Ann Oper Res (2018) 269:693–707
https://doi.org/10.1007/s10479-017-2551-y

RAOTA-2016

Intuitionistic fuzzy multi-objective linear programming problem with various membership functions

Sujeet Kumar Singh[1] · Shiv Prasad Yadav[2]

Published online: 31 May 2017

Abstract This study addresses intuitionistic fuzzy multi-objective linear programming problems using triangular intuitionistic fuzzy numbers with mixed constraints. We convert the problem into single objective fuzzy programming problem. Then using different types of membership functions (linear and nonlinear), we transform the problem into crisp linear/nonlinear programming problem, which is solved by suitable crisp programming approaches. The methodology is demonstrated with the help of a numerical example and the usefulness of various membership functions is discussed.

Keywords Intuitionistic fuzzy number · Membership function · Multi-objective linear programming problem

1 Introduction

Multi-objective linear programming problem (MOLPP) has important applications in many areas of engineering and management. A list of such applications can be viewed in Wu et al. (2016) and Xidonas et al. (2016). In applications, one of the major issues faced by experts and decision makers (DMs) is to determine the values of parameters. Since real world problems are very complex, experts and DMs frequently do not know the values of parameters precisely. So, considering the uncertainty, characterizing basic parameters of the model, which are the coefficients of objective functions and technical coefficients, might be more applicable. Therefore, it may be more realistic to take the descriptive knowledge of

✉ Sujeet Kumar Singh
sksinghma209@gmail.com

Shiv Prasad Yadav
spyorfma@gmail.com

1 The Logistics Institute-Asia Pacific, National University of Singapore, Singapore 119613, Singapore

2 Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India

experts or DMs about the parameters, which can be represented as fuzzy data. Thus the fuzzy multi-objective linear programming problems (FMOLPPs) with fuzzy parameters would be viewed as more effective than the conventional one in solving real physical problems. Even, in most of the cases of judgements, evaluation is done by human beings, i.e., by DMs where certainly there are limitations on availabilities and exactness of data. Naturally, every DM hesitates more or less on every evaluation activity. This gives the concept of intuitionistic fuzzy set (IFS) theory (Atanassov 1986). The major advantage of IFS over fuzzy set is that IFS separates the degree of acceptance and the degree of non-acceptance of a decision. Because of this advantage IFS theory has much wider scope of applicability than the usual fuzzy set theory in solving various kinds of real physical problems. The IFS theory is generalization of fuzzy theory, so any method for IFS theory is automatically applicable in fuzzy theory as a particular case. So, developing a method for IFS theory is more applicable than for ordinary fuzzy set theory and that is our intention for writing this paper.

It has been proposed by Bellman and Zadeh (1970) that a fuzzy decision might be defined as the fuzzy set formed by the intersection of fuzzy objective and constraint goals. From this point of view, Tanaka and Asai (1984) and Zimmermann (1978) introduced fuzzy linear programming problem (FLPP). Tong (1994) and Gasimov and Yenilmez (2002) among others, considered single objective mathematical programming with all fuzzy parameters. Tong (1994) considered the FLPP with fuzzy constraints. He solved the defuzzified problem by fuzzy decisive set method proposed by Sakawa and Yano (1985). Gasimov and Yenilmez (2002) considered single objective FLPP with constraints of less than type only with fuzzy parameters and solved it by fuzzy decisive set method and modified sub-gradient method. Ganesan and Veeramani (2006) studied fuzzy linear programs with trapezoidal fuzzy numbers. Lai and Hawng (1992) considered MOLPP with all parameters having triangular possibility distribution. They used an auxiliary model and it was solved by MOLPP methods. Chanas (1989) proposed a fuzzy programming problem as MOLPP and it was solved by parametric approach. Zimmermann (1978) proposed a fuzzy multi-criteria decision making (FMCDM) set which is defined as the intersection of all fuzzy goals and constraints. Singh and Yadav (2015a, c, 2016) developed the mathematical background for IFLPP and applied to transportation and manufacturing problems. There are many more literatures where fuzzy and IF theory have been applied successfully (Asuncin et al. 2007; Bit et al. 1992, 1993; Cascetta et al. 2006; Das et al. 1999; De and Sana 2015; Xu 1988). Although, literature is very rich for crisp MOLPP, a comprehensive review of such articles can be found in Wiecek et al. (2016) but there are few literatures for MOLPP in uncertain environment (Jana and Roy 2005, 2007).

