thus, δ_{-1} and δ_1 are neighbourhood attracting monomorphisms by Theorem 1.

Consider the monomorphism δ_{-1} . Suppose we introduce a small weight on δ_1 while considering the initial population state $Q(0)$, that is we take the initial population state to be

$$Q(0) = \epsilon\delta_1 + (1 - \epsilon)\delta_{-1}.$$

The replicator dynamics trajectory, $Q(t)$ does not converge to δ_{-1} even though it is neighbourhood attracting but converges to the dimorphism

$$P^* = \frac{1}{2} (\delta_{-1} + \delta_1).$$

P^* is a rest point of replicator dynamics by Lemma 1 in Hingu et al. (2016b) or Corollary 1 in Hingu et al. (2016a) with $\sigma(-1, 1) = \frac{1}{6} = \sigma(1, -1)$ and it is easily verified that it is also neighbourhood superior. Hence by Theorem 2, P^* is neighbourhood attracting. This shows that a small mass introduced far away from δ_{-1} destabilizes it. Thus, it is important to restrict the support of the initial population state $Q(0)$ to be close to -1 . The same thing can be observed when we consider δ_1 in place of δ_{-1} .

The above example illustrates the role of the support of initial population state $Q(0)$ in determining the weak convergence of the trajectory $Q(t)$ to P^* . Almost all existing literature regarding weak convergence of replicator trajectories may be divided into two types. One type of results talk about the attracting property of polymorphisms. The other type of results discuss the same for non-polymorphic states and most of these results assume Lyapunov stability of P^* . There is still a need of a general result which discusses the weakly attracting property for a population state without weak Lyapunov stability assumption. In the next section we study weakly attracting property by introducing the concept of superiority with respect to a given set.

3 Superiority and weak stability

Let $P^* \in \Delta$ and F be a closed subset of S containing the support of P^* . Before defining the concept of superiority of P^* with respect to F , we let $\Lambda(P^*, F)$ denote the set of all probability measures Q , satisfying the following three conditions.

- (i) $\text{supp}(Q) \subseteq F$,
- (ii) $P^* \ll Q$; that is, P^* is absolutely continuous w.r.t. Q ,
- (iii) $\log \left(\frac{dP^*}{dQ} \right)$ is integrable with respect to P^* .

Definition 6 A population state $P^* \in \Delta$ is said to be superior with respect to F if for every $Q \in \Lambda(P^*, F)$, $Q \neq P^*$,

$$E(P^*, Q) > E(Q, Q).$$

We first give necessary and sufficient conditions for a polymorphism P^* to be superior with respect to its support.

Lemma 1 Let $P^* \in \Delta$ be a polymorphic population state given by $P^* = \sum_{j=1}^k \alpha_j \delta_{x_j}$ where x_j 's are distinct points in S and $\alpha_j > 0$ for $j = 1, 2, \dots, k$ with $\sum_{j=1}^k \alpha_j = 1$. Let $F = \{x_1, x_2, \dots, x_k\}$. Then P^* is superior w.r.t. F if and only if P^* is an interior ESS of the finite game with pure strategy set F .

Proof The proof follows from the fact that the set $\Lambda(P^*, F)$ is the mixed strategy simplex corresponding to the finite game with pure strategies x_1, x_2, \dots, x_k . □

We are now ready to present the main result of this paper.

Theorem 4 Let $P^* \in \Delta$ be a rest point of the replicator dynamics and let $F \subset S$ be closed. If P^* is superior with respect to F , then the replicator dynamics trajectory $Q(t)$ converges weakly to P^* whenever $Q(0) \in \Lambda(P^*, F)$.

Proof Assume that P^* is superior with respect to F . This implies that, for every $Q \in \Lambda(P^*, F)$, $Q \neq P^*$,

$$E(P^*, Q) > E(Q, Q).$$

The Kullback–Leibler distance $V : \Lambda \rightarrow \mathbb{R}$ is given by

$$V(Q) = \int \ln \left(\frac{dP^*}{dQ} \right) dP^*. \tag{5}$$

The well definiteness of the Kullback–Leibler distance follows from the conditions (ii) and (iii) in the definition of Λ .

Clearly, $V(Q) \geq 0$ and $V(Q) = 0$ if and only if $Q = P^*$ (Bomze 1990).

By Lemma 2 in Bomze (1990), the replicator dynamics trajectories $Q(t) \in \Lambda$ for all $t > 0$ whenever $Q(0) \in \Lambda$. It also follows that,

$$\frac{d}{dt}V(Q(t)) = -E(P^*, Q(t)) + E(Q(t), Q(t)). \tag{6}$$

Therefore, whenever $Q \in \Lambda$,

$$\dot{V}(Q) = -E(P^*, Q) + E(Q, Q) < 0,$$

since P^* is superior with respect to F . This implies that V is strictly decreasing along any given replicator trajectory and hence it converges to a limit, say L , greater than or equal to its minimum value attained at $Q = P^*$, i.e.,

$$0 \leq L = \lim_{t \rightarrow \infty} V(Q(t)).$$

Now, if possible let $Q^* \neq P^*$ be a ω -limit of $Q(t)$. Then there exists a sequence $\{t_k\} \rightarrow \infty$ such that $Q(t_k) \rightarrow Q^*$ weakly. From (6), we get

$$\left. \frac{d}{dt}V(Q(t)) \right|_{t=t_k} = -E(P^*, Q(t_k)) + E(Q(t_k), Q(t_k)). \tag{7}$$

Taking limit as $k \rightarrow \infty$, L.H.S. of (7) converges to zero since L is the limit of the decreasing function $V(Q(t))$, where as R.H.S. converges to $-E(P^*, Q^*) + E(Q^*, Q^*)$ which is less than zero since $Q^* \in \Lambda$, giving us a contradiction. Thus, P^* is the only ω -limit of $Q(t)$ and the result follows from the compactness of Δ in the weak topology. \square

We illustrate the above theorem with the following example.

