Ann Oper Res (2019) 281:175–192
https://doi.org/10.1007/s10479-018-2828-9

S.I.: RISK IN FINANCIAL ECONOMICS

Revisiting generalized almost stochastic dominance

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Published online: 20 April 2018

Abstract We review the Generalized Almost Stochastic Dominance definition in Tsetlin et al. (Oper Res 63(2):363–377, 2015). We follow the classic concept of stochastic dominance in minimizing specific assumptions about decision makers' risk attitudes as highlighted by Levy (Manag Sci 38(4):555–593, 1992). We refine the definition of almost stochastic dominance by Leshno and Levy (Manag Sci 48(8):1074–1085, 2002) to secure economic intuition. We then present our definitions of almost stochastic dominance with an extension to the higher degrees. Our definition satisfies both expected utility maximization property and hierarchy property while minimizing assumption and securing economic intuition.

Keywords Stochastic dominance · Almost stochastic dominance · Expected utility maximization property · Hierarchy property

1 Introduction

Stochastic Dominance (SD) is one of the major rules for decision-making under uncertainty (Hadar and Russell 1969; Rothschild and Stiglitz 1970; Hanoch and Levy 1970; Levy 1990). SD rules refer to various partial orders on the space of distribution functions of random variables. SD stems from paradoxes that are sometimes revealed by the commonly employed

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mean–variance (MV) rule by Markowitz (1977). SD prevails over the MV rule, especially in the cases in which there is a clear-cut preference between two risky assets, as the MV rule fails to rank the two alternative prospects under consideration. See Levy (2016) for the updated survey.

In practice, SD is restrictive in conditions and thus limited in its function of making decisions. A small violation area in the cumulative distribution function (CDF) may cause SD to fail, as well as other criteria to reveal the preference. Alternatively, Leshno and Levy (2002) establish the Almost Stochastic Dominance (ASD) criterion to reveal choices that probably conform with "most" decision makers with an increasing and concave utility function. The ASD rules suggest eliminating some pathological preferences. This seminal work receives noticeable academic attention as the ongoing discussions of its redefinition and property indicate. The crucial theoretical articles and application literatures at least include Bali et al. (2009), Levy et al. (2010), Levy (2012), Tzeng et al. (2013), and Guo et al. (2013). Their respective versions of ASD reach a consensus on the Almost First-Degree Stochastic Dominance (AFSD), but not necessarily in the Almost Second-Degree Stochastic Dominance (ASSD).

ASD is recognized for both its expected utility maximization property (EUMP) and hierarchy property (HP), as summarized by Guo et al. (2013). EUMP refers to the property that ASD shows the equivalent ranking as that by expected utility maximization with the defined utility function. HP refers to the property that AFSD implies ASSD. However, the major previous ASD versions do not necessarily possess these two crucial properties. Specifically, Tzeng et al. (2013) point out that Leshno and Levy (2002) does not possess EUMP. Levy (1998, 2016) well study HP. However, Guo et al. (2013) point out that the ASD in Tzeng et al. (2013) does not possess HP. Guo et al. (2016) extend ASD definition in Tzeng et al. (2013) to discuss both risk seeker and risk averter. Accordingly, their ASD definition does not possess HP either. In short, those previous studies do not provide us satisfactory outcome.

Tsetlin et al. (2015) join the efforts to propose a generalized form of ASD, especially for the 2nd degree case which previous literatures debate most. The authors claim their definition for Generalized Almost Second-Degree Stochastic Dominance (GASSD) satisfies both HP and EUMP. In terms of HP, the authors first define the utility class which it is contained in the utility class of the 1st degree case under their selected conditions. In terms of EUMP, the authors further propose the theorem which is based on the 1st and 2nd degree of differentiation of the utility function. However, we notice that two major shortcomings that limits the application of their definition and lacks economic intuition. Accordingly, we minimize the specific assumptions about decision makers' risk attitudes and address those issues from the two perspectives in Sect. 3: defining utility function before CDF and the implementation. We show that the ASSD definition should avoid the two shortcomings. We need to provide the outcome intuitive and attain feasible implementation for its possible application. However, Tsetlin et al. (2015) admit that their version can only be investigated numerically. Their outcomes need additional relaxation of the conditions or assumption on probability shifts for SD relations to have a behavioral interpretation. We thus find it necessary to reexamine and redefine ASD.

We follow the classic concept of stochastic dominance in minimizing specific assumptions about decision makers' risk attitudes as highlighted by Levy (1992). Meanwhile, we need to secure the SD properties of EUMP and HP. We then propose and verify our ASD redefinition. We notice that our version avoid the two shortcomings in Tsetlin et al. (2015). We further reexamine the counterexamples in Tzeng et al. (2013) and Guo et al. (2013), which show the pitfalls of Leshno and Levy (2002) and Tzeng et al. (2013), respectively. In essence, these two counterexamples are not valid under our ASD redefinition. It accordingly validates that our version prevails.

We contribute to revise the Generalized Almost Stochastic Dominance definition in Tsetlin et al. (2015) in three major aspects. First, we minimize the specific assumption about decision makers' risk attitudes (Levy 1992), and satisfy both HP and EUMP properties (Guo et al. 2013). Second, contrary to their treatment, we define utility function before CDF and the implementation to avoid the possible shortcomings. Third, our definition can provide the economic intuition and attain feasible implementation for its possible application, while Tsetlin et al. (2015) cannot.

The remaining part of the paper is structured as follows. Section 2 discusses the ASD definition by Leshno and Levy (2002) and Tzeng et al. (2013). Section 3 discusses the GASSD in Tsetlin et al. (2015). Section 4 presents our ASD redefinition. Section 5 reexamines the counterexamples under our ASD definition. Section 6 concludes.

2 Discussion ASD on Leshno and Levy (2002) and Tzeng et al. (2013)

We focus our discussion on the versions of ASD definition by Leshno and Levy (2002) and Tzeng et al. (2013). The former initiates the topic of ASD definition and the latter presents a counterexample to the version by the former.

We first briefly review the ASD definitions by Leshno and Levy (2002). The authors denote F and G as the two CDFs of X and Y, respectively. Denoted by U_1 the set of all non-decreasing differentiable utility function ($u \in U_1(z)$ if $u' \geq 0$) and by U_2 the set of all non-decreasing and concave function, which is twice differentiable ($u \in U_2(z)$ if $u' \geq 0$ and $u'' \leq 0$). The authors impose restrictions on the utility functions, and the subsets of U_1 and U_2 are defined as follows. For every $0 < \varepsilon < 0.5$,

$$U_1^*(\varepsilon) = \left\{ u \in U_1 : u'(z) \leq \inf\left[u'(z)\right] \left[\frac{1}{\varepsilon} - 1\right] \forall z \in \left[\underline{z}, \overline{z}\right] \right\} \tag{P1}$$

and

$$U_2^*(\varepsilon) = \left\{ u \in U_2 : -u''(z) \leq \inf\left[-u''(z)\right] \left[\frac{1}{\varepsilon} - 1\right] \forall z \in \left[\underline{z}, \overline{z}\right] \right\}. \tag{P2}$$

Both refer to the respective minimums of the degrees of the non-decreasingness and concavity, respectively.

The ASD definitions are specified as follows. For every $0 < \varepsilon < 0.5$,

1. AFSD. F dominates G by ε-Almost FSD if and only if

$$\int_{S_1} \left[F(z) - G(z)\right] dz \leq \varepsilon \|F - G\|, \tag{1}$$

2. ASSD. F dominates G by ε-Almost SSD if and only if

$$\int_{S_2} \left[F(z) - G(z)\right] dt \leq \varepsilon \|F - G\| \quad \text{and} \quad \mathrm{E}_F(X) \geq \mathrm{E}_G(Y), \tag{2}$$

$$\text{where } S_1(F, G) = \left\{ z \in \left[\underline{z}, \overline{z}\right] : G(z) < F(z) \right\}, \tag{3}$$

$$S_2(F, G) = \{ z \in S_1(F, G) : \int_{\underline{z}}^{z} G(t)\, dt < \int_{\underline{z}}^{z} F(t)\, dt, \tag{4}$$

and $\|F - G\| = \int_{\underline{z}}^{\overline{z}} |F(z) - G(z)|\, dz$.

