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S.I.: GAME THEORY AND OPTIMIZATION

Transboundary pollution control and environmental absorption efficiency management

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Abstract In this paper, we suggest a two-player differential game model of transboundary pollution that accounts for time-dependent environmental absorption efficiency, which allows the biosphere to switch from a carbon sink to a source. We investigate the impact of negative externalities resulting from a transboundary pollution non-cooperative game wherein countries are dynamically involved. Based on a linear-quadratic specification for the instantaneous revenue function, we assess differences related to both transient path and steady state between cooperative solution, open-loop and Markov perfect Nash equilibria (MPNE). Regarding the methodological contribution of the paper, we suggest a particular structure of the conjectured value function to solve MPNE problems with multiplicative interaction between state variables in one state equation, so that third-order terms that arise in the Hamilton–Jacobi—Bellman equation are made negligible. Using a collocation procedure, we confirm the validity of the particular structure of the conjectured value function. The results suggest unexpected contrasts in terms of pollution control and environmental absorption efficiency management: (i) in the long run, an open-loop Nash equilibrium (OLNE) allows equivalent emissions to the social optimum but requires greater restoration efforts; (ii) although an MPNE is likely to end up with lower emissions and greater restoration efforts than an OLNE, it has a much greater chance of falling in the emergency area; (iii) the absence of cooperation and or precommitment becomes more costly as the initial absorption efficiency decreases; (iv) more heavily discounted MPNE strategies are less robust than OLNE to prevent irreversible switching of the biosphere from a carbon sink to a source.

Keywords Transboundary pollution · Environmental absorption efficiency · Hamilton–Jacobi–Bellman approximation

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1 Introduction

Because natural CO_2 sinks absorb 55% of all anthropogenic carbon emissions, they represent "a huge subsidy to the global economy worth half a trillion US$ annually" (Canadell et al. 2007). However, the efficiency of natural sinks at absorbing CO_2 has decreased by 5% in the last 50 years, and this pattern is expected to continue (Canadell et al. 2007). One illustration of this trend is provided by the oceanic carbon sink, the decreased efficiency of which is believed to have caused a decline of up to 30% in the efficiency of the Southern Ocean sink over the last 20 years (Le Quéré et al. 2007). As consequences of this trend, greenhouse warming and the associated release of soil carbon favor a transition of the biosphere from a carbon sink to a source of pollution in the long run (Cox et al. 2000; Cramer et al. 2001; Joos et al. 2001; Lenton et al. 2006; Piao et al. 2008). According to Joos et al. (2001), "in the most extreme cases, the terrestrial biosphere becomes a source of carbon during the second half of the century."

In the absence of a transnational institution that can impose an environmental policy, and due to the lack of enforceable global coordination, it is likely that decentralized decision-making processes at the national level related to polluting emissions will aggravate the decline in the efficiency of carbon sinks.

With rare exceptions, the problem of declining environmental absorption efficiency has been disregarded in dynamic game models of transboundary pollution, which are typically based on the assumption of a linear environmental absorption function (e.g., Li 2014; Benchekroun and Martín-Herrán 2016; Fünfgelt and Schulze 2016; Huang et al. 2016). More recently, a nonlinear absorption function has been proposed to account for declining environmental absorption efficiency (Mäler et al. 2003; Kossioris et al. 2008). This formulation, which combines the linear negative influence of the pollution stock with nonlinear positive feedback related to the release of past accumulated pollution, involves multiple steady state equilibria with different stability properties. Although this change in perspective allows more accurate handling of potentially irreversible environmental degradation, it relies on the assumption of an instantaneous change in the absorption efficiency that precludes investing in a restoration effort whose impact is, indeed, not instantaneous.

Economic study of these issues is of prime importance because, just like emissions reduction, restoration of carbon sinks' efficiency is also a public good. With two superimposed free-rider problems, the failure of reduction policy might also impede restoration policy.

In this paper, we suggest a two-player differential game model of transboundary pollution that accounts for time-dependent environmental absorption efficiency, which allows switching of the biosphere from a carbon sink to a source, and the ability to invest in the restoration of environmental absorption efficiency. In this setup, we investigate the impact of negative externalities resulting from the transboundary pollution control non-cooperative game wherein countries are dynamically involved. To do so, we assess differences related to both transient and steady states between cooperative and non-cooperative pollution control and environmental absorption efficiency management.

To date, the theoretical debate on transboundary pollution control has led to the emergence of two main streams (cf. Jørgensen et al. 2010; Long 2010; Benchekroun and Long 2011). The first stream recommends that polluters commit to a predetermined plan of action over time whenever environmental absorption efficiency is linear (van der Ploeg and de Zeeuw 1991, 1992) or non-linear (Mäler et al. 2003; Kossioris et al. 2008). The second stream of literature prescribes that, in the context of linear environmental absorption efficiency, polluters use softly discounted, non-linear strategies to make decisions contingent on the current pollution level over time (Dockner and Long 1993).[1]

In practice, only voluntary "contributions" to global emissions reduction, rather than commitments, were agreed to by the countries attending the successive International Conferences on the Climate since the implementation of the Kyoto Protocol in 2005. A recent conference, which took place in Warsaw in November 2013, illustrated how difficult it is to reach a global, enforceable agreement on emissions reduction. It is therefore important to analyze whether the absence of commitment at an international level on pollution control is likely to aggravate the decline in the efficiency of carbon sinks.

In this perspective, we compare the outcomes related to an open-loop Nash equilibrium (OLNE), which reflects the situation where polluters commit to a predetermined plan of action, and a Markov perfect Nash equilibrium (MPNE), which corresponds to the case where polluters make decisions contingent on the current state of the biosphere (Dockner et al. 2000; Long 2010). To characterize an MPNE solution near its steady state, we use both a carefully designed local quadratic approximation, and, to get a comparison, a numerical approximation based on the collocation method for PDEs (Judd 1998; Doraszelski 2003). Setting the cooperative solution as a benchmark, we identify which decision rule, OLNE or MPNE, better prevents the durable or transient switching of the biosphere from a pollution sink to a source, if any. The results show that the players' transient behavior has a critical impact on the resulting steady state in terms of pollution stock and environmental absorption efficiency. More importantly, unexpected contrasts are found between the cooperative and non-cooperative steady states.

The paper proceeds as follows. Section 2 develops the differential game model and its properties. In Sects. 3 and 4, we derive the cooperative and open-loop Nash equilibria, respectively. In Sect. 5, we study the Markov perfect Nash equilibrium for a locally quadratic value function and evaluate its accuracy. In Sect. 6, we compare the results. Section 7 concludes the paper.

2 Model formulation

We extend the non-linear quadratic model in El Ouardighi et al. (2014) to a differential game of transboundary pollution. In the setup of a two-player game where players are, e.g., nations, we interpret the biosphere as the place of interaction between pollution and environmental absorption efficiency. The pollution stock is supposed to increase with anthropogenic emissions and to depreciate at an environmental absorption efficiency rate. The pollution stock has a destructive impact on absorption efficiency, which in turn increases the pollution stock. In contrast with the existing literature, we model environmental absorption efficiency as a state variable, which allows for negative absorption efficiency that would stimulate pollution even in the absence of anthropogenic emissions. This is in line with scientific evidence that the biosphere may switch from a carbon sink to a source (e.g., Cox et al. 2000; Canadell et al. 2007). To counteract such a switch, we consider two policy instruments: anthropogenic emissions and efforts to restore environmental absorption efficiency.

The rate of change of the pollution stock over time is written as:

$$\dot{P}(t) = \sum_i e_i(t) - A(t)P(t) \quad P(0) = P_0 > 0 \tag{1}$$

[1] This prescription, which results in multiple equilibria, is confirmed by Rowat (2006), moderated by Rubio and Casino (2002) and opposed by Wirl (2007).

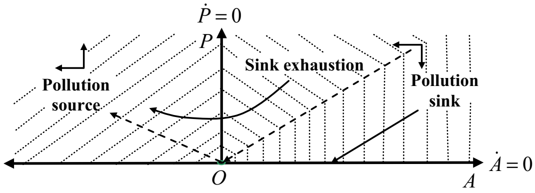

Fig. 1 Pollution-absorption dynamics without human interference

where $P(t) \geq 0$ denotes the pollution stock at time t, $e_i(t) \geq 0$, $i = 1, 2$, player i's current emissions, and $A(t)$ environmental absorption efficiency.[2]

Further, the rate of change of environmental absorption efficiency over time is:

$$\dot{A}(t) = \sum_i w_i(t) - \gamma P(t) \quad A(0) = A_0 \geq 0 \tag{2}$$

$w_i(t) \geq 0$ being player i's effort to restore environmental absorption efficiency, $i = 1, 2$, and $\gamma > 0$ the marginal destructive impact of pollution on the evolution of absorption efficiency. The efforts to restore environmental absorption efficiency may include reforestation (UNFCC 2008), and geo-engineering technologies (such as iron fertilization, biochar), which aim at accelerating natural processes of carbon removal (Vaughan and Lenton 2011).

The formulation in (1)–(2) includes the pollution accumulation model with linear absorption efficiency as a special case if environmental absorption efficiency is totally insensitive to the destructive impact of the pollution stock, $\gamma = 0$, and there is no restoration effort of environmental absorption efficiency, $w(t) = 0$, $\forall t$.[3] The formulation in (1)–(2) also extends the pollution accumulation model with zero absorption efficiency if environmental absorption efficiency is zero over the whole planning horizon, $A(t) = 0$, $\forall t$ (El Ouardighi et al. 2015b).

Assuming that there is no anthropogenic intervention (which corresponds to the pre-industrial era), over an infinite horizon, i.e., $e_i(t) = w_i(t) = 0$, $\forall t \geq 0$, $i = 1, 2$, the properties of the system (1)–(2) can be described in the phase diagram below. The system admits infinite steady states on the isocline $\dot{A} = 0$. The eigenvalues associated with such system are $-A_\infty$ and 0, which means that the positive horizontal axis is a sink, and the negative horizontal axis is a source.

Based on the steady state $(A_\infty, P_\infty) = (0, 0)$, the phase diagram can be divided into three regions:

- A sustainability area, where the initial environmental absorption capacity acts as a sink for any pollution stock starting on the right side of the (dashed) saddle path.
- An emergency area, where any path starting on the left side of the (dashed) saddle path and the right side of the $\dot{P} = 0$ isocline, though declining, tends to exhaust the pollution sink.
- An irreversible area (the so-called *Soylent green scenario*), on the left side of the $\dot{P} = 0$ isocline, where the environment becomes a source of formerly stored pollution (Fig. 1).

The phase diagram shows that a large initial pollution stock combined with sufficiently low initial environmental absorption efficiency is a sufficient condition for switching from a pollution sink to a source.

[2] By an application of Gronwall's lemma to the differential inequality $\dot{P}(t) \geq -A(t)P(t)$ which is implied by (1), one obtains, indeed, $P(t) \geq 0$, $\forall t$..

[3] For more details, see El Ouardighi et al. (2014).