In this paper, IFMOLPP with mixed constraints is proposed in which the coefficients of objective as well as constraint functions and right hand sides of constraints are TIFNs. Then accuracy is utilized to transfer the IFMOLPP into equivalent crisp MOLPP. There is broad literature for ranking the TIFNs with different recommendation levels. However, we use the accuracy function for convenience. Using Bellman and Zadeh's (1970) fuzzy decision-making process, the MOLPP is converted into an equivalent crisp convex programming problem using various types of linear and non-linear membership functions. A linear membership function is most commonly used because it is simple and it is defined by fixing two points: the upper and lower levels of acceptability. However, a linear membership function is not a suitable representation in many practical situations. Furthermore, if the membership function is interpreted as the fuzzy utility of the decision maker, used for describing levels of indifference, preference or aversion towards uncertainty, then a nonlinear membership function provides a better representation than a linear membership function. Moreover, it should be emphasized that unlike linear membership functions, for nonlinear membership functions the marginal rate of increase (or decrease) of membership values as a function of model parameters is not constant and hence this technique reflects reality better than the linear case (Bector and Chandra 2002; Gupta and Mehlawat 2009).

The paper is organized as follows: In Sect. 2, some basic definitions are provided from literature (Singh and Yadav 2015c, 2016). In Sect. 3, the IFMOLPP model has been formulated. Section 4 is the most important section, here we have developed state of the art of the solution method. In Sect. 5, the entire solution procedure is summarized as an algorithm. In Sect. 6, we have provided a numerical example for justification followed by conclusion in Sect. 7.

2 Preliminaries

Definition 1 Let X be a universe of discourse. Then an IFS $\tilde{A}^I$ in X is defined by a set of ordered triples $\tilde{A}^I = \{< x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) >: x \in X\}$, where $\mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} : X \to [0, 1]$ are functions such that $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1, \forall x \in X$. The value $\mu_{\tilde{A}^I}(x)$ represents the degree of membership and $\nu_{\tilde{A}^I}(x)$ represents the degree of non- membership of the element $x \in X$ being in $\tilde{A}^I$. $h(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x)$ represents the degree of hesitation for the element x being in $\tilde{A}^I$.

Definition 2 A TIFN $\tilde{A}^I$ is an IFS with the membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ given by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 < x \leq a_2 \\ 1, & x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x < a_3 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1^{'}}, & a_1^{'} < x \leq a_2 \\ 0, & x = a_2 \\ \dfrac{x - a_2}{a_3^{'} - a_2}, & a_2 \leq x < a_3^{'} \\ 1, & \text{otherwise.} \end{cases}$$

where $a_1^{'} \leq a_1 \leq a_2 \leq a_3 \leq a_3^{'}$. This TIFN is denoted by $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$. The set of all TIFNs is denoted by $IF(\mathbb{R})$.

Definition 3 Arithmetic operations on TIFNs
Let $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ and $\tilde{B}^I = (b_1, b_2, b_3; b_1^{'}, b_2, b_3^{'})$ then
Addition: $\tilde{A}^I \oplus \tilde{B}^I = (a_1 + b_1, a_2 + b_2, a_3 + b_3; a_1^{'} + b_1^{'}, a_2 + b_2, a_3^{'} + b_3^{'})$.
Subtraction: $\tilde{A}^I \ominus \tilde{B}^I = (a_1 - b_3, a_2 - b_2, a_3 - b_1; a_1^{'} - b_3^{'}, a_2 - b_2, a_3^{'} - b_1^{'})$.
Multiplication: $\tilde{A}^I \otimes \tilde{B}^I = (l_1, l_2, l_3; l_1^{'}, l_2, l_3^{'})$
where $l_1 = min\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}, l_3 = max\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}$
$l_1^{'} = min\{a_1^{'}b_1^{'}, a_1^{'}b_3^{'}, a_3^{'}b_1^{'}, a_3^{'}b_3^{'}\}, l_3^{'} = max\{a_1^{'}b_1^{'}, a_1^{'}b_3^{'}, a_3^{'}b_1^{'}, a_3^{'}b_3^{'}\}, l_2 = a_2b_2$.