The authors allege the following theorem for ASD:

1. AFSD. F dominates G by ε-Almost FSD if and only if for all u in $U_1^*(\varepsilon)$, $\mathrm{E}_F(u) \geq \mathrm{E}_G(u)$.
2. ASSD. F dominates G by ε-Almost SSD if and only if for all u in $U_2^*(\varepsilon)$, $\mathrm{E}_F(u) \geq \mathrm{E}_G(u)$.

Tzeng et al. (2013) present a counterexample and claim that pitfall of the ASSD definition by Leshno and Levy (2002) is that for all u in $U_2^*(\varepsilon)$, the relation $\mathrm{E}_F(u) \geq \mathrm{E}_G(u)$ does not hold. That is, the definition by Leshno and Levy (2002) does not possess EUMP.

Further, Tzeng et al. (2013) treat AFSD as unchanged and revisit the ASSD definition by Leshno and Levy (2002). The authors propose a redefinition of ASD and claim its extension to Almost Nth-Degree Stochastic Dominance. First, the authors define the set of $\hat{S}_2$ (the violation area of SSD) as $\hat{S}_2(F, G) = \{z \in [\underline{z}, \overline{z}] : G^{(2)}(z) < F^{(2)}(z)\}$, where $F^{(2)}(z) = \int_{\underline{z}}^{z} F(t)\,dt$ and $G^{(2)}(z) = \int_{\underline{z}}^{z} G(t)\,dt$. The superscript of F and G denote the nth integration of their respective CDFs. It is noteworthy that $S_2(F, G) \subset \hat{S}_2(F, G)$ because $\hat{S}_2$ is not necessarily included in S_1 as defined in (3). Further, the authors follow the restrictions on the utility function by Leshno and Levy (2002). The authors claim that they provide the correct, necessary, and sufficient condition and define ASSD as follows. For $0 < \varepsilon < 0.5$, F dominates G by ε-Almost SSD if and only if

$$\int_{\hat{S}_2} \left[F^{(2)}(z) - G^{(2)}(z)\right] dx \leq \varepsilon \| F^{(2)} - G^{(2)} \| \quad \text{and} \quad \mathrm{E}_F(X) \geq \mathrm{E}_G(Y), \tag{5}$$

where $\| F^{(2)} - G^{(2)} \| = \int_{\underline{z}}^{\overline{z}} |F^{(2)}(z) - G^{(2)}(z)| dz$.

The authors thus define the theorem for ASSD as follows. For all u in $U_2^*(\varepsilon)$, $\mathrm{E}_F(u) \geq \mathrm{E}_G(u)$ if and only if $\int_{\hat{S}_2} \left[F^{(2)}(z) - G^{(2)}(z)\right] dx \leq \varepsilon \| F^{(2)} - G^{(2)} \|$ and $\mathrm{E}_F(X) \geq \mathrm{E}_G(Y)$. That is, ASSD is defined in terms of both area difference and expected utility inequality.

However, Guo et al. (2013) present a counterexample and claim that the pitfall of the ASSD definition by Tzeng et al. (2013) is that AFSD is not a sufficient condition for ASSD. That is, the definition by Tzeng et al. (2013) does not possess HP.

In brief, the ASD definitions by Leshno and Levy (2002) and Tzeng et al. (2013) have their respective pitfalls. Neither version has both crucial properties, EUMP and HP, respectively.

3 Discussion on the GASSD in Tsetlin et al. (2015)

Tsetlin et al. (2015) first define the utility class for GASSD:

$$\underline{U}_2(\varepsilon_1^*, \varepsilon_2^*) = \Big\{ u \,|\, u^{(1)} > 0,\ u^{(2)} < 0 \text{ and } \sup\left[(-1)^{k+1} u^{(k)}(z)\right] \leq \inf\left[(-1)^{k+1} u^{(k)}(z)\right]\left(\frac{1}{\varepsilon_k^*} - 1\right),\ k = 1, 2 \Big\} \tag{6}$$

Thus, GASSD does satisfy the hierarchy property because $\underline{U}_2(\varepsilon_1^*, \varepsilon_2^*) \subset \underline{U}_1(\varepsilon_1^*)$, where $\underline{U}_1(\varepsilon_1^*)$ is the utility class for AFSD defined in all previous literatures on ASD.

The authors claim that they provide the correct, necessary, and sufficient condition. $\underline{U}_2$ includes the 1st and the 2nd degrees of differentiation of utility function. They define GASSD as follows. For $0 \leq \varepsilon_k^* \leq 0.5$, $k = 1, 2$, F dominates G by $(\varepsilon_1^*, \varepsilon_2^*)$-GASSD if and only if

$$\mathrm{E}_F(u) \geq \mathrm{E}_G(u), \quad \text{for all } u \in \underline{U}_2(\varepsilon_1^*, \varepsilon_2^*) \tag{7}$$

The authors thus define the following theorem for GASSD as follows. For $0 \le \varepsilon_k^* \le 0.5$, $k = 1, 2$, F dominates G by $\left(\varepsilon_1^*, \varepsilon_2^*\right)$-GASSD if and only if $F^{(2)}(\bar{z}) - G^{(2)}(\bar{z}) \le 0$ and

$$max_C \left\{ \frac{1}{\left(1 - 2\varepsilon_2^*\right)|C| + \varepsilon_2^* \left(\bar{z} - \underline{z}\right)} \left[\left(1 - 2\varepsilon_2^*\right) \int_C (F^{(2)}(z) - G^{(2)}(z))\, dz + \varepsilon_2^* \int_S (F^{(2)}(z) - G^{(2)}(z))\, dz \right] \right\}$$
$$\le \frac{\varepsilon_1^*}{1 - 2\varepsilon_1^*} \left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right], \quad \text{where } C \subset \left[\underline{z}, \bar{z}\right] \text{ and } |C| = \int_C dz. \tag{8}$$

The condition $F^{(2)}(\bar{z}) - G^{(2)}(\bar{z}) \le 0$ in (8) is equivalent to the $E_F(X) \ge E_G(Y)$ in (5). In addition, if $\varepsilon_2^* = 0$, Tsetlin et al. (2015) express $(\varepsilon_1^*, 0)$-GASSD as follows: F dominates G by $(\varepsilon_1^*, 0)$-GASSD if and only if

$$F^{(2)}(\bar{z}) - G^{(2)}(\bar{z}) \le 0 \text{ and } max_{z \in [\underline{z}, \bar{z}]}[F^{(2)}(z) - G^{(2)}(z)] \le \frac{\varepsilon_1^*}{1 - 2\varepsilon_1^*} \left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right]. \tag{9}$$

However, their definition of GASSD have two major issues. First, they do not define GASSD based on distributions and violation areas that can provide economic intuitions. Stochastic dominance theory should be defined based on distribution, not on utility as shown in their study. In practice, utility function is not necessarily known or observable a priori. If we define based on an arbitrary utility function, the outcome of stochastic dominance analysis most likely lacks economic intuition. Furthermore, Guo et al. (2013) highlight both HP and EUMP as essential properties of SD. However, Tzeng et al. (2013) does not satisfy HP. The ASSD utility class in their article [$k = 2$ in (6)] is identical to equation P2 in this paper. Tsetlin et al. (2015) requires both equations of P1 and P2 [that is, $k = 1, 2$ in (6)] to satisfy both HP and EUMP. However, we follow Leshno and Levy (2002) to define ASD based on violation areas which provides economic intuition. The derived utility class can both minimize the assumption. Specifically, our utility class takes only equation P1 [$k = 1$ in (6)] and $u^{(2)} < 0$ (a component of P2) to satisfy both HP and EUMP or treats equation P2 as redundant. In essence, our version minimizes the sufficient assumptions to secure both HP and EUMP and provides economic intuition.

Second, Theorem 2 in Tsetlin et al. (2015) exists a potential drawback of $\left(\varepsilon_1^*, \varepsilon_2^*\right)$-GASSD for many situations. The authors admit that it is not feasible (or not practical) to obtain analytical results. Consequently, their GASSD can only be investigated numerically. This drawback limits their definition's application or implementation.

Based on our discussion on those two major issues, we minimize the sufficient assumptions and redefine ASSD. We will also show and prove our definition to avoid those shortcomings.