Regarding the objective functions, we assume that each player seeks to maximize its intertemporal utility, U_i. In this regard, we make the following assumptions. First, each player's instantaneous revenue is a concave function of its own current rate of emissions, $e_i(t)\left(a_i - e_i(t)/2\right)$, where $2a_i > 0$ is an upper bound for emissions, $i = 1, 2$. This choice comes from the fact that the quadratic revenue function has been often used in dynamic game models with a linear absorption function (Rubio and Casino 2002; Wirl 2007).[4] The instantaneous social cost incurred by each player due to the pollution stock is a quadratic function, $c_i P(t)^2/2$, where $c_i > 0$ is a pollution cost coefficient, $i = 1, 2$ (Rubio and Casino 2002; Wirl 2007; Boucekkine et al. 2013). Finally, each player incurs an instantaneous increasing convex cost of effort to restore environmental absorption efficiency, $f_i w_i(t)^2/2$, where $f_i > 0$ is a restoration cost coefficient, $i = 1, 2$ (El Ouardighi et al. 2014; El Ouardighi et al. 2015a).

Letting $r_i > 0$ denote player i's discounting rate, and assuming an infinite planning horizon, player i's objective criterion is written as:

$$\underset{e_i(.),w_i(.)}{\text{Max}}\ U_i = \int_0^{+\infty} e^{-r_i t}\left[e_i(t)\left(a_i - e_i(t)/2\right) - c_i P(t)^2/2 - f_i w_i(t)^2/2\right]\mathrm{d}t \tag{3}$$

subject to (1)–(2), $e_i(t) \geq 0$ and $w_i(t) \geq 0$, $\forall t$, $i = 1, 2$. In the remainder of the paper, we set $f_i = 1$, $i = 1, 2$, without loss of generality. Further, invoking the symmetry assumption, which is standard in the literature on pollution control (e.g., Jørgensen et al. 2010; Long 2010), we use: $a_i \equiv a$, $c_i \equiv c$, and $r_i \equiv r$, $i = 1, 2$.

3 Cooperative solution

Cooperative solution strategies require that the players align their respective interests and maximize the overall utility over the planning horizon, that is:

$$U = \int_0^{+\infty} e^{-rt}\left[\sum\nolimits_i e_i(t)\left(a - e_i(t)/2\right) - cP(t)^2 - \sum\nolimits_i w_i(t)^2/2\right]\mathrm{d}t \tag{4}$$

Accordingly, the current-value Hamiltonian is written as:

$$H = \sum\nolimits_i e_i\left(a - e_i/2\right) - cP^2 - \sum\nolimits_i w_i^2/2 + \eta_1\left(\sum\nolimits_i e_i - AP\right) + \eta_2\left(\sum\nolimits_i w_i - \gamma P\right) \tag{5}$$

where $\eta_j \equiv \eta_j(t)$ are (current value) costate variables, $j = 1, 2$.

Assuming an interior solution and applying Pontryagin's principle in the discounted case, the necessary conditions for cooperative solution are:

$$H_{e_i} = a - e_i + \eta_1 = 0 \Rightarrow e_i = a + \eta_1 \tag{6}$$

$$H_{w_i} = -w_i + \eta_2 = 0 \Rightarrow w_i = \eta_2 \tag{7}$$

$i = 1, 2$, where the evolution of the costate variables is given by:

$$\dot{\eta}_1 = 2cP + (r + A)\,\eta_1 + \gamma\eta_2 \tag{8}$$

[4] Unlike most game studies with nonlinear absorption function, our study does not use a logarithmic revenue function (Mäler et al. 2003; Kossioris et al. 2008; Dockner and Wagener 2014). Note that a logarithmic function provides quite similar results to those obtained with our quadratic function.

$$\dot{\eta}_2 = r\eta_2 + \eta_1 P \tag{9}$$

Equations (8) and (9) confirm that the players' equilibrium strategies are symmetric.
Using (6)–(7) in (1)–(2), we obtain:

$$\dot{P} = 2(a + \eta_1) - AP \tag{10}$$

$$\dot{A} = 2\eta_2 - \gamma P \tag{11}$$

Let us assume a steady state to study optimal control at this state. Assuming that $\dot{P} = \dot{A} = 0$ gives $\sum_i e_i = AP$ and $\sum_i w_i = \gamma P$. Differentiating these equations with respect to time and accounting for $\dot{P} = \dot{A} = 0$ yields steady efforts $\dot{e}_i = \dot{w}_i = 0$ $i = 1,\ 2$. Differentiating (6)–(7) over any time interval and accounting for $\dot{e}_i = \dot{w}_i = 0, i = 1,\ 2$, the costate variables are time invariant. This implies that if the pollution stock and the absorption efficiency are both steady, then the related costate variables are also steady, and each player performs time-invariant efforts.

Proposition 1 *There exists a non-trivial cooperative steady state that is unique, given by:*

$$[P_\infty \quad A_\infty \quad e_{i\infty} \quad w_{i\infty}]^T = \left[\frac{r^2\gamma + h}{2\left(4c + \gamma^2\right)} \quad \frac{2(2a - r\gamma)\left(4c + \gamma^2\right)}{r^2\gamma + h} \quad \frac{2a - r\gamma}{2} \quad \frac{\gamma\left(r^2\gamma + h\right)}{4\left(4c + \gamma^2\right)} \right]^T \tag{12}$$

where $h \equiv \sqrt{r^4\gamma^2 + 4r\gamma(2a - r\gamma)\left(4c + \gamma^2\right)} > 0, i = 1, 2.$

Proof See "Appendix A1".

The cooperative solution allows for a feasible steady state emissions rate compatible with the preservation of a strictly positive environmental absorption rate if $a > r\gamma/2$. In (12), we observe that a lower discounting rate should result in a higher steady state emissions rate $e_{i\infty}$ and environmental absorption efficiency A_∞ as well as lower pollution stock P_∞ and restoration efforts $w_{i\infty}$. Note that there exists a trivial steady state solution characterized by zero emissions, pollution stock and restoration efforts and negative absorption efficiency, i.e., $\left[P'_\infty \quad A'_\infty \quad e'_{i\infty} \quad w'_{i\infty}\right]^T = [0 \quad -r \quad 0 \quad 0]^T$. This solution is different from the irreversible solution found in Tahvonen and Withagen (1996), which implies 'high' pollution concentration that can be optimal. In our model, this second steady state is unstable and therefore cannot be optimal.

Corollary 1 *At the steady state, the instantaneous cooperative utility is:*

$$u_\infty = \frac{4a^2 - r^2\gamma^2}{8} - \frac{\left(r^2\gamma + h\right)^2}{16\left(4c + \gamma^2\right)} \tag{13}$$

Proof See "Appendix A2".

The expression in (13) characterizes the utility of future generations in the cooperative case. It can be shown that $\partial u_\infty / \partial r < 0$, which implies that a larger discounting rate results in a lower utility of future generations under cooperation. Similarly, as $\partial u_\infty / \partial \gamma < 0$, more vulnerable environmental absorption efficiency negatively affects the utility of future generations.

Proposition 2 *The cooperative steady state is a saddle-point and the optimal path to the steady state is either monotonic or oscillatory.*

Proof See "Appendix A3".

That is, the saddle-path can either converge monotonically or oscillate toward the steady state. This result is consistent with previous findings where a clockwise spiraling path can go through the *Soylent Green area*, with negative environmental absorption efficiency during a finite time interval (El Ouardighi et al. 2014; El Ouardighi et al. 2015a).

Lemma 1 *The saddle paths of the control and state variables near the cooperative steady state are:*

$$e_i(t) = a + \eta_{1\infty} + B_1 e^{\chi_1 t} + B_2 e^{\chi_3 t} \tag{14}$$

$$w_i(t) = w_{i\infty} + B_3 e^{\chi_1 t} + B_4 e^{\chi_3 t} \tag{15}$$

$$P(t) = P_\infty + B_5 e^{\chi_1 t} + B_6 e^{\chi_3 t} \tag{16}$$

$$A(t) = A_\infty + B_7 e^{\chi_1 t} + B_8 e^{\chi_3 t} \tag{17}$$

where $B_1, \ldots, B_8$ are constants of integration and χ_1, χ_3 are eigenvalues with negative real part.

Proof See "Appendix A4".

Equations (14)–(17) are useful to characterize the transient paths for the control and state variables in the cooperative solution.

4 Open-loop Nash equilibrium

An analysis of the OLNE strategy, which can be justified whenever the players are capable of precommitting themselves to announced time paths, offers valuable insights into the impact of negative externalities resulting from the transboundary non-cooperative game of pollution control and environmental absorption efficiency management. In addition, open-loop strategies provide a useful benchmark for subsequent assessment of the strategic effects related to Markovian strategies (Fudenberg and Levine 1988).

Using the objective function in (3), the current-value Hamiltonian of player i is written as:

$$H^i = e_i\left(a - e_i/2\right) - cP^2/2 - w_i^2/2 + \lambda_{i1}\left(e_1 + e_2 - AP\right) + \lambda_{i2}\left(w_1 + w_2 - \gamma P\right) \tag{18}$$

where $\lambda_{ij} \equiv \lambda_{ij}(t)$ are player i's (current value) costate variables, $i = 1, 2$, $j = 1, 2$.

Assuming an interior solution and applying Pontryagin's principle in the discounted case, player i's necessary conditions for OLNE are:

$$H^i_{e_i} = a - e_i + \lambda_{i1} = 0 \Rightarrow e_i^{on} = a + \lambda_{i1} \tag{19}$$

$$H^i_{w_i} = -w_i + \lambda_{i2} = 0 \Rightarrow w_i^{on} = \lambda_{i2} \tag{20}$$

$i = 1, 2$, where the superscript "*on*" stands for open-loop Nash equilibrium.

The evolution of the costate variables is given by:

$$\dot{\lambda}_{i1} = cP + (r + A)\lambda_{i1} + \gamma\lambda_{i2} \tag{21}$$

$$\dot{\lambda}_{i2} = r\lambda_{i2} + \lambda_{i1}P \tag{22}$$

In the present case, the costate variables of both players coincide; that is, $\lambda_{i1} \equiv \lambda_1$ and $\lambda_{i2} \equiv \lambda_2$, $i = 1, 2$, and Eqs. (21) and (22) confirm that the players' equilibrium strategies are symmetric. Therefore, we are left with only two costate equations.

Using (21)–(22), (1)–(2) we obtain:

$$\dot{P} = 2(a + \lambda_{i1}) - AP \tag{23}$$

$$\dot{A} = 2\lambda_2 - \gamma P \tag{24}$$

Note that here also, if the pollution stock and the absorption efficiency are both steady, then each player performs time-invariant control efforts.

Proposition 3 *There exists a non-trivial OLNE steady state that is unique, given by:*

$$\begin{bmatrix} P_{\infty}^{on} & A_{\infty}^{on} & e_{i\infty}^{on} & w_{i\infty}^{on} \end{bmatrix}^T = \begin{bmatrix} \frac{r^2\gamma + k}{2(2c+\gamma^2)} & \frac{2(2a - r\gamma)(2c+\gamma^2)}{r^2\gamma + k} & \frac{2a - r\gamma}{2} & \frac{\gamma(r^2\gamma + k)}{4(2c+\gamma^2)} \end{bmatrix}^T \tag{25}$$

where $k \equiv \sqrt{r^4\gamma^2 + 4r\gamma(2a - r\gamma)(2c+\gamma^2)} > 0, i = 1, 2.$

Proof See "Appendix A5".