Scalar multiplication:

1. $k\tilde{A}^I = (ka_1, ka_2, ka_3; ka_1^{'}, ka_2, ka_3^{'}), k > 0.$
2. $k\tilde{A}^I = (ka_3, ka_2, ka_1; ka_3^{'}, ka_2, ka_1^{'}), k < 0.$

Definition 4 (*Accuracy function*) The concept of Yager (1981) for defuzzifying a TFN is extended to defuzzify a TIFN. Yager (1981) considered the expected value for membership function by finding the expected interval for a TFN. In the same fashion, we also have considered the expected value for non-membership function. The expected values are called score functions in this paper, as also in some recent articles. Then the two expected values are forming an expected interval again to represent a single quantity and so considered the average to get a better approximation as a single quantity for comparison purpose following the process of Yager (1981) for finding an expected value from an expected interval.
Let $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ be a TIFN. The score function for the membership function $\mu_{\tilde{A}^I}$ is denoted by $S(\mu_{\tilde{A}^I})$ and is defined by $S(\mu_{\tilde{A}^I}) = \frac{a_1+2a_2+a_3}{4}$. The score function for the non-membership function $\nu_{\tilde{A}^I}$ is denoted by $S(\nu_{\tilde{A}^I})$ and is defined by $S(\nu_{\tilde{A}^I}) = \frac{a_1^{'}+2a_2+a_3^{'}}{4}$. The accuracy function of $\tilde{A}^I$ is denoted by $f(\tilde{A}^I)$ and defined by $f(\tilde{A}^I) = \frac{S(\mu_{\tilde{A}^I})+S(\nu_{\tilde{A}^I})}{2} = \frac{(a_1+2a_2+a_3)+(a_1^{'}+2a_2+a_3^{'})}{8}$.

Theorem 1 *(Singh and Yadav 2016) The accuracy function $f : IF(\mathbb{R}) \rightarrow \mathbb{R}$ is a linear function.*

Definition 5 (*Ordering of TIFNs*) Let $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ and $\tilde{B}^I = (b_1, b_2, b_3; b_1^{'}, b_2, b_3^{'})$. Then

(i) $f(\tilde{A}^I) \geq f(\tilde{B}^I) \Rightarrow \tilde{A}^I \geq \tilde{B}^I$
(ii) $f(\tilde{A}^I) \leq f(\tilde{B}^I) \Rightarrow \tilde{A}^I \leq \tilde{B}^I$
(iii) $f(\tilde{A}^I) = f(\tilde{B}^I) \Rightarrow \tilde{A}^I \approx \tilde{B}^I$ (Numerically equivalent)
(iv) $min(\tilde{A}^I, \tilde{B}^I) = \tilde{A}^I$, If $\tilde{A}^I \leq \tilde{B}^I$ or $\tilde{B}^I \geq \tilde{A}^I$

Theorem 2 *Let $g : \mathbb{S} \rightarrow \mathbb{R}, \mathbb{S} \subseteq \mathbb{R}^n$ be a real valued function. If g is a convex function, then $\{x : g(x) \leq c, \forall c \in \mathbb{R}\}$ is a convex set and if g is a concave function, then $\{x : g(x) \geq c, \forall c \in \mathbb{R}\}$ is a convex set.*

Remark It is clear that if g is a convex function, then $\{x : g(x) \geq c, \forall c \in \mathbb{R}\}$ need not be a convex set and if g is a concave function, then $\{x : g(x) \leq c, \forall c \in \mathbb{R}\}$ need not be a convex set.

3 Problem formulation

The general multi-objective linear programming problem (MOLPP) with mixed constraints can be described by:

$$
\begin{aligned}
& Min\ Z = [Z_1, Z_2, Z_3, ..., Z_K] \\
s.t.\ & \sum_{j=1}^{n} a_{ij} x_j \ \geq \ b_i,\ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij} x_j \ \leq \ b_i,\ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij} x_j \ = \ b_i,\ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \ \geq \ 0, j = 1, 2, 3, ..., n,
\end{aligned} \tag{3.1}
$$

where $Z_p = \sum_{j=1}^{n} c_{pj} x_j$, $p = 1, 2, 3, ..., K$.

Definition 6 Let S_F be the feasible region for (3.1). A point $\bar{x}$ is said to be efficient or Pareto optimal solution of (3.1) if there does not exist any $x \in S_F$ such that, $Z_p(\bar{x}) \geq Z_p(x)\ \forall p$ and $Z_p(\bar{x}) > Z_p(x)$ for at least one p.

Definition 7 A point $\bar{x} \in S_F$ is said to be weak Pareto optimal solution of (3.1) if there does not exist any $x \in S_F$ such that $Z_p(\bar{x}) \geq Z_p(x)\ \forall p$, $p = 1, 2, ..., K$.