4 New definition of ASD

We choose not to follow Leshno and Levy (2002), Tzeng et al. (2013), and Levy (2016). Alternatively, we propose to define the various degrees of ASD in terms of violation ratio and U^*, in contrast with the framework in Tsetlin et al. (2015). We proceed to present our redefinition, lemma, and proposition of ASD in terms of AFSD and ASSD.

4.1 AFSD

Consider two alternative investments, X and Y with cumulative distribution functions $F(z)$ and $G(z)$, respectively. Define the range of possible outcomes by S:

$$S = \{z : \underline{z} < z < \overline{z}\}.$$

F Almost FSD dominates G means that $F(z) < G(z)$ for most of the range S, except for a relatively small segment that violates the dominance. Define the violation range of AFSD by S_1:

$$S_1 = \{z \in [\underline{z}, \overline{z}] : F(z) > G(z)\}$$

Further, we define the ratio between the area of the AFSD violation area and the total area of difference between the two cumulative distributions by ε_1:

$$\varepsilon_1 = \frac{\int_{S_1} [F(z) - G(z)]\, dz}{\int_{S_1} [F(z) - G(z)]\, dz + \int_{\bar{S}_1} [G(z) - F(z)]\, dz} = \frac{\int_{S_1} [F(z) - G(z)]\, dz}{\int_{S} |F(z) - G(z)|\, dz} \tag{10}$$

Definition 1 For $0 < \varepsilon < 0.5$, F ε-Almost FSD dominates G if and only if $\varepsilon_1 \leq \varepsilon$. The smaller ε_1, the stronger the AFSD. When $\varepsilon_1 = 0$, no violation area exists, the Almost FSD reduces to the standard FSD criterion.

Proposition 1 *$F \succ_{AFSD} G \Rightarrow c$. That is, if F ε-Almost FSD dominates G, then* $\mathrm{E}_F(X) > \mathrm{E}_G(Y)$.

See "Appendix A" for the proof. Proposition 1 states a sufficient but not necessary condition for $F \succ_{AFSD} G$. It requires that the S set for the specific order to be equal to zero to be a sufficient and necessary condition. For example, $S_1 \notin \varnothing$, then $\mathrm{E}_F(X) > \mathrm{E}_G(Y) \Rightarrow F \succ_{AFSD} G$. Similarly, it requires $S_2 \notin \varnothing$ to attain $F \succ_{ASSD} G$.

We define $\hat{U}_1^*(\varepsilon)$ as the set of utility functions, given by:

$$\hat{U}_1^*(\varepsilon) = \left\{u \in U_1 : u'(z) \leq \inf\left\{u'(z)\right\}\left[\frac{1}{\varepsilon} - 1\right] \forall z \in S\right\}. \tag{11}$$

Theorem 1 *F ε_1-Almost FSD dominates G if and only if* $\mathrm{E}_F[u(X)] \geq \mathrm{E}_G[u(Y)]$ *for all* $u \in \hat{U}_1^*(\varepsilon)$.

Proof See "Appendix B".

4.2 ASSD

Consider two alternative investments, X and Y with cumulative distribution functions F(z) and G(z), respectively. Define the Almost SSD dominance of F over G means that $\int_{\underline{z}}^{z} F(t)\, dt < \int_{\underline{z}}^{z} G(t)\, dt$ for most of the range S, except for a relatively small segment that violates the dominance. Define the area over which SSD is violated by S_2:

$$S_2 = \left\{z \in S_1 : F^{(2)}(z) > G^{(2)}(z)\right\},$$

where $F^{(2)}(z) = \int_{\underline{z}}^{z} F(t)\, dt$ and $G^{(2)}(z) = \int_{\underline{z}}^{z} G(t)\, dt$.

Define the ratio between the area of SSD violation area and the total area between the cumulative distributions by ε_2:

$$\varepsilon_2 = \frac{\int_{S_2} [F(z) - G(z)]\, dz}{\int_{S_2} [F(z) - G(z)]\, dz + \int_{\bar{S}_2} [G(z) - F(z)]\, dz} \neq \frac{\int_{S_2} [F(z) - G(z)]\, dz}{\int_{S} |F(z) - G(z)|\, dz} \tag{12}$$

We differentiate the difference area into two parts over S_2 and $\bar{S}_2$, while Leshno and Levy (2002) and Levy (2016) treat the integration of the whole area as one term, as indicated by (2). We expect our differentiation treatment can accurately depict the violation area because

S_2 handles $F > G$ with $(F - G)$ term, and $\bar{S}_2$ covers both $F > G$ and $G > F$ with $(G - F)$ term. Our treatment prevails, especially for the area over $\bar{S}_2$. This is because the cases of $F > G$ and $G > F$ occur and the integration over $\bar{S}_2$ may be offset and become smaller. The offset of individual integration parts usually occurs in the left parts. Consequently, the denominator in (12) will be smaller than that treated by the direct absolute value arrangement by Leshno and Levy (2002) and Levy (2016). Tsetlin et al. (2015) admit that the necessary and sufficient conditions for $\left(\varepsilon_1^*, 0\right)$-GASSD is not the same as the integral condition that is used for ASSD by Levy (2016). The outcomes in Leshno and Levy (2002) and Levy (2016) are thus not consistent with that in Tsetlin et al. (2015). We had better seek a better alternative and redefine ASSD. In addition, based on the calculated ε values for AFSD and ASSD, we can further determine the relation between AFSD and ASSD. The inequality sign in (12) indicates our ASSD definition is different from that in Leshno and Levy (2002) and Levy (2016).

Definition 2 For $0 < \varepsilon < 0.5$, F ε-Almost SSD dominates G if and only if $\varepsilon_2 \leq \varepsilon$. The smaller ε_2, the stronger the ASSD. When $\varepsilon_2 = 0$, no violation area exists, the Almost SSD reduces to the standard SSD criterion.

We notice that our definition of violation ratio is closely connected to the version in Tsetlin et al. (2015). Specifically,

$$
\begin{aligned}
\varepsilon_2 &= \frac{\int_{S_2} [F(z) - G(z)]\,\mathrm{dz}}{\int_{S_2} [F(z) - G(z)]\,\mathrm{dz} + \int_{\bar{S}_2} [G(z) - F(z)]\,\mathrm{dz}} \\
&= \frac{\int_{S_2} [F(z) - G(z)]\,\mathrm{dz}}{\int_{S} [G(z) - F(z)]\,\mathrm{dz} + 2\int_{S_2} [F(z) - G(z)]\,\mathrm{dz}} \\
&= \frac{\int_{S_2} [F(z) - G(z)]\,dz}{\mathrm{G}^{(2)}(\bar{z}) - \mathrm{F}^{(2)}(\bar{z}) + 2\int_{S_2} [F(z) - G(z)]\,\mathrm{dz}} \leq \varepsilon
\end{aligned}
$$

Then,

$$
\int_{S_2} [F(z) - G(z)]\,\mathrm{dz} \leq \frac{\varepsilon}{1 - 2\varepsilon} \left[\mathrm{G}^{(2)}(\bar{z}) - \mathrm{F}^{(2)}(\bar{z})\right],
$$

where $\int_{S_2} [F(z) - G(z)]\,\mathrm{dz} \geq \max_{z \in [\underline{z}, \bar{z}]} [\mathrm{F}^{(2)}(z) - \mathrm{G}^{(2)}(z)]$ is the violation area of SSD in the definition of ASSD. Therefore, our definition of violation ratio and Definition 2 implies the second condition in (9) [Eq. (2) in Tsetlin et al. 2015]. However, the second condition in (9) does not imply our violation ratio as defined in (12) and Definition 2 unless two more conditions are imposed: $0 < \varepsilon < 0.5$ and the max term in (9) should be positive. That is, our structure of the violation ratio definition is different from that in Tsetlin et al. (2015). Most of the difference essentially lies in that our version cannot contain those do not violate SSD definition and make ASSD definition meaningful.

Proposition 2 $F \succ_{ASSD} G \Rightarrow \mathrm{E}_F(X) > \mathrm{E}_G(Y)$. *If F ε-Almost SSD dominates G, then* $\mathrm{E}_F(X) > \mathrm{E}_G(Y)$.

Proof See "Appendix C".

Both Leshno and Levy (2002) and Tzeng et al. (2013) require $\mathrm{E}_F(X) \geq \mathrm{E}_G(Y)$ as the definition, as indicated by (2) and (5). Alternatively, we treat this condition as a proposition or theorem.