Therefore, precommitment strategies also allow a feasible steady state emissions rate compatible with the preservation of a strictly positive environmental absorption rate under the feasibility condition $a > r\gamma/2$. In (25), a lower discounting rate also increases the steady state emissions rate $e_{i\infty}^{on}$ and environmental absorption efficiency A_{∞}^{on}, and lowers the pollution stock P_{∞}^{on} and restoration efforts $w_{i\infty}^{on}$. Here also, there exists a trivial steady state solution such that $\begin{bmatrix} P_{\infty}^{\prime on} & A_{\infty}^{\prime on} & e_{i\infty}^{\prime on} & w_{i\infty}^{\prime on} \end{bmatrix}^T = [0 \quad -r \quad 0 \quad 0]^T$, which is diverging and therefore not optimal.

Corollary 2 *At the steady state, player i's OLNE instantaneous utility is:*

$$u_{i\infty}^{on} = \frac{4a^2 - r^2\gamma^2}{8} - \frac{(4c+\gamma^2)(r^2\gamma + k)^2}{32(2c+\gamma^2)^2} \tag{26}$$

Proof See "Appendix A6".

The expression in (26) represents the utility of future generations from the perspective of player i in the OLNE case. It has the same properties as in the cooperative case regarding the influence of the discounting rate, r, and the marginal impact of pollution on the evolution of absorption efficiency, γ.

Proposition 4 *The OLNE steady state has the saddle-point property and the optimal path to the steady state is either monotonic or oscillatory.*

Proof See "Appendix A7".

As in the cooperative setting, the saddle-path can either converge monotonically or follow an oscillatory trajectory toward the steady state.

Lemma 2 The saddle paths of the control and state variables near the OLNE steady state are:

$$e_i^{on}(t) = a + \lambda_{1\infty} + D_1 e^{\xi_1 t} + D_2 e^{\xi_3 t} \tag{27}$$

$$w_i^{on}(t) = w_{i\infty}^{on} + D_3 e^{\xi_1 t} + D_4 e^{\xi_3 t} \tag{28}$$

$$P^{on}(t) = P^{on}_{\infty} + D_5 e^{\xi_1 t} + D_6 e^{\xi_3 t} \tag{29}$$

$$A^{on}(t) = A^{on}_{\infty} + D_7 e^{\xi_1 t} + D_8 e^{\xi_3 t} \tag{30}$$

where $D_1, \ldots, D_8$ are constants of integration and ξ_1, ξ_3 are eigenvalues with negative real part.

Proof See "Appendix A8".

Equations (27)–(30) enable the computation of transient paths for the control and state variables in an OLNE.

5 Markov perfect Nash equilibrium

The MPNE has the desirable property of subgame perfectness because current actions depend on the current state vector and time, and not only on time, as OLNE does. However, the non-linear quadratic structure of the game makes the search for an analytical solution of MPNE more difficult than for usual linear quadratic models. The main reason for this is that the two state variables interact multiplicatively in one state equation, which involves intricate third-order terms in the resolution process. Thus, the design of an approximation for the MPNE requires particular attention.

Using the objective function in (3), the Hamilton–Jacobi–Bellman (HJB) equations are:

$$rV^i = \underset{e_i, w_i}{\text{Max}} \left[e_i \left(a - e_i / 2 \right) - cP^2/2 - w_i^2/2 + V^i_P \left(e_1 + e_2 - AP \right) + V^i_A \left(w_1 + w_2 - \gamma P \right) \right] \tag{31}$$

where $V^i(A, P)$ is player i's value function, $i = 1, 2$.

Player i's necessary conditions for Markov perfect Nash equilibrium in the case of an interior solution are written as:

$$e_i^{mp} = a + V^i_P(A, P) \tag{32}$$

$$w_i^{mp} = V^i_A(A, P) \tag{33}$$

After substituting the necessary conditions of optimality into the HJB equations, we obtain:

$$rV^i = \left(a + V^i_P \right)^2 \Big/ 2 - cP^2/2 + \left(V^i_A \right)^2 \Big/ 2 + V^i_P \left(V^j_P - AP \right) + V^i_A \left(V^j_A - \gamma P \right) + aV^i_P \tag{34}$$

$i, j = 1, 2,\ \ i \neq j$. Invoking the symmetry assumption, we write: $V^i = V$, $V^i_P = V_P$ and $V^i_A = V_A$, $i = 1, 2$, which leads us to rewrite (34) as follows:

$$rV^i = a^2/2 + \left(3V_P/2 + 2a \right) V_P + 3\left(V_A \right)^2/2 - cP^2/2 - \gamma V_A P - V_P AP \tag{35}$$

In linear-quadratic transboundary pollution models, a quadratic approximation of the value function for the evaluation of the MPNE is usually considered (see, e.g., Long 2010, Section 1.2.4). In the context of our non-linear quadratic model, assuming a local quadratic approximation of the form, with real parameters $\alpha_1, .., \alpha_6$, that is:

$$V = \alpha_1 + \alpha_2 A + \alpha_3 P + \alpha_4 A^2 + \alpha_5 P^2 + \alpha_6 AP$$

may however not be relevant because it would lead to inclusion of two third-order terms, respectively proportional to $A^2 P$ and AP^2, in the approximation of the HJB Eq. (35), that are likely non-negligible near the steady state. So, some care is needed to design the local quadratic approximation, to make such third-order terms locally negligible.

Assumption 1 We consider for the value function a local quadratic approximation with real parameters $\alpha_1, .., \alpha_6$, that is:

$$V = \alpha_1 + \alpha_2 \left(A - A_\infty^{mp}\right) + \alpha_3 \left(P - P_\infty^{mp}\right) + \alpha_4 \left(A - A_\infty^{mp}\right)^2 + \alpha_5 \left(P - P_\infty^{mp}\right)^2 + \alpha_6 \left(A - A_\infty^{mp}\right)\left(P - P_\infty^{mp}\right) \tag{36}$$

where A_∞^{mp} and P_∞^{mp} are the steady state values (to be determined) of A and P for the MPNE. We assume that the parameters of the problem are such that the steady state values of all the state and control variables are positive. Note that the steady state utility, or social welfare of future generations, is approximated by $r\alpha_1$.

By substituting the expression in (36) and its respective partial derivatives in (35), two third-order terms, respectively proportional to $\left(A - A_\infty^{mp}\right)^2\left(P - P_\infty^{mp}\right)$ and $\left(A - A_\infty^{mp}\right)\left(P - P_\infty^{mp}\right)^2$, appear in the approximation of the HJB equation, that are clearly negligible near the steady state and can therefore be neglected. Though this is sufficient to ensure the accuracy of our local quadratic approximation in (36), additional justifications based on a numerical procedure are provided hereafter; they compare this local approximation with the one obtained by a different procedure.

Proposition 5 *Under Assumption 1, the MPNE strategies near the steady state are given by:*

$$e^{mp} = a + \alpha_3 + 2\alpha_5 \left(P - P_\infty^{mp}\right) + \alpha_6 \left(A - A_\infty^{mp}\right) \tag{37}$$

$$w^{mp} = \alpha_2 + 2\alpha_4 \left(A - A_\infty^{mp}\right) + \alpha_6 \left(P - P_\infty^{mp}\right) \tag{38}$$

and the MPNE steady state is given by:

$$\left[P_\infty^{mp}, A_\infty^{mp}, e_\infty^{mp}, w_\infty^{mp}\right]^T = \left[2\alpha_2/\gamma, \gamma\left(a + \alpha_3\right)/\alpha_2, a + \alpha_3, \alpha_2\right]^T \tag{39}$$

Proof See "Appendix A9".

From (38) and (39), we note that both emissions and restoration effort MPNE strategies are locally linear. However, $\alpha_2 > 0$ and $-a < \alpha_3 < 0$ are required for a feasible steady state solution. The steady state solution involved by (39), if any, should be globally asymptotically stable for the linearized dynamical system presented in the proof of Proposition 5 (see "Appendix A9"). The Jacobian matrix associated with the MPNE controlled system is:

$$J^{mp} = \begin{bmatrix} 4\alpha_5 - A_\infty^{mp} & 2\alpha_6 - P_\infty^{mp} \\ 2\alpha_6 - \gamma & 4\alpha_4 \end{bmatrix} \tag{40}$$

Using (39), we get the trace and determinant of the Jacobian matrix, that is, respectively:

$$Tr\left(J^{mp}\right) = 4\left(\alpha_4 + \alpha_5\right) - \frac{\gamma\left(a + \alpha_3\right)}{\alpha_2}, \quad \left|J^{mp}\right| = 2\left\{2\alpha_4\left[4\alpha_5 - \frac{\gamma\left(a + \alpha_3\right)}{\alpha_2}\right] - \left(2\alpha_6 - \gamma\right)\left(\alpha_6 - \frac{\alpha_2}{\gamma}\right)\right\}$$

Table 1 Coefficient values of the quadratic approximation of the value function for the higher equilibrium

α_1	α_2	α_3	α_4	α_5	α_6
$-2.321435 \cdot 10^{-5}$	$7.200114 \cdot 10^{-4}$	$-3.151574 \cdot 10^{-3}$	$-1.602299 \cdot 10^{-3}$	$-2.798305 \cdot 10^{-2}$	$1.093377 \cdot 10^{-2}$

The global asymptotic stability property imposes that $Tr\left(J^{mp}\right) < 0$ and $\left|J^{mp}\right| > 0$, and that both the eigenvalues $\psi_{1,2}$ of the Jacobian matrix, that is:

$$\psi_{1,2} = \frac{\gamma\left[4\alpha_2\left(\alpha_4+\alpha_5\right)-\gamma\left(a+\alpha_3\right)\right] \pm \sqrt{\gamma^2\left[4\alpha_2\left(\alpha_4-\alpha_5\right)+\gamma\left(a+\alpha_3\right)\right]^2+8\gamma\alpha_2^2\left(2\alpha_6-\gamma\right)\left(\gamma\alpha_6-\alpha_2\right)}}{2\gamma\alpha_2}$$

have negative real parts.

Because our feedback solution is approximated around a small neighborhood of the steady state, convergence to the actual solution, if it exists, should be quite satisfactory. To confirm the accuracy of the quadratic model adopted in Assumption 1, we also use an alternative approach based on the collocation method for PDEs (e.g., Judd 1998; Doraszelski 2003; Dawid et al. 2015; Jaakkola 2015) to compute the MPNE strategies numerically (see "Appendix A10"). For the following constellation of parameters, that is:

$$r = 0.075, \quad \gamma = 0.05, \quad a = 0.006, \quad c = 0.035,$$

we used the approach described in "Appendix A9" and identified two MPNE strategies, each of them resulting into a globally asymptotically stable steady state for the associated linearized system. We selected the most profitable Nash equilibrium strategy according to the objective criterion in (3). The selected MPNE is a more profitable (higher) equilibrium because it is characterized by greater environmental absorption efficiency and lower pollution stock at the steady state than the less profitable (lower) MPNE. Note that, in practice, convergence to the higher equilibrium is not granted, as the lower equilibrium may well arise as a poverty-trap (e.g., Azariadis and Stachurski 2005) if the players are unable to coordinate. Therefore, a coordination device on the higher equilibrium is needed to escape from the poverty-trap and therefore allow for an increase in welfare to both players. The coefficient values of the local quadratic approximation of the value function for the higher equilibrium are given in Table 1.