If the coefficients of the objective functions, decision variables and right hand sides of constraints are uncertain, which are represented by TIFNs in particular, then (3.1) becomes fully IFMOLPP as:

$$
\begin{aligned}
& Min \tilde{Z}^I = [\tilde{Z_1}^I, \tilde{Z_2}^I, \tilde{Z_3}^I, ..., \tilde{Z_K}^I] \\
s.t.\ & \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j \ \geq \ \tilde{b_i}^I,\ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j \ \leq \tilde{b_i}^I,\ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j = \tilde{b_i}^I,\ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \geq 0, j = 1, 2, 3, ..., n,
\end{aligned} \tag{3.2}
$$

where $\tilde{Z_p}^I = \sum_{j=1}^{n} (\tilde{c_{pj}})^I x_j$, $p = 1, 2, 3, ..., K$.

Let us assume that $\tilde{b_i}^I$, $i = 1, 2, 3, ..., m$ is of the following form:
The left TIFN $\tilde{b_i}^I = (b_i^l, b_i, b_i; b_i^{l'}, b_i, b_i)$, $i = 1, 2, 3, ..., m_1$, as in constraints having inequalities of "$\geq$" type the fuzziness is only on left side. The right TIFN $\tilde{b_i}^I = (b_i, b_i, b_i^r; b_i, b_i, b_i^{r'})$, $i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2$, as in constraints having inequalities of "$\leq$" type the fuzziness is only on right side and $\tilde{b_i}^I = (b_i^l, b_i, b_i^r; b_i^{l'}, b_i, b_i^{r'})$, $i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m$, because in equality type constraints the fuzziness may be on both sides.

Using accuracy function which is linear, Problem 3.2 is transformed to the following crisp MOLPP:

$$
\begin{aligned}
& Min\ \ Z^{'} = [Z_1^{'}, Z_2^{'}, Z_3^{'}, ..., Z_K^{'}] \\
s.t\ \ & \sum_{j=1}^{n} a_{ij}^{'} x_j \ \geq\ b_i^{'},\ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \ \leq\ b_i^{'},\ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \ =\ b_i^{'},\ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \ \geq\ 0,\ j = 1, 2, 3, ..., n,
\end{aligned}
\tag{3.3}
$$

where $Z_p^{'} = f(\tilde{Z_p}^I)$, $p = 1, 2, 3, ..., K$; $b_i^{'} = f(\tilde{b_i}^I)$ and $a_{ij}^{'} = f(\tilde{a_{ij}}^I)$, $i = 1, 2, 3, ..., m$, $j = 1, 2, ..., n$.

Theorem 3 (Singh and Yadav 2015b) *An efficient solution for* (3.3) *is an efficient solution for* (3.2).

Thus solving the IFMOLPP model (3.2) is equivalent to solve the crisp MOLPP model (3.3).

3.1 IFLPP

Note that a single objective IFLPP is given by

$$
\begin{aligned}
& Min\ \ \tilde{Z}^I, \\
s.t.\ & \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j \ \geq\ \tilde{b_i}^I,\ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j \ \leq \tilde{b_i}^I,\ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} \tilde{a_{ij}}^I x_j = \tilde{b_i}^I,\ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \geq 0,\ j = 1, 2, 3, ..., n,
\end{aligned}
\tag{3.4}
$$

Using accuracy function which is linear, Problem 3.4 is transformed to the following crisp LPP:

$$
\begin{aligned}
& Min \;\; Z^{'}, \\
s.t \quad & \sum_{j=1}^{n} a_{ij}^{'} x_j \;\; \geq \;\; b_i^{'}, \; i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \;\; \leq \;\; b_i^{'}, \; i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \;\; = \;\; b_i^{'}, \; i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \;\; \geq \;\; 0, j = 1, 2, 3, ..., n,
\end{aligned}
\tag{3.5}
$$

Theorem 4 *An optimal solution for* (3.5) *is an optimal solution for* (3.4).

Thus the optimal solution for an IFLPP can be easily obtained by transforming the IFLPP into crisp LPP by using accuracy function. So, we focus on solving IFMOLPP which is the important aspects for this paper.