Define $\hat{U}_2^*(\varepsilon)$ is the set of utility functions, given by:

$$\hat{U}_2^*(\varepsilon) = \left\{ u \in U_2 : u'(z) \leq \inf\left[u'(z)\right]\left[\frac{1}{\varepsilon} - 1\right] \forall z \in S \right\}. \tag{13}$$

Our definition requires the 1st degree differentiation of the utility function and ε_2. In contrast, Leshno and Levy (2002) rely on a single ε value, as indicated by (2).

Theorem 2 *F is said to -Almost SSD dominates G if and only if* $\mathrm{E}_F\left[u\left(X\right)\right] \geq \mathrm{E}_G\left[u\left(Y\right)\right]$ *for all* $u \in \hat{U}_2^*(\varepsilon)$.

Proof See "Appendix D".

Proposition 3 *AFSD* $\Longrightarrow$ *ASSD. That is, F ε-Almost FSD dominates G, then F ε-Almost SSD dominates G.*

Proof See "Appendix E".

Our proposed redefinition of ASD differentiates the ε values for AFSD and ASSD. This framework also helps calibrate and connect the relationship between AFSD and ASSD.

4.3 Almost Nth-degree stochastic dominance

We extend our ASD redefinition to a higher order in this section. Consider two alternative investments, X and Y, with CDFs $F(\mathrm{z})$ and $G(\mathrm{z})$, respectively. Define the area over which Nth-Degree SD is violated by $\mathrm{S_N}$:

$$S_N = \left\{ \mathrm{z} \in S_{N-1} : F^{(N)}(\mathrm{z}) > G^{(N)}(\mathrm{z}) \right\},$$

where $F^{(N)}(z) = \int_{\underline{z}}^{z} F^{(N-1)}\left(t\right)dt$ and $G^{(N)}(\mathrm{z}) = \int_{\underline{z}}^{z} G^{(N-1)}dt$.

Define the ratio between the area of Nth-Degree SD violation and the total area between the cumulative distributions by ε_N:

$$\varepsilon_N = \frac{\int_{S_N}\left[F(z) - G(z)\right]dz}{\int_{S_N}\left[F(z) - G(z)\right]dz + \int_{\bar{S}_N}\left[G(z) - F(z)\right]dz} \neq \frac{\int_{S_N}\left[F(z) - G(z)\right]dz}{\int_{\mathrm{S}}\left|F(z) - G(z)\right|dz} \tag{14}$$

Definition 3 For $0 < \varepsilon < 0.5$, F ε-Almost Nth- Degree SD dominates G if and only if $\varepsilon_N \leq \varepsilon$. The smaller ε_N, the stronger the ANSD. When $\varepsilon_N = 0$, no violation area exists, Almost SD reduces to the standard Nth- Degree SD criterion.

Proposition 4 $F \succ_{ANSD} G \Rightarrow \mathrm{E}_F\left(X\right) > \mathrm{E}_G\left(Y\right)$. *If F ε-Almost Nth- Degree SD dominates G, then* $\mathrm{E}_F\left(X\right) > \mathrm{E}_G\left(Y\right)$.

Proof The proof is similar to the proof for Proposition 2.

Define $U_\mathrm{N} = \left\{\mathrm{u} : (-1)^{\mathrm{n}+1}\,\mathrm{u}^{(\mathrm{n})} \geq 0,\ \ \mathrm{n} = 1, 2, \ldots, \mathrm{N}\right\}$, where $u^{(\mathrm{n})}$ denotes the nth degree derivative of the utility function u. In addition, define $\hat{U}_N^*(\varepsilon)$ is the set of utility functions, given by:

$$\hat{U}_N^*(\varepsilon) = \left\{ u \in U_N : u'(z) \leq \inf\left[u'(z)\right]\left[\frac{1}{\varepsilon} - 1\right] \forall z \in S \right\}. \tag{15}$$

We present Lemma 1 as the prerequisite to prove Theorem 3.

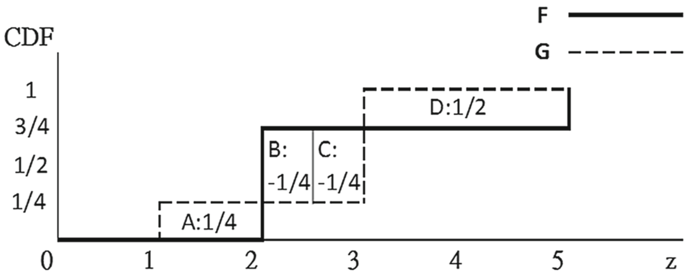

Fig. 1 The Payoff CDFs of F and G as defined in (16) and (17)

Lemma 1 *If* $u \in U_N$ *and* $z \in \bar{S}_N$*, then we have* $\int_{\bar{S}_N} u'(z)\,[G(z) - F(z)]\,dz \geq 0,\ \ for\ N = 1, 2, \ldots$

Proof See "Appendix F".

Theorem 3 *F ε-Almost Nth-Degree SD dominates G if and only if* $\mathrm{E}_F\,[u\,(X)] \geq \mathrm{E}_C\,[u\,(Y)]$ *for all* u $\in \hat{U}_N^*(\varepsilon)$.

Proof See "Appendix G".

Proposition 5 *Almost Nth-Degree SD* $\Longrightarrow$ *Almost N+1th-Degree SD. That is, F ε-Almost NSD dominates G, then F ε-Almost (N+1) SD dominates G.*

Proof The proof is similar to the proof for Proposition 3.

In terms of implementation, all the ε values in our definition are based on direct integration of violation and non-violation areas. Further numerical analysis can be avoided in our version but required in Tsetlin et al. (2015).

5 Reexamining the counterexamples under our redefinition of ASD

We reexamine two counterexamples in this section: the counterexample in Tzeng et al. (2013) to the ASD definition in Leshno and Levy (2002) and the counterexample by Guo et al. (2013) to the definition in Tzeng et al. (2013).

First, we outline the counterexample in Tzeng et al. (2013) to the version in Leshno and Levy (2002) as follows.

Let x ∈ [0, 5] . Assume the two payoff distributions are (Fig. 1)

$$\mathrm{F(z)} = \begin{cases} 0, & \text{if } 0 \leq z < 2, \\ \frac{3}{4}, & \text{if } 2 \leq z < 5, \\ 1, & \text{if } z = 5, \end{cases} \tag{16}$$

$$\mathrm{G(z)} = \begin{cases} 0, & \text{if } 0 \leq z < 1 \\ \frac{1}{4}, & \text{if } 1 \leq z < 3 \\ 1, & \text{if } 3 \leq z \leq 5 \end{cases} \tag{17}$$

We follow (3)–(4) and define $S_1 = \{\mathrm{z{:}F(z)} > G(\mathrm{z})\} = \{\mathrm{z : z} \in [2, 3]\}$ and $S_2 = \left\{\mathrm{z} \in S_1{:}\ \int_0^z \mathrm{F\,(t)}\,dt > \int_0^z \mathrm{G\,(t)}\,dt\right\} = \{\mathrm{z : z} \in [5/2, 3]\}$. If we follow (2) as defined in Leshno and Levy (2002), the value $\frac{1/4}{|1/4|+|-1/4|+|-1/4|+|1/2|} = \frac{1}{5} \leq \varepsilon < 0.5$ satisfies ASSD condition.

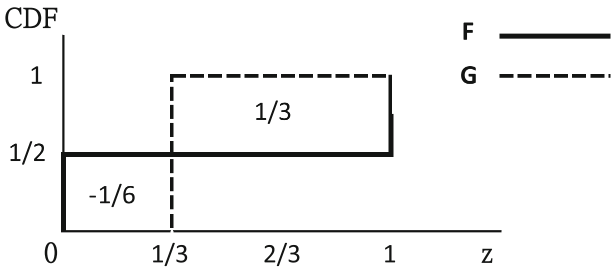

Fig. 2 The Payoff CDFs of F and G as defined in (18) and (19)

Their absolute value treatment inflates the denominator and deliberately decreases the ratio. Accordingly, it helps ASSD to hold.