The corresponding steady state is given by:

$$\begin{aligned}&\left(P_\infty^{mp}, A_\infty^{mp}, e_\infty^{mp}, w_\infty^{mp}\right)\\&\quad = \left(2.880045 \cdot 10^{-2}, 1.978042 \cdot 10^{-1}, 2.848426 \cdot 10^{-3}, 7.200114 \cdot 10^{-4}\right)\end{aligned}$$

along with the corresponding negative eigenvalues:

$$\left(\psi_1, \psi_2\right) = \left(-3.103781 \cdot 10^{-1}, -5.767552 \cdot 10^{-3}\right)$$

By numerically comparing the local quadratic approximation of the value function on a neighborhood of the steady state solution with its local approximation obtained using the collocation method on the same neighborhood (see "Appendix A10") with *Matlab*, a very small difference between the two approximations is obtained, as shown in Fig. 2.

Figure 3 compares the two approximations in terms of the relative error of the quadratic approximation with respect to the one obtained by the collocation method. Such a relative error is defined as the absolute value of their difference, divided by the average on the neighborhood of the absolute value of the approximation obtained by the collocation method

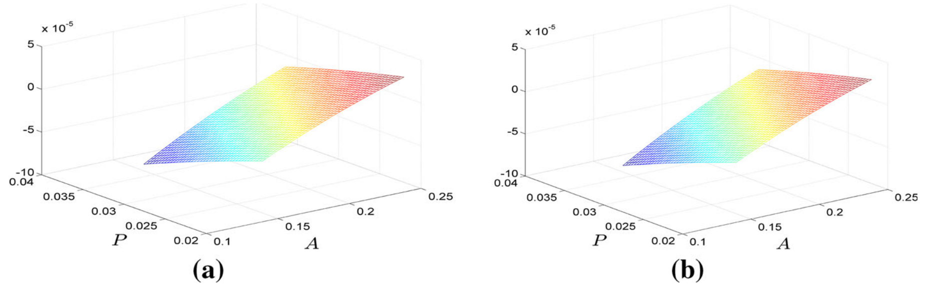

Fig. 2 Compared approximations of the value function with local quadratic conjecture and collocation method. **a** Local quadratic conjecture **b** collocation method

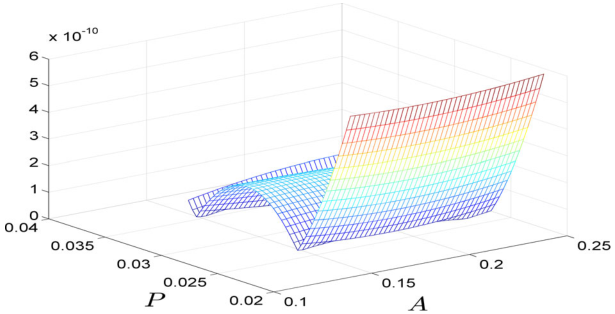

Fig. 3 Relative error of the local quadratic approximation of the value function with respect to the local approximation obtained with the collocation method

(the average of the absolute value is introduced to avoid an infinite relative error in case of zero crossings of the approximation). As one can see from the figure, a negligible relative error is obtained, which suggests that the initial conjectured quadratic approximation was so good that only a negligible update was performed by the collocation method to reach the default tolerance.[5]

For the local quadratic approximation, Fig. 4 shows the Bellman residual error, defined as:

$$rV - \left[a^2/2 + (3V_P/2 + 2a)\,V_P + 3\,(V_A)^2/2 - cP^2/2 - \gamma V_A P - V_P A P\right]$$

and provides a measure of the quality of the approximation, as it expresses the pointwise violation of the HJB Eq. (35). The figure shows a very small Bellman residual error. Moreover, for the local quadratic approximation, when it is computed in $(A_\infty^{mp}, P_\infty^{mp})$, the absolute value of the ratio between the Bellman residual error and the left-hand side of the HJB Eq. (35) is $3.4766 \cdot 10^{-7}$. Similar results hold for the approximation by the collocation method.

Finally, for the same choice of the problem parameters, we compare the value α_1 assumed in $(A_\infty^{mp}, P_\infty^{mp})$ by the local quadratic approximation of the value function, with the expression:

$$\left[e_\infty^{mp}\left(a - e_\infty^{mp}/2\right) - c\left(P_\infty^{mp}\right)^2/2 - \left(w_\infty^{mp}\right)^2/2\right]\Big/\, r$$

that is obtained by replacing in (3) the resulting approximation of the steady state solution for the MPNE: the two values so obtained ($-2.321435 \cdot 10^{-5}$ and $-2.321434 \cdot 10^{-5}$, respec-

[5] The result depends partly on the neighborhood size. For larger sizes, a larger relative error is usually obtained.

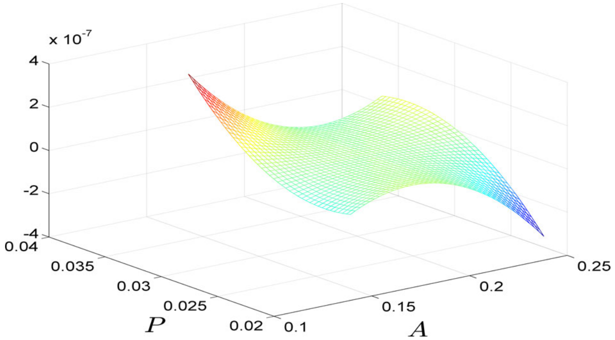

Fig. 4 Bellman residual error for the local quadratic approximation of the value function

tively) are practically coincident. The results obtained clearly support the validity of the local quadratic approximation in Assumption 1.[6]

6 Comparative analysis

In this section, we determine the main differences between the cooperative, OLNE and approximated MPNE strategies in terms of steady state and transient path. To do so, we use both analytical and numerical results.

Based on our analytical results for the cooperative and OLNE cases, we first set some important comparisons.

Proposition 6 It holds that:

$$e_{i\infty}^{on} = e_{i\infty} \tag{41}$$

$$P_{\infty}^{on} > P_{\infty} \tag{42}$$

$$w_{i\infty}^{on} > w_{i\infty} \tag{43}$$

$$A_{\infty}^{on} < A_{\infty} \tag{44}$$

$$\sum_i u_{i\infty}^{on} < u_{\infty} \tag{45}$$

Proof See "Appendix A10".

That is, the steady state global emissions in the cooperative and the OLNE settings are equal. The same conclusion can be obtained with a logarithmic revenue function of current rate of emissions, that is, $a_i \ln e_i(t)$, $a_i > 0$, $i = 1, 2$, and with a linear pollution cost, that is, $cP(t)$, which makes it a robust result in that it is driven neither by the functional form of revenue function nor by that of pollution costs. Actually, this result is due to the particular transition equations assumed, which allow switching of the biosphere from a carbon sink to a source, and the ability to invest in the restoration of environmental absorption efficiency. The main implication of (41) is that the OLNE steady state imposes no sacrifice on future

6 Figures 2, 3 and 4, as the final check of the validity of the local quadratic approximation, refer to a specific choice of the parameter values inside the ranges considered in Table 1, for which it was possible to find a solution of the non-linear system in **Proposition 7 that also satisfies the feasibility requirements provided in **Appendix A9. However, the robustness of our approach was also confirmed by the results obtained for other choices of the parameter values inside such ranges, for which a feasible solution was obtained.

Table 2 Base case and range values

Parameter	r	γ	a	c
Base case	0.05	0.05	0.006	0.01
Range	[0.05, 0.1]	[0.05, 0.1]	[0.001, 0.006]	[0.01, 0.06]

generations in terms of emissions compared with the cooperative steady state. This result departs from other transboundary pollution game models in which environmental absorption efficiency is represented as an instantaneous function of the pollution stock in that these models suggest that the steady state emissions in the cooperative setting should be lower than in an OLNE.

On the other hand, the result in (42) states that the steady state pollution is lower in the cooperative solution than in the OLNE. Intuitively, the difference between the two outcomes results from the fact that the players in the OLNE game do not take into account the *social* cost of a marginal increment in the pollution stock as in the cooperative game, but only their *private* marginal user cost. The result in (43) indicates that, compared with the cooperative solution, an OLNE leads each player to over-invest in environmental absorption efficiency. Note that the dominance of the OLNE investment in absorption efficiency over the cooperative investment effort refers to the steady state, which by definition is an asymptotic outcome where player i's investment is needed only to make up for the destructive impact of the pollution stock on environmental absorption efficiency. Relatedly, in (44), steady state absorption efficiency is greater in the cooperative case than in the OLNE case. The intuition behind this result is that over-investment in absorption efficiency involved by the OLNE solution at the steady state compared to the cooperative solution does not suffice to negate the impact of the excess emissions during the transient path. This point will be checked on a numerical basis below. At the steady state, cooperative and OLNE emissions are equivalent, which implies equivalent revenues. However, the non-cooperative steady state involves not only a greater cost of pollution than the cooperative setting but also a greater cost of investment effort in environmental absorption efficiency. Therefore, future generations are better off if the players decide to cooperate rather than to precommit, as shown in (45).

Similar analytical comparisons involving the MPNE case are unavailable. To compare the cooperative, OLNE and MPNE solutions on a numerical basis, we select the base case parameter values in Table 2.

The base case in Table 2 reflects a favorable configuration which is characterized *in relative terms* by players' patience (i.e., low discounting rate, r), strong environmental resilience (i.e., low destructive impact of pollution on absorption efficiency, γ), high marginal incentive for emissions (i.e., high upper bound on emissions, $2a$) and low marginal pollution cost (i.e., low pollution cost function coefficient, c). The numerical solutions of the cooperative, OLNE and MPNE cases were computed with *Maple* 18.0.

To assess the sensitivity of the steady states, a broad range of values were used for all parameters (Table 2). In the MPNE case, the solutions were calculated for 10,000 combinations of the parameters with 10 values for each parameter r, γ, a, c. Among the 10,000 trials, 141 combinations of parameters gave rise to a single locally asymptotically stable solution and 162 combinations resulted in two locally asymptotically stable solutions. In the remaining 9697 combinations of parameters, no MPNE stable solution could be found.

We proceed in four steps. We start by identifying the stable steady states respectively associated with the cooperative, OLNE and MPNE solutions in the base case. Then we assess

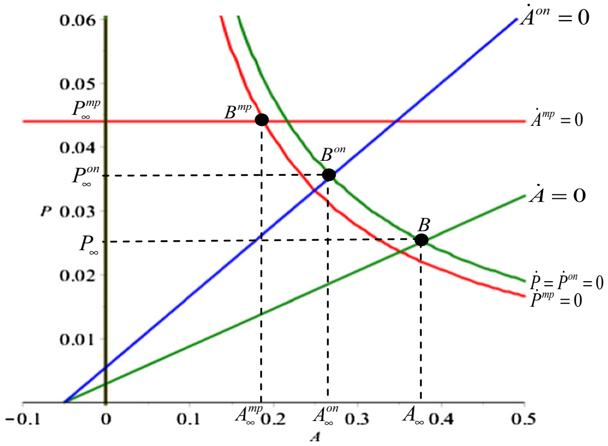

Fig. 5 Comparison between cooperative, OLNE and MPNE steady states

the sensitivity of these steady states to the various parameters in terms of comparative statics, whenever possible. We also represent the phase diagram in the state space, and the transient paths of the emissions rate and restoration effort for each case. Finally, we characterize the nature of convergence to the steady state as a function of the magnitude of the discounting rate for each case.