4 Solution method

There are various methods to solve a MOLPP. These methods are classified into two general classes: scalarization methods and nonscalarization methods. These approaches convert the MOLPP into a single objective programming program (SOPP), a sequence of SOPPs, or another MOLPP. Under some assumptions, solution sets of these new programs yield solutions of the original problem. Scalarization methods explicitly employ a scalarizing function to accomplish the conversion while nonscalarizing methods use other means. Solving the SOPP typically yields one solution of the MOLPP so that a repetitive solution scheme is needed to obtain a subset of solutions of the MOLPP. A comprehensive review of such methods can be found in Wiecek et al. (2016).
Here, we apply the scalarization technique which involves formulating a MOLPP related SOPP by means of a real-valued scalarizing function typically being a function of the objective functions of the MOLPP. First of all, we assign a suitable goal for individual objective functions. The best way to assign a goal is to find the optimal value of each objective function subject to the same set of constraints and call it the desired or the most acceptable level denoted by L_p. In this way, we find k different solutions for k SOLPPs say $\mathbb{S} = \{X_1, X_2, ..., X_k\}$. Find the values of each objective for all elements of $\mathbb{S}$. Find the maximum of each objective $Z_p^{'}$ on $\mathbb{S}$. Let $U_p = max\{Z_p^{'}(X); X \in \mathbb{S}, p = 1, 2, ..., K\}$. U_p is the worst acceptable level of achievement for the pth objective function. Then Model (3.3) transformed to a goal programming problem, where the goal is to attain the individual optimum L_p for the pth objective function. However, some tolerance is allowed and that tolerance level is given by the maximum value U_p. The obtained fuzzy goal programming (FGP) model is given by:

$$
\begin{aligned}
& Find \;\; \{x_j, j = 1, 2, 3, ..., n\}, \\
s.t. \quad & Z_p^{'} \sim L_p, \;\; p = 1, 2, 3, ..., K \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \;\; \geq \;\; b_i^{'}, \; i = 1, 2, 3, ..., m_1,
\end{aligned}
$$

$$\sum_{j=1}^{n} a_{ij}^{'} x_j \leq b_i^{'}, \; i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \tag{4.1}$$
$$\sum_{j=1}^{n} a_{ij}^{'} x_j = b_i^{'}, \; i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m,$$
$$x_j \geq 0, \; j = 1, 2, 3, ..., n,$$

where "$\sim$" is fuzzy goal, which means some deviation or tolerance is allowed in strict goal. To change the FGP model (4.1) into a crisp programming model, we define different types of linear and non-linear membership functions.

In the literature, one of the major assumptions in solving fuzzy mathematical programming problem involves the use of linear membership functions for all fuzzy sets utilized in a decision making process. A linear approximation is most commonly used because of its simplicity. It is defined by fixing two points, the upper and lower levels of acceptability of the decision variable. If general fuzzy set theory is considered, then such type of assumption is not justified always. Thus a justification in the assumption is desirable according to fuzziness of the data. If fuzzy set theory is used to model real decision making processes and an assertion is made that the resulting models are the real models, then some kind of empirical justification for this assumption is necessary. From this point of view, we have considered several linear/nonlinear shapes for membership functions.

4.1 Linear membership function

A linear membership function μ_L (Zangiabadi and Maleki 2013) can be defined as follows (Fig. 1).

$$\mu_L(Z_p(x)) = \begin{cases} 1, & if \quad Z_p \leq L_p, \\ \frac{U_p - Z_p}{U_p - L_p}, & if \quad L_p \leq Z_p < U_p, \\ 0, & if \quad Z_p \geq U_p. \end{cases}$$

This function is a strictly decreasing concave and convex function for $Z_p(x)$.

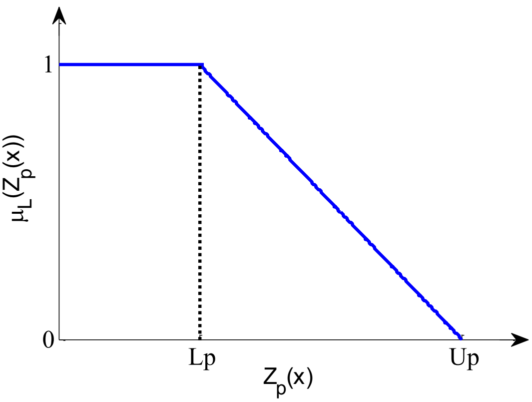
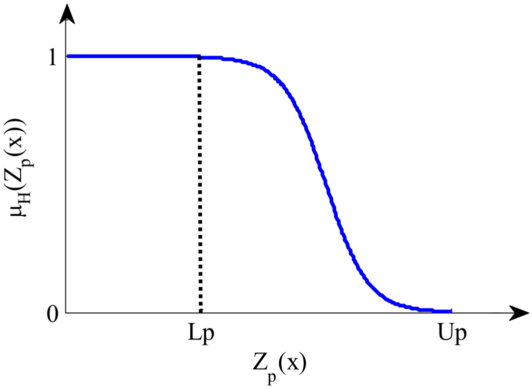