Tzeng et al. (2013) further define a utility function whose marginal utility satisfies the following conditions:

$$\mathrm{u}'(\mathrm{z}) = \begin{cases} \frac{21}{2} - z, & \text{if } 0 \le z \le \frac{5}{2}, \\ 18 - 4z, & \text{if } \frac{5}{2} \le z \le 4, \\ 6 - z, & \text{if } 4 \le z \le 5. \end{cases}$$

The authors show that the defined utility function lies in U_2^* (ε) as defined in (13), but $\mathrm{E}_F\left[u\left(X\right)\right] < \mathrm{E}_G\left[u\left(Y\right)\right]$. This violates the theorem for ASSD in Leshno and Levy (2002).

Furthermore, we follow our ASD redefinition and reexamine the counterexample in Tzeng et al. (2013). Actually, the defined utility function does not lie in the utility set $\hat{U}_2^*(\varepsilon)$ even though $\varepsilon_2 = \frac{1/4}{1/4+1/4+(-1/4)+1/2} = 1/3 \le \varepsilon < 0.5$ satisfies our ASSD condition. That is, the $\sup\left[u'(z)\right]$ occurs when z = 0 and $\frac{21}{2} = \sup\left[u'(z)\right] > u'(z) > inf\left[u'(z)\right]\left[\frac{1}{\varepsilon_2} - 1\right] = 1 * 2$. Yet, we expect the former to be less than or equal to the latter, as indicated in (13). Thus, the utility used in Tzeng et al. (2013) does not lie in the utility set $\hat{U}_2^*(\varepsilon)$, as defined in (13). Accordingly, we do not need to further discuss the relation between $\mathrm{E}_F\left[u\left(X\right)\right]$ and $\mathrm{E}_G\left[u\left(X\right)\right]$. Since we cannot compare $\mathrm{E}_F\left[u\left(X\right)\right]$ and $\mathrm{E}_G\left[u\left(X\right)\right]$, this specific counterexample is valid to Leshno and Levy (2002), but not applicable to our Theorem 2.

Next, Guo et al. (2013) do not propose their ASD redefinition but present a counterexample to the version in Tzeng et al. (2013). We outline the counterexample in Guo et al. (2013) as follows. Assume the two payoff distributions are defined and depicted (Fig. 2) as:

$$\mathrm{F(z)} = \begin{cases} \frac{1}{2} \ if \ 0 \le z < 1, \\ 1 \ if \ z = 1, \end{cases} \tag{18}$$

$$G(z) = \begin{cases} 0 \ if \ 0 \le z < 1/3 \\ 1 \ if \ 1/3 \le z \le 1 \end{cases} \tag{19}$$

$$S_1 = \{\mathrm{z{:}F(z)} > G\mathrm{(z)}\} = \left\{\mathrm{z} : \mathrm{z} \in \left[0, \frac{1}{3}\right]\right\}$$

Guo et al. (2013) show that their counterexample satisfies AFSD but not the ASSD definition in Tzeng et al. (2013). That is, the version in Tzeng et al. (2013) does not possess HP.

Alternatively, we follow our ASD redefinition and calculate the respective $\varepsilon_i (\forall i = 1, 2)$ values for AFSD and ASSD. ε_1 value indicates that F ε-Almost FSD dominates G for $\varepsilon_1 = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{3} \le \varepsilon < 0.5$. ε_2 value also shows the positive outcome since $\varepsilon_2 = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} =$

$\frac{1}{3} \leq \varepsilon_1 \leq \varepsilon < 0.5$; that is, F ε-Almost SSD dominates G. Thus, AFSD implies ASSD and HP holds. We confirm that HP holds in the counterexample in Guo et al. (2013) if we follow our ASD definition. That is, our version of the ASD definition does not violate HP.

In short, our ASD definition possesses both HP and EUMP.

6 Conclusion

Almost Stochastic Dominance has considerable room for application, but the previous studies do not reach a generally accepted definition, especially on the second-degree case. We contribute with our version of ASD, which can even be extended to Nth-Degree case.

We review the GASSD definition in Tsetlin et al. (2015) and address its shortcomings in two aspects: defining utility function before CDF and the implementation. We also show that our version contributes to avoid the two shortcomings and secure economic intuition. Further, we follow our ASSD definition that minimizes the assumption and reexamine the respective counterexamples in Tzeng et al. (2013) and Guo et al. (2013). The former to the ASD definition in Leshno and Levy (2002) is valid, but this version does not apply to our ASD theorem. The latter in Guo et al. (2013) possesses HP under our ASD redefinition. These investigations prove that our ASD redefinition prevails.

Appendix A: Proof to Proposition 1

1.

$$\begin{aligned}\varepsilon_1 &= \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_S |F(z)-G(z)|\,dz} = \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_{S_1}[F(z)-G(z)]\,dz + \int_{\bar{S}_1}[G(z)-F(z)]\,dz}\\ &= \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_{S_1}[F(z)-G(z)]\,dz + \int_{\bar{S}_1}[G(z)-F(z)]\,dz + \int_{S_1}[G(z)-F(z)]\,dz - \int_{S_1}[G(z)-F(z)]\,dz}\\ &= \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_S [G(z)-F(z)]\,dz + 2\int_{S_1}[F(z)-G(z)]\,dz} = \frac{VA_1}{E_F - E_G + 2VA_1},\end{aligned}$$

where VA_1 denotes the FSD violation area and E_F and E_G denote $E_F(X)$ and $E_G(Y)$, respectively.

For $S_1 \neq \emptyset$, $VA_1 > 0$, if $E_F(X) \leq E_G(Y)$, then $\varepsilon_1 \geq 0.5$. Therefore, if F ε-Almost FSD dominates G, then $E_F(X) > E_G(Y)$.

2. If you add the condition, $S_1 \notin \emptyset$, then $E_F(X) > E_G(Y) \Rightarrow F \succ_{AFSD} G$.
For $S_1 \notin \emptyset$, $VA_1 > 0$, if $E_F > E_G$, then $\varepsilon_1 \leq \varepsilon < 0.5$, F ε-Almost FSD dominates G.

Appendix B: Proof to Theorem 1

1. "Only if" part: We show that if

$$\varepsilon_1 \leq \varepsilon \tag{B1}$$

then $E_F[u(X)] \geq E_G[u(Y)]$ for all $u \in \hat{U}_1^*(\varepsilon)$. By integration by parts, we have

$$\mathrm{E_F}\left[\mathrm{u}\left(\mathrm{X}\right)\right] = \int_{\underline{z}}^{\bar{z}} \mathrm{u(z)d}F(\mathrm{z}) = \left[\mathrm{u(z)}F(\mathrm{z})\right]_{\underline{z}}^{\bar{z}} - \int_{\underline{z}}^{\bar{z}} \mathrm{u}'(\mathrm{z})F(\mathrm{z})\mathrm{dz} = \mathrm{u}\left(\bar{\mathrm{x}}\right) - \int_{\underline{z}}^{\bar{z}} \mathrm{u}'(\mathrm{z})F(\mathrm{z})\mathrm{dz}.$$

and

$$\begin{aligned}\mathrm{E_F}\left[\mathrm{u}\left(\mathrm{X}\right)\right] - \mathrm{E_G}\left[\mathrm{u}\left(\mathrm{Y}\right)\right] &= \int_{\underline{z}}^{\bar{z}} u'(\mathrm{z})\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} \\ &= \int_{\mathrm{S_1}} u'(\mathrm{z})\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} \\ &\quad + \int_{\bar{S}_1} u'(\mathrm{z})\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz},\end{aligned} \tag{B2}$$

where over S_2, $F(\mathrm{z}) > G(\mathrm{z})$ and $\bar{S}_1$ is the compliment of S_1 in $\left[\underline{\mathrm{x}}, \bar{\mathrm{x}}\right]$. If $u \in U_1$, then the first integral part is non-positive, $\int_{\mathrm{S_1}} u'(\mathrm{z})\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} \leq 0$ and we could have FSD of F over G under $\bar{S}_1$, that is $G(\mathrm{z}) - F(\mathrm{z}) > 0$ for all $\mathrm{z} \in \bar{S}_1$. Note that as $u \in U_1$ ($u' > 0$) the integral over $\bar{S}_1$ is nonnegative, $\int_{\bar{S}_1} u'(\mathrm{z})\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} \geq 0$. Denote that $\inf\left[\mathrm{u}'\left(\mathrm{x}\right)\right] = \underline{\theta}$ and $\sup\left[u'\left(\mathrm{x}\right)\right] = \bar{\theta}$. Thus we have

$$\begin{aligned}&\mathrm{E_F}\left[u\left(\mathrm{X}\right)\right] - \mathrm{E_G}\left[u\left(\mathrm{Y}\right)\right] \geq \bar{\theta}\int_{\mathrm{S_1}}\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} + \underline{\theta}\int_{\bar{S}_1}\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz} \\ &\quad = -\left(\bar{\theta} + \underline{\theta}\right)\int_{\mathrm{S_1}}\left[F(\mathrm{z}) - G(\mathrm{z})\right]\mathrm{dz} + \underline{\theta}\left\{\int_{\mathrm{S_1}}\left[F(\mathrm{z}) - G(\mathrm{z})\right]\mathrm{dz} + \int_{\bar{S}_1}\left[G(\mathrm{z}) - F(\mathrm{z})\right]\mathrm{dz}\right\}\mathrm{c}\end{aligned} \tag{B3}$$