To compare the transient behaviors associated with the cooperative, OLNE and MPNE solutions, we assumed in all the subsequent cases a large initial pollution stock, i.e., significantly larger than the maximum corresponding stable steady state value resulting from the three solutions ($P_0 = 0.1$). In contrast, to assess the sensitivity of the transient paths to the magnitude of the initial absorption efficiency, we used two initial values, one being greater than the maximum stable steady state value among the three solutions, that is, $A_0 = 0.45$, and the other being lower than the minimum stable steady state value among the three solutions, that is, $A_0 = 0.15$.

We are primarily interested in stable interior solutions. As for the cooperative and OLNE solutions, the system of 8 non-linear equations derived for the MPNE gave rise to a single Nash equilibrium with globally asymptotically stable steady state. In the cooperative, OLNE and MPNE cases, the respective stability conditions were all satisfied. That is, the saddle-point property was fulfilled with two positive and two negative eigenvalues with no imaginary part in the cooperative case, $(\chi_1, \chi_3) = (-0.427, -0.004)$, and in the OLNE case, $(\xi_1, \xi_3) = (-0.305, -0.008)$. In the MPNE case, the global asymptotic stability criterion was granted with two negative eigenvalues with no imaginary part, $(\psi_1, \psi_2) = (-0.01, -0.235)$.

The main results are illustrated in Fig. 5. A unique (non-trivial) steady state is observed in each case, that is, B (i.e., intersection of the isoclines $\dot{P} = 0$ and $\dot{A} = 0$) for the cooperative solution with $(A_\infty, P_\infty) = (0.3776, 0.025)$, B^{on} (i.e., $\dot{P}^{on} = 0$ and $\dot{A}^{on} = 0$) for the OLNE with $\left(A_\infty^{on}, P_\infty^{on}\right) = (0.2684, 0.035)$, and B^{mp} (i.e., $\dot{P}^{mp} = 0$ and $\dot{A}^{mp} = 0$) for the MPNE with $\left(A_\infty^{mp}, P_\infty^{mp}\right) = (0.1883, 0.0439)$.

For the chosen constellation of parameters, the phase diagram for the MPNE (Fig. 6) confirms that all paths starting from a reasonable neighborhood around the steady state, B^{mp}, converge to the steady state. From (slightly) negative initial absorption efficiency, the convergence to the steady state is also granted with initially increasing then decreasing pollution stock. This pattern is driven by the fact that emissions reduction, though useful, is insufficient under negative absorption efficiency and only heavy restoration efforts can turn

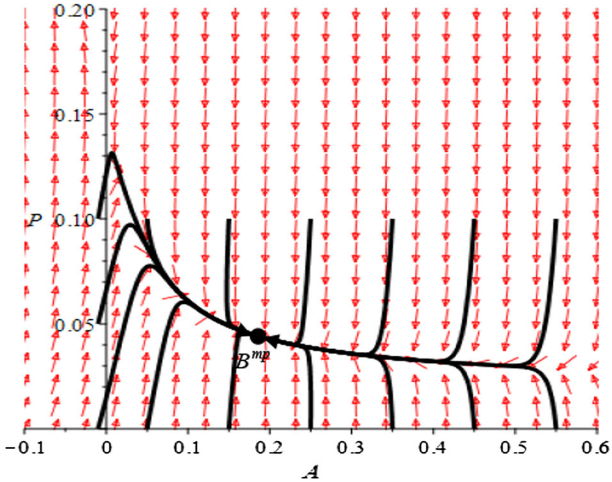

Fig. 6 Phase diagram in the state space (A, P) for Markov perfect Nash equilibrium

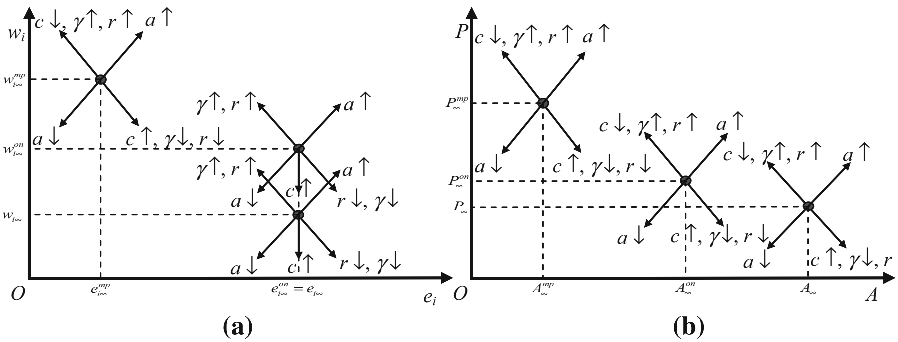

Fig. 7 Sensitivity analysis for cooperative, OLNE and MPNE stable steady states. **a** Control variables **b** state variables

the biosphere to pollution sink. Therefore, the pollution stock can never decrease before a strengthening of the absorption efficiency. A similar requirement is observed in a cooperative solution with clockwise oscillatory convergence to the steady state (e.g., El Ouardighi et al. 2015a).

In Fig. 5, we obtain $P_\infty^{mp} > P_\infty^{on} > P_\infty$. This result is consistent with other differential game models that represent environmental absorption efficiency as an instantaneous function of the pollution stock because our model shows that the steady state pollution stock is lower in an OLNE than in a MPNE. Moreover, we also get $A_\infty^{mp} < A_\infty^{on} < A_\infty$, so that the absorption efficiency in an MPNE is substantially lower than in an OLNE in spite of lower emissions and greater restoration efforts at the steady state. Therefore, the MPNE strategy leads to a stable steady state, which is more likely to be located in the emergency area, whereas the cooperative and OLNE strategies result in a steady state that has a better chance of falling into the sustainability area.

The figures below describe the sensitivity of the steady state to the parameters' values for the control variables (Fig. 7a) and the state variables (Fig. 7b) in the three equilibria.

In the cooperative, OLNE and MPNE cases, a lower discounting rate, r, moves the steady state further toward the sustainability area (7b). This suggests that players' patience enhances the long run efficiency of the biosphere as a pollution sink because it leads to the substitution of lower emissions during the transient path for increased emissions at the steady state. Correlatively, the restoration effort should be reduced at the steady state (7a). A greater pollution cost coefficient, c, also moves the steady state further toward the sustainability area in the cooperative, OLNE and MPNE cases (7b). Though it reduces the restoration effort at the steady state, it does not affect the emissions in the cooperative and OLNE cases, which suggests that emissions are actually reduced during the transient path only. Regarding the influence of an increase in the destructive impact of pollution on absorption efficiency, γ, it results in lower emissions and greater restoration efforts at the steady state in all equilibria (7a). A move of the steady state toward the emergency area is observed in all equilibria, with reduced absorption efficiency and an increased pollution stock (7b). Finally, an increase in the revenue function coefficient, a, leads to a complementary increase in the cooperative, OLNE and MPNE steady state pollution stock and absorption efficiency (7a), which results in both greater emissions and restoration efforts (7b).

In Fig. 8, the phase plane in the state space is depicted for the cooperative and non-cooperative solutions. In the case of (relatively) high initial absorption efficiency, i.e., $A_0 = 0.45$, the cooperative, OLNE and MPNE strategies consist in decreasing the absorption efficiency slightly and the pollution stock dramatically in a finite time, and then decreasing the absorption efficiency significantly and increasing the pollution stock slightly until the corresponding steady state is reached, that is, B for the cooperative solution, B^{on} for the OLNE, and B^{mp} for the MPNE. In the case of (relatively) low initial absorption efficiency, i.e., $A_0 = 0.15$, the cooperative, OLNE and MPNE strategies also consist in substantially decreasing the pollution stock first, and then in increasing the absorption efficiency while converging to the steady state. Overall, the cooperative, OLNE and MPNE transient paths are quite similar and differ only in their steady state respective values. Whatever the initially (positive) efficiency of pollution sinks, it is optimal in all cases to reduce the pollution stock substantially before strengthening the biosphere as a pollution sink. However, although a contingent strategy preserves the long run absorption efficiency somewhat, OLNE is more effective than an MPNE in managing pollution sinks' efficiency.

The patterns in Fig. 8 are explained by the time paths of emissions rate and restoration effort shown in Fig. 9, where time is expressed as a logarithmic function along the X-axis. We first observe that in all equilibria, the greater the initial absorption efficiency, the greater the emissions rate and the lower the restoration effort.

In the case of (relatively) high initial absorption efficiency, i.e., $A_0 = 0.45$, the OLNE strategy exhibits greater emissions and initially lower and then greater restoration efforts than the cooperative strategy. In contrast, the MPNE strategy shows initially greater then lower emissions, and initially lower then greater restoration efforts than the cooperative strategy. The non-cooperative equilibria under high initial absorption efficiency are consistent with the tragedy of the commons in that they reflect both excessive emissions and underinvestment in absorption efficiency over the most economically influential part of the planning horizon, i.e., the time interval during which the discounted instantaneous profits are still strictly positive.

In the case of (relatively) low initial absorption efficiency, i.e., $A_0 = 0.15$, the optimal policy in all equilibria consists of initially small and increasing emissions in gradual stages and initially positive and decreasing restoration efforts. The first jump in emissions occurs quickly, immediately after the dramatic decrease in the pollution stock. Along the time path, cooperative emissions are lower than OLNE emissions over a wide time interval, until the steady state where the emissions are the same in the cooperative solution and

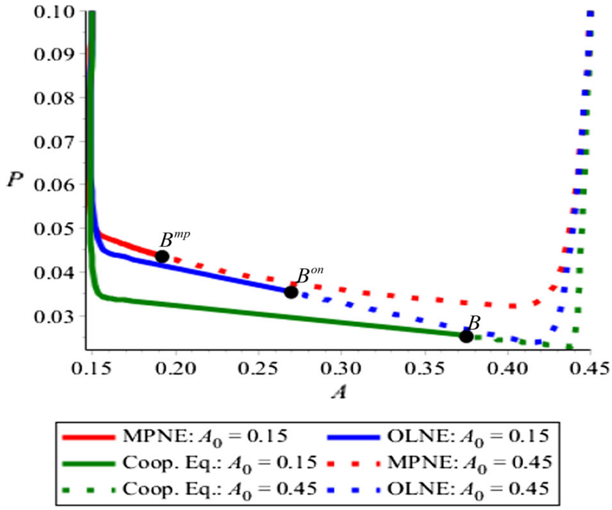

Fig. 8 Phase plane in the state space for the cooperative, open-loop and Markov perfect Nash cases from low and high absorption efficiency

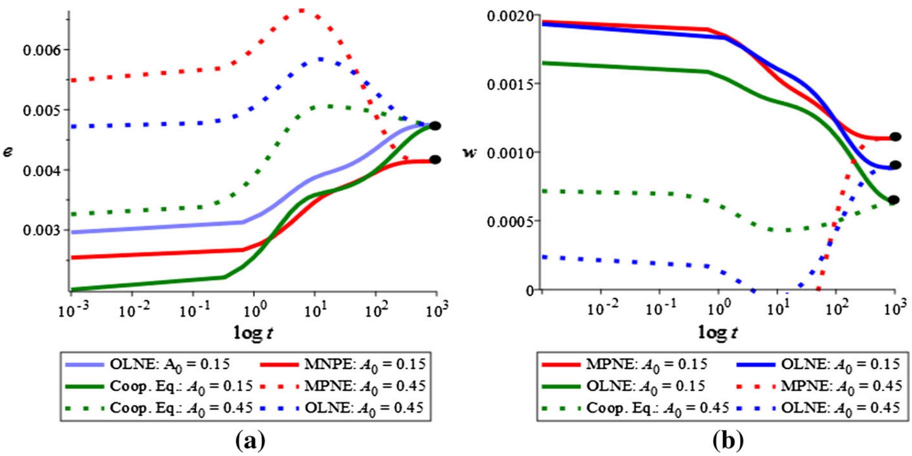

Fig. 9 Transient paths of emissions rate and restoration effort for the cooperative, OLNE and MPNE cases from low and high absorption efficiency. **a** Emissions rate **b** restoration effort

OLNE ($e_\infty^{on} = e_\infty = 0.0047$). Lower transient emissions in the cooperative case require less restoration efforts than in an OLNE during the transient path and in the steady state ($w_\infty = 0.0006 < w_\infty^{on} = 0.0008$). In an MPNE, emissions alternate between greater and smaller values than in the cooperative solution until the steady state ($e_\infty^{mp} = 0.0041$) is reached. Restoration efforts are always greater than in the cooperative solution until the steady state ($w_\infty^{mp} = 0.001$) is attained. The emissions in an MPNE are lower and restoration efforts alternate between greater and smaller values than in the OLNE until the steady state.