Example 2 Let $S = [0, 2]$ and $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers with $\sum_{n=1}^\infty a_n = a < \infty$. Let $F = [0, \bar{z}]$ where $1 < \bar{z} < 2$, and let $f : [0, 2] \rightarrow \mathbb{R}$ be a continuous function satisfying $f \equiv 1$ on $[0, 1]$, $f < 1$ on $(1, \bar{z}]$ and $f(2) > 1 + 2a$. Define $P^* \in \Delta$ by $P^*(B) = \lambda(B \cap [0, 1])$, where λ is the Lebesgue measure on $[0, 1]$. We take g to be a continuous function on $[0, 2]$ and let

$$\langle g, P^* \rangle := \int_S g(z) P^*(dz).$$

We also consider $f_n(z) := \cos(n\pi z)$; $n = 1, 2, 3, \dots$ Now, the payoff function is defined as

$$u(z, w) = f(z) + g(w) - \sum_{n=1}^\infty a_n f_n(z) f_n(w).$$

Since $\langle f_n, P^* \rangle = 0$ and $\langle f, P^* \rangle = 1$, it follows that, for $Q \in \Delta$

$$E(P^*, Q) - E(Q, Q) = [1 - \langle f, Q \rangle] + \sum_{n=1}^\infty a_n \langle f_n, Q \rangle^2. \tag{8}$$

As $f < 1$ on $(1, \bar{z}]$, we have $1 - \langle f, Q \rangle \geq 0$ for every $Q \in \Lambda(P^*, F)$ with strict inequality when $[0, 1] \subsetneq \text{supp}(Q)$. Moreover, $\sum_{n=1}^\infty a_n \langle f_n, Q \rangle^2 \geq 0$ for every $Q \in \Lambda(P^*, F)$ with strict inequality when $\text{supp}(Q) = [0, 1]$, since $\sum_{n=1}^\infty a_n \langle f_n, Q \rangle^2 = 0$ implies that $\langle f_n, Q \rangle = 0 = \langle f_n, P^* \rangle$, for every n which in turn gives $P^* = Q$.

Therefore, we obtain the fact that P^* is superior w.r.t. F . By Theorem 4, P^* weakly attracts all replicator trajectories in $\Lambda(P^*, F)$.

To show that P^* is not weakly Lyapunov stable, consider $Q(0) = (1 - \epsilon_0)P^* + \epsilon_0\delta_2$ which is close to P^* when ϵ_0 is small. By Lemma 2 in Bomze (1990), the replicator trajectory $Q(t)$ is of the form,

$$Q(t) = (1 - \epsilon(t))P(t) + \epsilon(t)\delta_2,$$

where $\text{supp}(P(t)) = [0, 1]$. Putting $B = \{2\}$ in (2), we get

$$\epsilon'(t) = \epsilon(t) \sigma(2, Q(t)), \quad \epsilon(0) = \epsilon_0.$$

Note that,

$$\begin{aligned} \sigma(2, Q(t)) &= \epsilon(t)(1 - \epsilon(t))[u(2, 2) - E(P(t), \delta_2)] \\ &\quad + (1 - \epsilon(t))^2[E(\delta_2, P(t)) - E(P(t), P(t))]. \end{aligned}$$

It can be seen that

$$\begin{aligned} u(2, 2) - E(P(t), \delta_2) &= f(2) - 1 + \sum_{n=1}^{\infty} a_n[\langle f_n, P(t) \rangle - 1] \\ &\geq f(2) - 1 - 2a > 0. \end{aligned}$$

Similarly we can show that $E(\delta_2, P(t)) - E(P(t), P(t)) > 0$. Therefore, $\sigma(2, Q(t)) > 0$ for every $t > 0$. This implies that $Q(t)$ stays away from P^* weakly, and hence P^* is not weakly Lyapunov stable.

Remark 2 Note that, in Example 2, although P^* is not weakly Lyapunov stable w.r.t. $S = [0, 2]$, it is w.r.t. $F = [0, \bar{z}]$. This illustrates again the importance of restricting the support of Q to sets close to the support of P^* (as is also discussed for Example 1) when dealing with stability of population states.

4 Concluding remarks

In this paper, we introduced the notion of superiority of a population state P^* with respect to a given closed set F . The main result (Theorem 4), proves that such a superior population state P^* attracts the replicator dynamics trajectories $Q(t)$ weakly whenever, $Q(0) \in \Lambda(P^*, F)$.

Note that, P^* being neighbourhood superior (as in Cressman 2005) is equivalent to P^* being superior with respect to all F 's which are sufficiently close to support of P^* , when P^* is polymorphic. In view of this, our main result generalizes the weak stability of polymorphic states proved in (Cressman 2005, Theorem 2).

By considering $F = S$, our main result also includes (Shaiju and Bernhard 2009, Theorem 4.8), which says that a global \mathcal{ERS} rest point of replicator dynamics is weakly attracting. When we take $F = S$, the superiority property with respect to the set becomes nothing but the global \mathcal{ERS} property (see also Cressman and Hofbauer 2005).

To conclude, our main theorem (Theorem 4), unifies various weak stability results in the literature, and provides a more general perspective.

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