Since $\in \hat{U}_1^*(\varepsilon)$, by definition, we have $\bar{\theta} \leq \underline{\theta}\left(1/\varepsilon - 1\right)$; i.e., $\varepsilon \leq \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. By (B1), we have

$$\varepsilon_1 = \frac{\int_{\mathrm{S_1}}\left[\mathrm{F(z) - G(z)}\right]\mathrm{dz}}{\int_{\mathrm{S_1}}\left[\mathrm{F(z) - G(z)}\right]\mathrm{dz} + \int_{\bar{S}_1}\left[\mathrm{G(z) - F(z)}\right]\mathrm{dz}} \leq \varepsilon \leq \frac{\underline{\theta}}{\left(\bar{\theta} + \underline{\theta}\right)} \tag{B4}$$

By (B3) and (B4), we prove that $\mathrm{E_F}\left[u\left(X\right)\right] - \mathrm{E_G}\left[u\left(Y\right)\right] \geq 0$ for all $u \in U_1^*(\varepsilon)$.

2. "If" part: We show that if

$$\varepsilon_1 > \varepsilon \tag{B5}$$

then there exist a $u \in \hat{U}_1^*(\varepsilon)$ such $\mathrm{E_F}\left[u\left(X\right)\right] - \mathrm{E_G}\left[u\left(Y\right)\right] < 0$.

It is obvious that $\mathrm{u} \in \hat{U}_1^*(\varepsilon)$, $\varepsilon = \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. With no loss of generality, we assume that $\mathrm{S_1}$ is an interval and denote $S_1 = [a, b]$, and $\bar{S}_1 = \overline{[a, b]}$ (the complement of $[a, b]$ in $\left[\underline{z}, \bar{\mathrm{z}}\right]$. Define

$$u(z) = \begin{cases}\underline{\theta}z & \text{if } \underline{z} \leq z \leq a \\ \bar{\theta}\left(z - a\right) + \underline{\theta}a & \text{if } a \leq z \leq b \\ \underline{\theta}\left(z - b\right) + \bar{\theta}b + \underline{\theta}a & \text{if } b \leq z \leq \bar{\mathrm{z}}\end{cases}.$$

We have

$$\begin{aligned}&\mathrm{E_F}\left[u\left(X\right)\right] - \mathrm{E_G}\left[u\left(Y\right)\right] = \int_{\mathrm{S_1}} u'(z)\left[G(z) - F(z)\right]\mathrm{dz} + \int_{\bar{S}_1} u'(z)\left[G(z) - F(z)\right]\mathrm{dz} \\ &\quad = \bar{\theta}\int_{\mathrm{S_1}}\left[G(z) - F(z)\right]\mathrm{dz} + \underline{\theta}\int_{\bar{S}_1}\left[G(z) - F(z)\right]\mathrm{dz} \\ &\quad = -\left(\bar{\theta} + \underline{\theta}\right)\int_{\mathrm{S_1}}\left[F(z) - G(z)\right]\mathrm{dz} + \underline{\theta}\left\{\int_{\mathrm{S_1}}\left[F(z) - G(z)\right]\mathrm{dz} + \int_{\bar{S}_1}\left[G(z) - F(z)\right]\mathrm{dz}\right\}\end{aligned}$$

We assume that

$$\varepsilon_1 = \frac{\int_{S_1} [F(z) - G(z)]\,dz}{\int_{S_1} [F(z) - G(z)]\,dz + \int_{\bar{S}_1} [G(z) - F(z)]\,dz} > \varepsilon > \frac{\underline{\theta}}{(\bar{\theta} + \underline{\theta})},$$

thus we have $E_F[u(X)] - E_G[u(Y)] < 0$. This completes the proof of the if part of Theorem 1.

Appendix C: Proof to Proposition 2

1.
$$\begin{aligned}\varepsilon_2 &= \frac{\int_{S_2} [F(z) - G(z)]\,dz}{\int_{S_2} [F(z) - G(z)]\,dz + \int_{\bar{S}_2} [G(z) - F(z)]\,dz} \\ &= \frac{\int_{S_2} [F(z) - G(z)]\,dz}{\int_{S_2} [F(z) - G(z)]\,dz + \int_{\bar{S}_2} [G(z) - F(z)]\,dz + \int_{S_2} [G(z) - F(z)]\,dz - \int_{S_2} [G(z) - F(z)]\,dz} \\ &= \frac{\int_{S_2} [F(z) - G(z)]\,dz}{\int_{S} [G(z) - F(z)]\,dz + 2\int_{S_2} [F(z) - G(z)]\,dz} = \frac{VA_2}{E_F - E_G + 2VA_2},\end{aligned}$$

   where VA_2 denotes the SSD violation area and E_F and E_G denote $E_F(X)$ and $E_G(Y)$, respectively.
   For $S_2 \neq \emptyset$, $VA_2 > 0$, if $E_F(X) < E_G(Y)$, then $\varepsilon_2 > 0.5$. Therefore, if F ε-Almost SSD dominates G, then $E_F(X) > E_G(Y)$.
2. If you have additional condition, $S_2 \notin \emptyset$, then $E_F(X) > E_G(Y) \Rightarrow F \succ_{ASSD} G$.
   For $S_2 \notin \emptyset$, $VA_2 > 0$, if $E_F > E_G$, then $\varepsilon_2 \leq \varepsilon < 0.5$, F ε-Almost SSD dominates G.

Appendix D: Proof to Theorem 2

1. "Only if" part: We show that if

$$\varepsilon_2 \leq \varepsilon \tag{D1}$$

   then $E_F[u(X)] \geq E_G[u(Y)]$ for all $u \in \hat{U}_2^*(\varepsilon)$. By integration by parts, we have

$$E_F[u(X)] = \int_{\underline{z}}^{\bar{z}} u(z)dF(z) = [u(z)F(z)]_{\underline{z}}^{\bar{z}} - \int_{\underline{z}}^{\bar{z}} u'(z)F(z)dz = u(\bar{z}) - \int_{\underline{z}}^{\bar{z}} u'(z)F(z)dz.$$

   and

$$\begin{aligned}E_F[u(X)] - E_G[u(Y)] &= \int_{\underline{z}}^{\bar{z}} u'(z)[G(z) - F(z)]\,dz \\ &= \int_{S_2} u'(z)[G(z) - F(z)]\,dz \\ &\quad + \int_{\bar{S}_2} u'(z)[G(z) - F(z)]\,dz,\end{aligned} \tag{D2}$$

   where over S_2, $F(z) > G(z)$ and $\bar{S}_2$ is the compliment of S_2 in $[\underline{z}, \bar{z}]$. If $u \in U_2$, then the first integral part is non-positive, $\int_{S_2} u'(z)[G(z) - F(z)]\,dz \leq 0$ and we could have SSD of F over G under $\bar{S}_2$, that is $\int_{\underline{z}}^{z} [G(t) - F(t)]\,dt > 0$ for all $z \in \bar{S}_2$. Note that as $u \in U_2$