Because transient emissions in an OLNE are greater than in a cooperative solution as a result of greater transient restoration efforts, the OLNE strategy under low initial absorption efficiency is consistent with the assumption of a voracity effect in that it transiently

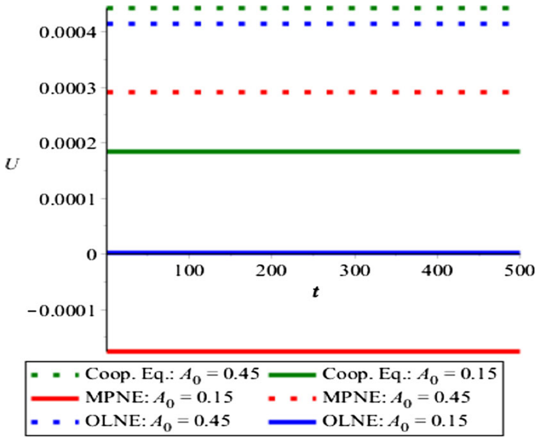

Fig. 10 Overall utility for the cooperative, OLNE and MPNE cases from low and high initial absorption efficiency

promotes both excessive emissions and overinvestment in absorption efficiency over the planning horizon (Benchekroun and Chaudhuri 2014). By comparison, an MPNE promotes lumpier substitutability between the two policy instruments than an OLNE does. At the steady state, the magnitude of emissions in an MPNE is limited by low absorption efficiency.

Figure 10 reports the overall utility associated with the cooperative, OLNE and MPNE cases under (relatively) high and low initial absorption efficiency, respectively.

In the context of high initial absorption efficiency, the players' joint overall utility is positive for all cases. Note that the overall utility under cooperation and the joint overall utility under non-cooperative equilibrium with commitment are almost similar. That is, the welfare cost for players with OLNE strategies that are not able to cooperate is limited. In comparison, the welfare cost for MPNE strategies regarding cooperation or commitment is greater. Under low initial absorption efficiency, the players' overall utility is positive only for cooperation, while it is zero for OLNE and negative for MPNE. This decline in joint overall utility of non-cooperative strategies might be seen as a confirmation of the presence of a voracity effect. Here, the welfare cost for non-cooperative players that are unable to cooperate, and the welfare cost for players with MPNE strategies that are unable to commit are both significantly greater than previously. In terms of elasticity of the overall joint utility with respect to the initial absorption efficiency, denoted by ψ and defined as the percentage of decrease of the overall utility divided by the percentage of decrease of the initial absorption efficiency, we get $\psi_{U/A_0} = 0.876$ in the cooperative case, $\psi^{on}_{\sum_i U_i/A_0} = 1.49$ in the OLNE case and $\psi^{mp}_{\sum_i V^i/A_0} = 2.399$ in the MPNE case. These results imply that contingent strategies are more vulnerable in terms of social welfare to a decrease in initial absorption efficiency than commitment strategies are, which in turn are more affected than cooperative strategies are. That is, the absence of cooperation or commitment becomes more costly as the initial absorption efficiency of pollution sinks tends to exhaustion. This result explains why nations have been urged in recent years to commit to pollution control at an international level (Stern 2006; IPCC 2007; Stern 2015).

Finally, the steady state utility, or social welfare of future generations, is positive in the cooperative, OLNE and MPNE cases and its ranking is consistent with that reported in Fig. 10 from high initial absorption efficiency. The welfare cost of future generations is limited for OLNE strategies regarding cooperative strategy and significant for MPNE strategies regarding cooperation or commitment. In all cases, the decrease in initial absorption efficiency

Table 3 Threshold of impact of the discounting rate on the nature of convergence to the steady state

Discounting rate	0.1	0.15	0.2	0.25
Cooperative solution	Monotonic	Monotonic	Monotonic	Diverging
OLNE	Monotonic	Monotonic	Monotonic	Diverging
MPNE	Monotonic	Diverging	Diverging	Diverging

is more costly to current generations than to future ones because the future generations neither benefit from the revenues nor incur the costs related to the transient path toward the steady state.

Table 3 provides insights into the nature of convergence to the steady state depending on the discounting rate value for the set of lower bound values. The value of the discounting rate above, with which the path to the steady state becomes diverging in the case of the MPNE, is relatively low ($r = 0.15$). In comparison with this threshold, the value of the discounting rate above, with which the saddle-path to the steady state becomes diverging, is greater both in cooperative solution and OLNE (i.e., $r = 0.24$). As a result, more heavily discounted MPNE strategies are less robust to a definitive switching of pollution sinks to a source, while such switching would not occur in an OLNE. This implies that relatively impatient players should not opt for contingent strategies because they are more likely to make the evolution of the biosphere irreversibly uncontrollable. It is noteworthy that the threshold value of the discounting rate from which cooperative and OLNE paths are diverging is still monotonic, which departs from Tahvonen and Withagen (1996), who argue that an optimal path leading to irreversible pollution is typically non-monotonic.

Figure 11 confirms the irreversible impact of a higher discounting rate ($r = 0.15$) on the MPNE trajectory. Along the MPNE diverging path, which is also monotonic, restoration efforts are initially low and convexly increasing, but less rapidly than emissions. This pattern suggests that MPNE strategies where transient emissions and restoration efforts act as complements rather than substitutes are likely to result in a definitive switching of pollution sinks to a source. In comparison, although both cooperative and OLNE paths remain monotonic, they are more likely to end up in the emergency area. This result confirms that although commitment and cooperative strategies are more robust than MPNE strategies to a definitive switching of pollution sinks to a source, they are nevertheless vulnerable to players' impatience, which directly affects future generations' social welfare.

7 Conclusion

In this paper, we suggest a non-linear quadratic differential game model of pollution control with a time-dependent environmental absorption efficiency that allows the biosphere to switch from a carbon sink to a source.

Using a cooperative solution as a benchmark, we showed that the fact that the players do not properly internalize the social cost of the pollution stock in the non-cooperative cases incites them to excessively substitute their transient path for steady state utility. This leads them to a disproportionate use of strategic substitutability between polluting emissions and restoration of environmental absorption efficiency during the transient path, which reduces the efficiency of carbon sinks and increases the pollution stock at the steady state. In both cooperative and non-cooperative steady states, the magnitude of emissions increases with

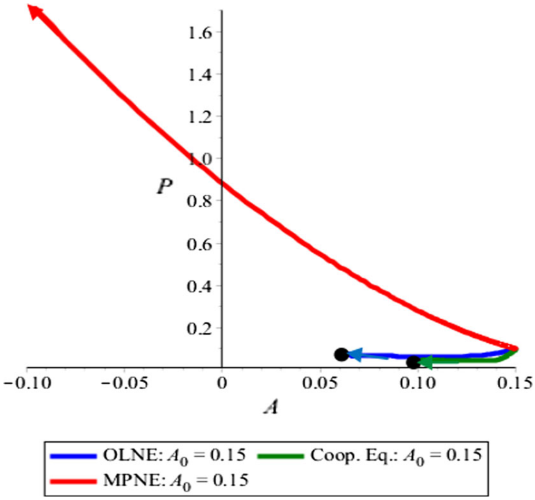

Fig. 11 Phase plane in the state space for the cooperative, OLNE and MPNE cases from low absorption efficiency and higher discounting rate

absorption efficiency, while restoration efforts increase with the pollution stock, and vice versa.

We were then able to identify which decision rule, open-loop or Markov perfect Nash strategy, best prevents the biosphere from switching from a pollution sink to a source. Although it fails to reach the same environmental absorption efficiency as a cooperative solution, an OLNE imposes limited economic sacrifice at the steady state notably because it allows equivalent polluting emissions to the social optimum. This result justifies the optimistic tone of the call by some authors for rapid action against climate change (Stern 2015). However, an OLNE requires a greater restoration effort at the steady state than a cooperative solution does to partly compensate for the excessive emissions released during the transient path.

If the biosphere is still an efficient pollution sink, it is optimal in all cases to reach the steady state both by an overshooting emissions trajectory and an undershooting restoration efforts trajectory. If the efficiency of the biosphere as a pollution sink is already low, it is optimal to set initially low emissions and high restoration efforts, and then to increase emissions gradually as absorption efficiency increases, and decrease restoration efforts as the pollution stock declines. Our results suggest that the absence of cooperation and/or precommitment becomes more costly as the initial efficiency of pollution sinks tends to exhaustion. Also, a decrease in initial efficiency of pollution sinks is more costly to current generations than to future ones in both cooperative and non-cooperative strategies.

When both players are sufficiently patient, an MPNE and an OLNE do not differ much in terms of transient path and steady state. In this context, both strategies can prevent pollution sinks from turning into a source during the transient path and the steady state, including in the case where the absorption efficiency is low. However, although an MPNE may generate lower emissions and greater restoration efforts than an OLNE in the case of low absorption efficiency, it has a much greater chance of falling into the emergency area. In contrast, when both players are relatively impatient, MPNE strategies are less robust than OLNE strategies at preventing irreversible switching of the biosphere from a carbon sink to a source. Although more heavily discounted OLNE strategies can prevent pollution sinks from transiently turning into a source, these strategies are more likely to end up in the emergency area.

Regarding the methodological contribution of the paper, we suggest a particular structure of the conjectured value function to solve MPNE problems with multiplicative interaction between state variables in one state equation, so that third-order terms that arise in the Hamilton–Jacobi-Bellman equation are made negligible. Using a collocation procedure, we confirm the validity of the particular structure of the conjectured value function.

Overall, our results provide an update in the debate on pollution control in that they suggest that the absence of commitment at an international level on pollution control can preserve both current and future generations from experiencing switching of carbon sinks to a source only if the players remain sufficiently patient. More importantly, regardless of whether polluters commit to a plan of action or make contingent decisions, our study emphasizes the need for environmental policies where efforts to restore environmental absorption efficiency are an essential policy instrument.

The scope of our model could be extended to the case where the marginal destructive impact of the pollution stock on the evolution of absorption efficiency can be reduced by an effort to improve environmental resilience. Another possible extension of the model presented here could assume that the impact of restoration efforts on the evolution of absorption efficiency depends concavely on the level of absorption efficiency. These assumptions would require multiplicative separability between control and state variables, which would allow the emergence of a Skiba threshold (Grass et al. 2008) with multiple steady states and lead to the determination of history-dependent policies. Other future directions of research include an analysis of the Pareto optimality of the obtained equilibria and an extension of the model to the case of taxation rules.

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Appendix

A1. To determine the (non-trivial) cooperative steady state, we solve the canonical system (8)–(11) in the state-costate space. Using (6) and (7) leads to the system's steady state in (12), $i = 1,\ 2$. Note that since:

$$\lim_{t\mapsto+\infty} e^{-rt}\eta_1(t)P(t) = \lim_{t\mapsto+\infty}\left[-r\gamma\left(r^2\gamma+h\right)e^{-rt}\Big/4\left(4c+\gamma^2\right)\right]=0$$
$$\lim_{t\mapsto+\infty} e^{-rt}\eta_2(t)A(t) = \lim_{t\mapsto+\infty}\left[\gamma\left(2a-r\gamma\right)e^{-rt}\Big/2\right]=0$$

the cooperative steady-state solution satisfies the limiting transversality conditions. □

A2. Plugging the expressions of $e_{i\infty}$, P_∞ and $w_{i\infty}$ from (12) in the integrand of (4) and simplifying gives (13). □

A3. Writing the variables in the order P, A, η_1, η_2, the Jacobian matrix associated with the canonical system (8)–(11) is:

$$J=\begin{bmatrix}-A & -P & 2 & 0\\ -\gamma & 0 & 0 & 2\\ 2c & \eta_1 & r+A & \gamma\\ \eta_1 & 0 & P & r\end{bmatrix}$$

The determinant of the Jacobian matrix is:

$$|J| = \left(4c+\gamma^2\right) P_\infty^2 + r\gamma A_\infty P_\infty = \left(r^2\gamma + h\right)^2 \Big/ 4\left(4c+\gamma^2\right) + r\gamma\,(2a - r\gamma)$$

Using Dockner's formula (Dockner 1985), the sum of the principal minors of J of order 2 minus the squared discount rate, denoted by K, is:

$$\begin{aligned} K &= -A_\infty\,(r + A_\infty) - 4c - 2\gamma P_\infty \\ &= -\frac{\left[2\,(2a - r\gamma)\left(4c+\gamma^2\right) + r\left(r^2\gamma + h\right)\right]^2}{\left(r^2\gamma + h\right)^2} - \frac{\gamma h + 4c\left(4c + \gamma^2 - r^2\right)}{\left(4c+\gamma^2\right)} \end{aligned}$$

Because $|J| > 0$ and $K < 0$, the cooperative steady state is a saddle-point. We compute:

$$\Omega = K^2 - 4\,|J| = A_\infty\,(r + A_\infty)\,[A_\infty\,(r + A_\infty) + 8c] + 4\gamma A_\infty^2 P_\infty + 16c\,(c + \gamma P_\infty) - 16cP_\infty^2$$

The sign of Ω can be either negative or positive. In the first case, the steady state is a saddle-focus with transient oscillations. Otherwise, it is a saddle-node and the optimal solution monotonically converges to the steady state. □

A4. The linear approximation of the system (8)–(11) around its steady state is:

$$\begin{aligned} \dot{\eta}_1 &= 2cP + \eta_1\,(r + A_\infty) + \eta_{1\infty}\,(A - A_\infty) + \gamma\eta_2 \\ \dot{\eta}_2 &= r\eta_2 + \eta_{1\infty} P + P_\infty\,(\eta_1 - \eta_{1\infty}) \\ \dot{P} &= 2\,(a + \eta_1) - P_\infty\,(A - A_\infty) - A_\infty P \\ \dot{A} &= 2\eta_2 - \gamma P \end{aligned}$$

The four eigenvalues associated with the Jacobian matrix of the canonical system are:

$$\begin{aligned} {}^{3}_{1}\chi^{4}_{2} &= \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - 4\,|J|}} \\ &= \frac{r}{2} \pm \sqrt{\frac{(r+A_\infty)^2 + A_\infty^2}{4} + \gamma P_\infty + \frac{8ac}{r^2\gamma^2} \pm \sqrt{\left(A_\infty^2 + \frac{8ac}{r^2\gamma^2}\right)\left(\gamma P_\infty + \frac{8ac}{r^2\gamma^2}\right) + \left[\frac{(r+A_\infty)^2 A_\infty}{4} + \frac{8ac}{r\gamma^2}\right] A_\infty - 4cP_\infty^2}} \end{aligned}$$

where A_∞ and P_∞ are given in (12). As expected, two eigenvalues, χ_1 and χ_3, have negative real part and two have positive real part, χ_2 and χ_4. Choosing the roots with negative real part for convergence, the time paths of the costate and state variables are written as:

$$\begin{aligned} \eta_1(t) &= \eta_{1\infty} + B_1 e^{\chi_1 t} + B_2 e^{\chi_3 t} \\ \eta_2(t) &= \eta_{2\infty} + B_3 e^{\chi_1 t} + B_4 e^{\chi_3 t} \\ P(t) &= P_\infty + B_5 e^{\chi_1 t} + B_6 e^{\chi_3 t} \\ A(t) &= A_\infty + B_7 e^{\chi_1 t} + B_8 e^{\chi_3 t} \end{aligned}$$

These equations involve 10 unknowns $(B_1, \ldots, B_8, \eta_1(0),\ \eta_2(0))$ that can be solved with the 10 following equations, which are drawn from the above expressions and the linearized versions of (8)–(11):

$$\begin{aligned} \eta_1\,(0) &= \eta_{1\infty} + B_1 + B_2 \\ \eta_2\,(0) &= \eta_{2\infty} + B_3 + B_4 \end{aligned}$$

$$\begin{aligned} P_0 &= P_\infty + B_5 + B_6 \\ A_0 &= A_\infty + B_7 + B_8 \end{aligned}$$

$$\dot{\eta}_1(0) = 2cP_0 + \eta_1(0)(r + A_\infty) + \eta_{1\infty}(A_0 - A_\infty) + \gamma\eta_2(0) = B_1\chi_1 + B_2\chi_3$$
$$\dot{\eta}_2(0) = r\eta_2(0) + \eta_{1\infty}(P_0 - P_\infty) + \eta_1(0)P_\infty = B_3\chi_1 + B_4\chi_3$$
$$\dot{P}(0) = 2(a + \eta_1(0)) - P_\infty(A_0 - A_\infty) - A_\infty P_0 = B_5\chi_1 + B_6\chi_3$$
$$\dot{A}(0) = 2\eta_2(0) - \gamma P_0 = B_7\chi_1 + B_8\chi_3$$
$$2(B_3\chi_1 + B_4\chi_3) - \gamma(B_5\chi_1 + B_6\chi_3) = B_7\chi_1^2 + B_8\chi_3^2$$
$$2(B_1\chi_1 + B_2\chi_3) - A_\infty(B_5\chi_1 + B_6\chi_3) - P_\infty(B_7\chi_1 + B_8\chi_3) = B_5\chi_1^2 + B_6\chi_3^2$$

This system of equations can be solved numerically to determine real-valued solutions. Finally, using (6)–(7) yields (14)–(17). □

A5. To compute the (non-trivial) OLNE steady state, we solve the system in the state-costate space (21)–(24). Using (19) and (20) leads us to derive the steady state in (25), $i = 1, 2$. Note that since:

$$\lim_{t\mapsto+\infty} e^{-rt}\lambda_1(t)P(t) = \lim_{t\mapsto+\infty}\left[-r\gamma\left(r^2\gamma + k\right)e^{-rt}\big/4\left(2c+\gamma^2\right)\right] = 0$$
$$\lim_{t\mapsto+\infty} e^{-rt}\lambda_2(t)A(t) = \lim_{t\mapsto+\infty}\left[\gamma(2a - r\gamma)e^{-rt}\big/2\right] = 0$$

the OLNE steady-state solution satisfies the limiting transversality conditions. □

A6. Plugging the expressions of $e_{i\infty}^{on}$, P_∞^{on} and $w_{i\infty}^{on}$ from (25) in the integrand of (3) and simplifying gives (26). □

A7. Writing the variables in the order $P, A, \lambda_1, \lambda_2$, the Jacobian matrix associated with the canonical system (21)–(24) is:

$$J^{on} = \begin{bmatrix} -A & -P & 2 & 0 \\ -\gamma & 0 & 0 & 2 \\ c & \lambda_1 & r+A & \gamma \\ \lambda_1 & 0 & P & r \end{bmatrix}$$

The determinant of J^{on} is:

$$\left|J^{on}\right| = \left(2c+\gamma^2\right)P_\infty^{on2} + r\gamma A_\infty^{on}P_\infty^{on} = \frac{\left(r^2\gamma+k\right)^2}{4\left(2c+\gamma^2\right)} + r\gamma(2a - r\gamma)$$

Further, the sum of the principal minors of J^{on} of order 2 minus the squared discount rate is:

$$K^{on} = -A_\infty^{on}\left(r + A_\infty^{on}\right) - 2c - 2\gamma P_\infty^{on}$$
$$= -\frac{2(2a - r\gamma)\left(2c+\gamma^2\right)}{r^2\gamma + k}\left[r + \frac{2(2a - r\gamma)\left(2c+\gamma^2\right)}{r^2\gamma+k}\right] - \frac{\gamma\left(r^2\gamma+k\right)}{\left(2c+\gamma^2\right)} - 2c$$

which is strictly negative. Hence, the OLNE steady state is a saddle-point. We now compute: $\Omega^{on} = \left(K^{on}\right)^2 - 4\left|J^{on}\right| = A_\infty^{on2}\left[\left(r + A_\infty^{on}\right)^2 + 4\gamma P_\infty^{on}\right] + \frac{16ac}{r^2\gamma^2}\left[A_\infty^{on}\left(r + A_\infty^{on}\right) + \frac{4ac}{r^2\gamma^2}\right] + 8c\left(\frac{4a}{r^2\gamma} - P_\infty^{on}\right)P_\infty^{on}$ where A_∞^{on} and P_∞^{on} are given in (25). The sign of this expression, which is unclear, determines whether the saddle path converges monotonically or oscillates towards the steady state. □

A8. The linear approximation of system (21)–(24) around its steady state is:

$$\dot{\lambda}_1 = cP + \lambda_1(r + A_\infty) + \lambda_{1\infty}(A - A_\infty) + \gamma\lambda_2$$
$$\dot{\lambda}_2 = r\lambda_2 + \lambda_{1\infty}P + P_\infty(\lambda_1 - \lambda_{1\infty})$$

$$\dot{P} = 2(a + \lambda_1) - P_\infty (A - A_\infty) - A_\infty P$$
$$\dot{A} = 2\lambda_2 - \gamma P$$

The eigenvalues associated with the Jacobian of the non-cooperative canonical system are:

$${}^{3}_{1}\xi^{4}_{2} = r/2 \pm \sqrt{\left[\left(r + A_\infty^{on}\right)^2 + \left(A_\infty^{on}\right)^2\right]/4 + \gamma P_\infty^{on} + 4ac/r^2\gamma^2 \pm \sqrt{\left[A_\infty^{on}\left(r + A_\infty^{on}\right)/2 + 2ac/r^2\gamma^2 + \gamma P_\infty^{on}\right]^2 - \left(2c + \gamma^2\right)\left(P_\infty^{on}\right)^2 - 4a}}$$

where A_∞^{on} and P_∞^{on} are given in (25). Choosing roots with negative real part, ξ_1 and ξ_3, and using similar methods as in the cooperative case leads us to find $D_1, \ldots, D_8$, and $\lambda_1(0)$, $\lambda_2(0)$. □