($u'$is decreasing) the integral over $\bar{S}_2$ is nonnegative, $\int_{\bar{S}_2} u'(z)\,[G(z) - F(z)]\,\mathrm{dz} \geq 0$. Denote that $\inf\left[u'(x)\right] = \underline{\theta}$ and $\sup\left[u'(x)\right] = \bar{\theta}$. Thus, we have

$$\mathrm{E_F}\left[u(X)\right] - \mathrm{E_G}\left[u(Y)\right] \geq \bar{\theta} \int_{S_2} [G(z) - F(z)]\,\mathrm{dz} + \underline{\theta} \int_{\bar{S}_2} [G(z) - F(z)]\,\mathrm{dz}$$
$$= -\left(\bar{\theta} + \underline{\theta}\right) \int_{S_2} [F(z) - G(z)]\,\mathrm{dz} + \underline{\theta} \left\{ \int_{S_2} [F(z) - G(z)]\,\mathrm{dz} + \int_{\bar{S}_2} [G(z) - F(z)]\,\mathrm{dz} \right\} \tag{D3}$$

Since $u \in U_2^*(\varepsilon)$, by definition, we have $\bar{\theta} \leq \underline{\theta}\,(1/\varepsilon - 1)$; i.e., $\varepsilon \leq \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. By (D1), we have

$$\varepsilon_2 = \frac{\int_{S_2} [\mathrm{F(z) - G(z)}]\,\mathrm{dz}}{\int_{S_2} [\mathrm{F(z) - G(z)}]\,\mathrm{dz} + \int_{\bar{S}_2} [\mathrm{G(z) - F(z)}]\,\mathrm{dz}} \leq \varepsilon \leq \frac{\underline{\theta}}{\left(\bar{\theta} + \underline{\theta}\right)} \tag{D4}$$

By (D3) and (D4), we prove that $\mathrm{E_F}\left[\mathrm{u(X)}\right] - \mathrm{E_G}\left[\mathrm{u(Y)}\right] \geq 0$ for all $\mathrm{u} \in \hat{U}_2^*(\varepsilon)$.
The integration specified in Appendix D occurs over both $\bar{S}_2$ and S_2, but the dispute lies on the integration over $\bar{S}_2$; that is, $\int_{\bar{S}_2} u'(z)\,[G(z) - F(z)]\,\mathrm{dz}$. Take Fig. 1 for example, $\bar{S}_2$ includes A, B, and D. Thus, the integration from the origin should be positive. That is, when $z_1 \in \bar{S}_2$, we do not have $F(z_1) \leq G(z_1)$, but $F^{(2)}(z_1) > G^{(2)}(z_1)$.

2. “If” part: We show that if

$$\varepsilon_2 > \varepsilon \tag{D5}$$

then there exist a $\mathrm{u} \in \hat{U}_2^*(\varepsilon)$ such $\mathrm{E_F}\left[\mathrm{u(X)}\right] - \mathrm{E_G}\left[\mathrm{u(Y)}\right] < 0$.
It is obvious that $\mathrm{u} \in \hat{U}_2^*(\varepsilon)$, $\varepsilon = \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. With no loss of generality, we assume that S_2 is an interval and denote $S_2 = [a, b]$, and $\bar{S}_2 = \overline{[a, b]}$ (the complement of $[a, b]$ in $\left[\underline{z}, \bar{z}\right]$. Define

$$u(z) = \begin{cases} \underline{\theta} z & \text{if } \underline{z} \leq z \leq a \\ \bar{\theta}\,(z - a) + \underline{\theta} a & \text{if } a \leq z \leq b \\ \underline{\theta}\,(z - b) + \bar{\theta} b + \underline{\theta} a & \text{if } b \leq z \leq \bar{z} \end{cases}.$$

We have

$$\mathrm{E_F}\left[u(X)\right] - \mathrm{E_G}\left[u(Y)\right] = \int_{S_2} u'(z)\,[G(z) - F(z)]\,\mathrm{dz} + \int_{\bar{S}_2} u'(z)\,[G(z) - F(z)]\,\mathrm{dz}$$
$$= \bar{\theta} \int_{S_2} [G(z) - F(z)]\,\mathrm{dz} + \underline{\theta} \int_{\bar{S}_2} [G(z) - F(z)]\,\mathrm{dz}$$
$$= -\left(\bar{\theta} + \underline{\theta}\right) \int_{S_2} [F(z) - G(z)]\,\mathrm{dz} + \underline{\theta} \left\{ \int_{S_2} [F(z) - G(z)]\,\mathrm{dz} + \int_{\bar{S}_2} [G(z) - F(z)]\,\mathrm{dz} \right\}$$

We assume that

$$\varepsilon_2 = \frac{\int_{S_2} [\mathrm{F(z) - G(z)}]\,\mathrm{dz}}{\int_{S_2} [\mathrm{F(z) - G(z)}]\,\mathrm{dz} + \int_{\bar{S}_2} [\mathrm{G(z) - F(z)}]\,\mathrm{dz}} > \varepsilon > \frac{\underline{\theta}}{\left(\bar{\theta} + \underline{\theta}\right)},$$

We thus have $\mathrm{E_F}\left[u(X)\right] - \mathrm{E_G}\left[u(Y)\right] < 0$. This completes the proof of the if part of Theorem 2.

Appendix E: Proof to Proposition 3

$$\bar{S}_2 = \bar{S}_1 + (S_1 - S_2)$$

$$\varepsilon_1 = \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_S |F(z)-G(z)|\,dz} = \frac{\int_{S_1}[F(z)-G(z)]\,dz}{\int_{S_1}[F(z)-G(z)]\,dz + \int_{\bar{S}_1}[G(z)-F(z)]\,dz}$$

$$\varepsilon_2 = \frac{\int_{S_2}[F(z)-G(z)]\,dz}{\int_{S_2}[F(z)-G(z)]\,dz + \int_{\bar{S}_2}[G(z)-F(z)]\,dz}$$

$$= \frac{\int_{S_2}[F(z)-G(z)]\,dz}{\int_{S_2}[F(z)-G(z)]\,dz + \int_{\bar{S}_1}[G(z)-F(z)]\,dz + \left\{\int_{S_1}[G(z)-F(z)]\,dz - \int_{S_2}[G(z)-F(z)]\,dz\right\}}$$

Define $a = \int_{S_1}[F(z)-G(z)]\,dz > 0$, $b = \int_{\bar{S}_1}[G(z)-F(z)]\,dz > 0$, $c = \int_{S_2}[F(z)-G(z)]\,dz > 0$ $S_2 \subset S_1$, this means $a \geq c$.

Then, $\varepsilon_1 = \frac{a}{a+(b-a)+a}$ and $\varepsilon_2 = \frac{c}{c+(b-a)+c}$.

If $\varepsilon_1 < \varepsilon < 0.5$ and $a \geq c$, then $b > a$ and $\varepsilon_2 \leq \varepsilon_1 < \varepsilon < 0.5$. Thus, ASSD holds.

Appendix F: Proof to Lemma 1

When n = 1,

$$G(z) - F(z) > 0 \quad \text{for all } z \in \bar{S}_1.$$

If $u'(z) \geq 0$, then $\int_{\bar{S}_1} u'(z)[G(z)-F(z)]\,dz \geq 0$.

Thus, as $u \in U_1$ the integral over $\bar{S}_1$ is nonnegative.

When n = 2,

$$G^{(2)}(z) - F^{(2)}(z) = \int_{\underline{z}}^{z} [G(t) - F(t)]\,dt > 0 \quad \text{for all } z \in \bar{S}_2.$$

If $u'(z) \geq 0$ and $u''(z) \leq 0$ (this means $u'(z)$ is non-increasing), then $\int_{\bar{S}_2} u'(z)[G(z)-F(z)]\,dz \geq 0$.

Therefore, as $u \in U_2$ the integral over $\bar{S}_2$ is nonnegative.

When n = 3,

$$\int_{\bar{S}_3} u'(z)[G(z)-F(z)]\,dz = u'(\bar{z})\left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right] + \int_{\bar{S}_3} -\left[u''(z)\right]\left[G^{(2)}(z) - F^{(2)}(z)\right] dz$$

$$G^{(3)}(z) - F^{(3)}(z) = \int_{\underline{z}}^{z}\left[G^{(2)}(t) - F^{(2)}(t)\right] dt > 0 \quad \text{for all } z \in \bar{S}_3.$$

If $u''(z) \leq 0$ and $u'''(z) \geq 0$ (this means $-u''(z)$ is non-increasing), then $\int_{\bar{S}_3} -\left[u''(z)\right]\left[G^{(2)}(z) - F^{(2)}(z)\right] dz \geq 0$.