A9. Assuming a second order Taylor-polynomial approximation of the value function centered on $\left(A_\infty^{mp}, P_\infty^{mp}\right)$, i.e.,

$$V = \alpha_1 + \alpha_2\left(A - A_\infty^{mp}\right) + \alpha_3\left(P - P_\infty^{mp}\right) + \alpha_4\left(A - A_\infty^{mp}\right)^2 + \alpha_5\left(P - P_\infty^{mp}\right)^2 + \alpha_6\left(A - A_\infty^{mp}\right)\left(P - P_\infty^{mp}\right)$$

we obtain constant second order derivatives for V. We assume that its unknown coefficients $\alpha_1, .., \alpha_6$ and the problem parameters are such that the steady state is positive. Plugging (37) and (38) in (1)–(2), respectively, gives the MPNE controlled system, that is:

$$\dot{P} = 2\left[a + \alpha_3 + 2\alpha_5\left(P - P_\infty^{mp}\right) + \alpha_6\left(A - A_\infty^{mp}\right)\right] - AP$$
$$\dot{A} = 2\left[\alpha_2 + 2\alpha_4\left(A - A_\infty^{mp}\right) + \alpha_6\left(P - P_\infty^{mp}\right)\right] - \gamma P$$

The resulting approximation of the HJB Eq. (35) is:

$$\begin{aligned} & r\left[\alpha_1 + \alpha_2\left(A - A_\infty^{mp}\right) + \alpha_3\left(P - P_\infty^{mp}\right) + \alpha_4\left(A - A_\infty^{mp}\right)^2 + \alpha_5\left(P - P_\infty^{mp}\right)^2 + \alpha_6\left(A - A_\infty^{mp}\right)\left(P - P_\infty^{mp}\right)\right] \\ & = \left[\alpha_3 + 2\alpha_5\left(P - P_\infty^{mp}\right) + \alpha_6\left(A - A_\infty^{mp}\right)\right]\left\{\frac{3}{2}\left[\alpha_3 + 2\alpha_5\left(P - P_\infty^{mp}\right) + \alpha_6\left(A - A_\infty^{mp}\right)\right] - AP + 2a\right\} \\ & \quad + \left[\alpha_2 + 2\alpha_4\left(A - A_\infty^{mp}\right) + \alpha_6\left(P - P_\infty^{mp}\right)\right]\left\{3\left[\alpha_2 + 2\alpha_4\left(A - A_\infty^{mp}\right) + \alpha_6\left(P - P_\infty^{mp}\right)\right]/2 - \gamma P\right\} \\ & \quad - cP^2/2 + a^2/2 \end{aligned}$$

Rearranging its terms, this can be expressed as a third-order polynomial in $\left(A - A_\infty^{mp}\right)$ and $\left(P - P_\infty^{mp}\right)$. Then, the unknown parameters of the approximation of the value function are found by setting to 0 the coefficients, up to the second order, of such a third-order polynomial (its third-order terms are locally negligible, by construction). To do so, one solves the system:

$$\begin{aligned} r\alpha_1 - 3\alpha_2^2/2 - \alpha_3\left(3\alpha_3/2 + 2a\right) - a^2/2 + \gamma\alpha_2 P_\infty^{mp} + \alpha_3 A_\infty^{mp} P_\infty^{mp} + c\left(P_\infty^{mp}\right)^2/2 &= 0 \\ \alpha_2\left(\gamma - 3\alpha_6\right) + r\alpha_3 - 2\alpha_5\left(3\alpha_3 + 2a\right) + \alpha_3 A_\infty^{mp} + \left(c + \gamma\alpha_6\right) P_\infty^{mp} + 2\alpha_5 A_\infty^{mp} P_\infty^{mp} &= 0 \\ c/2 + \alpha_5\left(r - 6\alpha_5 + 2A_\infty^{mp}\right) + \alpha_6\left(\gamma - 3\alpha_6/2\right) &= 0 \\ \alpha_2\left(r - 6\alpha_4\right) - \alpha_6\left(3\alpha_3 + 2a\right) + \left(\alpha_3 + 2\gamma\alpha_4\right) P_\infty^{mp} + \alpha_6 A_\infty^{mp} P_\infty^{mp} &= 0 \\ \alpha_3 + 2\gamma\alpha_4 - 6\alpha_6\left(\alpha_4 + \alpha_5\right) + \alpha_6\left(r + A_\infty^{mp}\right) + 2\alpha_5 P_\infty^{mp} &= 0 \\ \alpha_4\left(r - 6\alpha_4\right) - \alpha_6\left(3\alpha_6/2 - P_\infty^{mp}\right) &= 0 \end{aligned}$$

We have 6 equations in 8 unknowns $\alpha_1, \ldots, \alpha_6$, A_∞^{mp}, P_∞^{mp}. To get two other equations, we use (37)–(38), imposing a steady state solution for the MPNE:

$$2\,(a + \alpha_3) - A_\infty^{mp} P_\infty^{mp} = 0$$

$$2\alpha_2 - \gamma P_\infty^{mp} = 0$$

Finally, we obtain a non-linear system of 8 equations in 8 unknowns, which in general might be solved numerically, but not analytically. When it can be solved, its solution provides:

$$\left[P_\infty^{mp}, A_\infty^{mp}, e_\infty^{mp}, w_\infty^{mp}\right]^T = \left[2\alpha_2/\gamma, \gamma\,(a + \alpha_3)\,/\alpha_2, a + \alpha_3, \alpha_2\right]^T$$

Clearly, the necessary conditions for the positivity of the steady state values and the positivity of each player's instantaneous revenue are $\alpha_2 > 0$ and $\alpha_3 > -a$. In general, the non-linear system above may admit multiple solutions (or even no solutions at all). To solve the system, we use the *Matlab* function `fsolve`, using default options, and starting with a random initialization of the 8 unknowns, initially chosen as realizations of independent and uniformly distributed random variables on the interval [0, 10]. To check whether a numerically found solution of the non-linear system above is likely also a steady state solution for the MPNE, we perform the following additional checks:

(1) Positivity of the state and control variables, and positivity of each player's instantaneous revenue: $\alpha_2 > 0$ and $-a < \alpha_3 < 0$;
(2) Global asymptotical stability for the linearized dynamical system: the eigenvalues of the Jacobian matrix (40) should have negative real parts;
(3) Comparison between the value of α_1 assumed in $\left(A_\infty^{mp}, P_\infty^{mp}\right)$ by the local quadratic approximation of the value function, and the one obtained by replacing the resulting steady state values in (3), the latter being equal to $\left[e_\infty^{mp}\left(a - e_\infty^{mp}/2\right) - c\left(P_\infty^{mp}\right)^2/2 - \left(w_\infty^{mp}\right)^2/2\right]/r$: in the case where $P_\infty^{mp}, A_\infty^{mp}, e_\infty^{mp}, w_\infty^{mp}$ is really an approximation of a steady state solution for the MPNE, and the local quadratic approximation is a good approximation of the value function, then the two values should be approximately the same. In practice, we check whether the relative error of the first expression with respect to the second one is smaller than 0.01.

In the case where multiple solutions are obtained that satisfy the requirements (1), (2), and (3) above, one can choose the one associated with the largest value of $\left[e_\infty^{mp}\left(a - e_\infty^{mp}/2\right) - c\left(P_\infty^{mp}\right)^2/2 - \left(w_\infty^{mp}\right)^2/2\right]\Big/ r$. □

A10. To perform an additional check for the validity of the local quadratic approximation of the value function, we also exploit an alternative method to approximate the HJB equation and then we check whether the two obtained approximations are similar. As the alternative approximation method,[7] we use the collocation method for PDEs, applied to the HJB Eq. (35), on a small neighborhood of the steady state solution determined by the local quadratic approximation (in our implementation, we chose the box $\left[0.75A_\infty^{mp}, 1.25A_\infty^{mp}\right] \times \left[0.75P_\infty^{mp}, 1.25P_\infty^{mp}\right]$ as such a neighborhood). The collocation method approximates locally on that neighborhood

[7] In an initial phase, we also tested the application of the method of successive approximations, using a discrete-time version of the HJB Eq. (39), and following a procedure similar to the one proposed in (Cacace et al. 2013) for another dynamic game model. However, that approach demonstrated to be much slower than the collocation method, and suffered from boundary effects, due to the need to discretize and bound all the state and control variables. A similar finding about such boundary effects was reported in (Cacace et al. 2013).

the unknown value function as a high-order polynomial in $\left(A - A_\infty^{mp}\right)$ and $\left(P - P_\infty^{mp}\right)$, with unknown coefficients. These are determined by imposing that the HJB Eq. (35) holds exactly on a finite number of fixed points ("collocation points") inside the neighborhood (or equivalently, that its Bellman residual error is 0 on such points). The number of these points is equal to the number of unknown coefficients of the high-order polynomial. In this way, to find the values of the coefficients, one is reduced to solving a non-linear system (due to the non-linearity of the HJB equation), where each equation is associated with one collocation point. For the implementation of the collocation method, we follow the procedure proposed in (Doraszelski 2003), that is:

– We construct a polynomial approximation of order $K = 12$, using a tensor-product basis of univariate Chebyshev polynomials. This choice of the approximation order is guided by the need to find a suitable trade-off between the accuracy of the approximation, and the computational effort needed to find it (as in the cited reference, the number of coefficients in the expansion is $(K + 1)^2$);
– Likewise in Doraszelski (2003), we employ a two-dimensional expanded Chebyshev array (as defined in Judd 1998) to define the $(K + 1)^2$ collocation points.

To initialize the collocation method, at first we fit the initial local quadratic approximation of the value function by a linear combination of the same basis functions used by the collocation method itself (in practice, we apply the collocation method itself to this function approximation problem, instead of to the PDE). Then, we apply the collocation method to the PDE, starting from the resulting vector of coefficients. Finally, to check for the validity of the resulting high-order polynomial approximation and for the validity of the original local quadratic approximation, we compute, for both approximations, the Bellman residual error on a larger uniform grid of N^2 points in the same neighborhood (with $N = 30$), which are different from the collocation points, checking whether this error is small. We also evaluate, on each grid point, the relative error of the quadratic approximation with respect to the high-order polynomial approximation obtained using the collocation method. The results of all these comparisons are reported in Sect. 5, for a specific choice of the problem parameters. We conclude by mentioning that, in contrast with Doraszelski (2003), it is not necessary to check a posteriori whether the partial derivatives of the approximation of the value function obtained by the collocation method are good approximations of the corresponding partial derivatives of the value function, which are needed to express the optimal controls [see the expressions (37) and (38)]. Indeed, in the case where the local quadratic approximation and the one found by the collocation method are similar, we can use the partial derivatives of the local quadratic approximation itself. In any case, it is worth mentioning that the numerical results have shown good agreement between the approximations of the partial derivatives obtained by the two methods, as one can infer from Fig. 2 in Sect. 5. □

A11. The results in (41)–(45) follow from the respective comparisons of (12) and (25). □

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