Since $u'(\bar{z})\left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right] > 0$ and $\int_{\bar{S}_3} -\left[u''(z)\right]\left[G^{(2)}(z) - F^{(2)}(z)\right] dz \geq 0$, thus $\int_{\bar{S}_3} u'(z)[G(z)-F(z)]\,dz \geq 0$

Therefore, as $u \in U_3$ the integral over $\bar{S}_3$ is nonnegative.

When n = N > 3,

$$\int_{\bar{S}_N} u'(z)[G(z)-F(z)]\,dz = u'(\bar{z})\left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right] + \int_{\bar{S}_N} -\left[u''(z)\right]\left[G^{(2)}(z) - F^{(2)}(z)\right] dz$$

$$= u'(\bar{z})\left[G^{(2)}(\bar{z}) - F^{(2)}(\bar{z})\right] + \left[-u''(\bar{z})\right]\left[G^{(3)}(\bar{z}) - F^{(3)}(\bar{z})\right] + \int_{\bar{S}_N} u''(z)\left[G^{(3)}(z) - F^{(3)}(z)\right] dz$$

$$= \cdots$$

$$= \sum_{n=2}^{N-1} (-1)^n u^{(n-1)}(\bar{z}) \left[G^{(n)}(\bar{z}) - F^{(n)}(\bar{z})\right] + \int_{\bar{S}_N} (-1)^N \left[u^{(N-1)}(z)\right]\left[G^{(N-1)}(z) - F^{(N-1)}(z)\right] dz$$

$G^{(N)}(z) - F^{(N)}(z) = \int_{\underline{z}}^{z} \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] dt > 0$ for all $z \in \bar{S}_N$.

If $(-1)^{N-1} u^{(N-1)}(z) \le 0$ and $(-1)^{N-1} u^N(z) \ge 0$ (this means $(-1)^N u^{(N-1)}(z)$ is non-increasing), then $\int_{\bar{S}_N} (-1)^N \left[u^{(N-1)}(z)\right]\left[G^{(N-1)}(z) - F^{(N-1)}(z)\right] dz \ge 0$.

Since $\sum_{n=2}^{N-1} (-1)^n u^{(n-1)}(\bar{z}) \left[G^{(n)}(\bar{z}) - F^{(n)}(\bar{z})\right] > 0$ and $\int_{\bar{S}_N} (-1)^N \left[u^{(N-1)}(z)\right] \left[G^{(N-1)}(z) - F^{(N-1)}(z)\right] dz \ge 0$, thus $\int_{\bar{S}_N} u'(z)\left[G(z) - F(z)\right] dz \ge 0$

Therefore, as $u \in U_N$ the integral over $\bar{S}_N$ is nonnegative.

Appendix G: Proof to Theorem 3

1. "Only if" part: We show that if

$$\varepsilon_N \le \varepsilon \tag{G1}$$

then $E_F[u(X)] \ge E_G[u(Y)]$ for all $u \in U_N^*(\varepsilon)$. By integration by parts, we have

$$E_F[u(X)] = \int_{\underline{z}}^{\bar{z}} u(z)dF(z) = [u(z)F(z)]_{\underline{z}}^{\bar{z}} - \int_{\underline{z}}^{\bar{z}} u'(z)F(z)dz = u(\bar{z}) - \int_{\underline{z}}^{\bar{z}} u'(z)F(z)dz.$$

and

$$\begin{aligned} E_F[u(X)] - E_G[u(Y)] &= \int_{\underline{z}}^{\bar{z}} u'(z)\left[G(z) - F(z)\right] dz \\ &= \int_{S_N} u'(z)\left[G(z) - F(z)\right] dz \\ &\quad + \int_{\bar{S}_N} u'(z)\left[G(z) - F(z)\right] dz, \end{aligned} \tag{G2}$$

where over S_N, $F(z) > G(z)$ and $\bar{S}_N$ is the compliment of S_N in $[\underline{z}, \bar{z}]$. If $u \in U_N$, then the first integral part is non-positive, $\int_{S_N} u'(z)\left[G(z) - F(z)\right] dz \le 0$ and we could have Nth-Degree SD of F over G under $\bar{S}_N$, that is $G^{(N)}(z) - F^{(N)}(z) > 0$ for all $z \in \bar{S}_N$. In addition, Lemma 1 specifies if $z \in \bar{S}_N$ and $(-1)^n u^{(n-1)} \ge 0$, $n = 2, 3, \ldots, N$, then the second part integral is also non-negative, $\int_{\bar{S}_N} u'(z)\left[G(z) - F(z)\right] dz \ge 0$. Denote that $\inf\left[u'(z)\right] = \underline{\theta}$ and $\sup[u'(z)] = \bar{\theta}$. Thus we have

$$E_F[u(X)] - E_G[u(Y)] \ge \bar{\theta} \int_{S_N} \left[G(z) - F(z)\right] dz + \underline{\theta} \int_{\bar{S}_N} \left[G(z) - F(z)\right] dz$$

$$= -\left(\bar{\theta} + \underline{\theta}\right) \int_{S_N} \left[F(z) - G(z)\right] dz + \underline{\theta} \left\{ \int_{S_N} \left[F(z) - G(z)\right] dz + \int_{\bar{S}_N} \left[G(z) - F(z)\right] dz \right\} \tag{G3}$$

Since $u \in U_N^*(\varepsilon)$, by definition, we have $\bar{\theta} \le \underline{\theta}(1/\varepsilon - 1)$; i.e., $\varepsilon \le \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. By (G1), we have

$$\varepsilon_N = \frac{\int_{S_N} \left[F(z) - G(z)\right] dz}{\int_{S_N} \left[F(z) - G(z)\right] dz + \int_{\bar{S}_N} \left[G(z) - F(z)\right] dz} \le \varepsilon \le \frac{\underline{\theta}}{\left(\bar{\theta} + \underline{\theta}\right)} \tag{G4}$$

By (G3) and (G4), we prove that $E_F[u(X)] - E_G[u(Y)] \ge 0$ for all $u \in U_N^*(\varepsilon)$.

2. "If" part: We show that if

$$\varepsilon_N > \varepsilon \tag{G5}$$

then there exist a $u \in U_N^*(\varepsilon)$ such $E_F[u(X)] - E_G[u(Y)] < 0$.
It is obvious that $u \in U_N^*(\varepsilon)$, $\varepsilon = \underline{\theta}/\left(\bar{\theta} + \underline{\theta}\right)$. With no loss of generality, we assume that S_N is an interval and denote $S_N = [a, b]$, and $\bar{S}_N = \overline{[a, b]}$ (the complement of $[a, b]$ in $\left[\underline{z}, \bar{z}\right]$. Define

$$u(z) = \begin{cases} \underline{\theta} z & \text{if } \underline{z} \le z \le a \\ \bar{\theta}(z-a) + \underline{\theta} a & \text{if } a \le z \le b \\ \underline{\theta}(z-b) + \bar{\theta} b + \underline{\theta} a & \text{if } b \le z \le \bar{z} \end{cases}.$$

We have

$$\begin{aligned} E_F[u(X)] - E_G[u(Y)] &= \int_{S_N} u'(z)[G(z) - F(z)]\,dz + \int_{\bar{S}_N} u'(z)[G(z) - F(z)]\,dz \\ &= \bar{\theta}\int_{S_N}[G(z) - F(z)]\,dz + \underline{\theta}\int_{\bar{S}_N}[G(z) - F(z)]\,dz \end{aligned}$$

$$= -\left(\bar{\theta} + \underline{\theta}\right)\int_{S_N}[F(z) - G(z)]\,dz + \underline{\theta}\left\{\int_{S_N}[F(z) - G(z)]\,dz + \int_{\bar{S}_N}[G(z) - F(z)]\,dz\right\}$$

We assume that

$$\varepsilon_N = \frac{\int_{S_N}[F(z) - G(z)]\,dz}{\int_{S_N}[F(z) - G(z)]\,dz + \int_{\bar{S}_N}[G(z) - F(z)]\,dz} > \varepsilon > \frac{\underline{\theta}}{\left(\bar{\theta} + \underline{\theta}\right)},$$

We thus have $E_F[u(X)] - E_G[u(Y)] < 0$. This completes the proof of the if part of Theorem 3.

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