Fig. 1 Linear membership function

Fig. 2 Hyperbolic membership function



4.2 Hyperbolic membership function

The hyperbolic membership function (Jana and Roy 2007; Zangiabadi and Maleki 2013) is a convex function over a part of the objective function values and is concave over the remaining part. When the DM is worse off with respect to a goal, he/she tends to have a higher marginal rate of satisfaction with respect to that goal. A convex shape captures this behavior in the membership function. On the other hand, when the DM is better off with respect to a goal, he/she tends to have a smaller marginal rate of satisfaction. Such behavior is modeled using the concave portion of the membership function. The complete function is as follows:

$$\mu_H(Z_p(x)) = \begin{cases} 1, & if \quad Z_p \leq L_p, \\ \frac{1}{2}\tanh\left(\left(\frac{U_p+L_p}{2} - Z_p(x)\right)\alpha_p\right) + \frac{1}{2}, & if \quad L_p \leq Z_p < U_p, \\ 0, & if \quad Z_p \geq U_p. \end{cases}$$

where $\alpha_p = \frac{6}{U_p - L_p}$
The possible figure of hyperbolic membership function may be as in Fig. 2. This membership function has the following properties:

- μ_H is strictly monotonically decreasing function of $Z_p(x)$;
- $\mu_H(Z_p(x)) = 1/2 \Leftrightarrow Z_p(x) = \frac{1}{2}(U_p + L_p)$;
- μ_H is strictly convex function of $Z_p(x)$ for $Z_p(x) \geq \frac{1}{2}(U_p + L_p)$ and strictly concave function of $Z_p(x)$ for $Z_p(x) \leq \frac{1}{2}(U_p + L_p)$;
- $\mu_H(Z_p(x))$ satisfies the condition that $0 < \mu_H(Z_p(x)) < 1$ for $L_p < Z_p(x) < U_p$ and approaches asymptotically 0 and 1 as $Z_p(x) \to \infty$ and $-\infty$ respectively.

4.3 Parabolic membership function

The parabolic membership function (Jana and Roy 2007; Tiwari et al. 2013) μ_P can be defined as follows (Fig. 3):

$$\mu_P(Z_p(x)) = \begin{cases} 1, & if \quad Z_p \leq L_p, \\ 1 - \frac{(Z_p - L_p)^2}{(U_p - L_p)^2}, & if \quad L_p \leq Z_p < U_p, \\ 0, & if \quad Z_p \geq U_p. \end{cases}$$

Lemma 1 *The function $\mu_P(Z_p(x))$ is a concave function.*

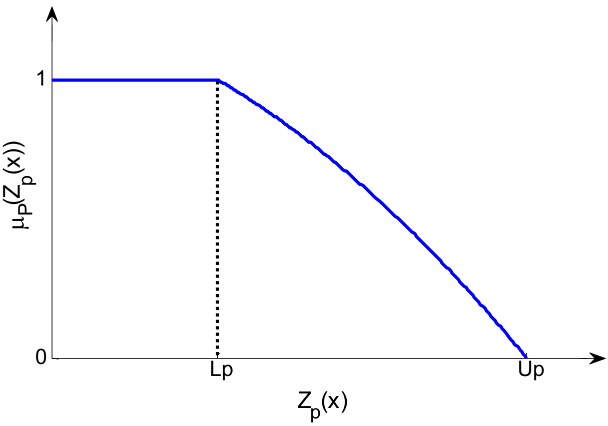


Fig. 3 Parabolic membership function

Proof Since $Z_p(x)$ is a linear function, let us assume that $Z_p(x) = ax + by$ in two variables. Then the Hessian matrix of $\mu_P(Z_p(x))$ is, $H(\mu_P) = -2\frac{Z_p - L_p}{U_p - L_p}\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$, which is negative semidefinite. This shows $\mu_P(Z_p(x))$ is a concave function. Similarly, it can be proved for higher number of variables. □

Consequently, the problem (4.1) may be described as how to make a reasonable plan so that the DM is most satisfied with fuzzy goals. That is, there should be the highest degree of balance among fuzzy goals. Let $\lambda = min\{\mu(Z_p(X)), p = 1, 2, ..., K\}$. Then according to Zimmermann (1978), this can be expressed as:

$$
\begin{aligned}
& Max \ \ \lambda \\
s.t. \ \ & \mu(Z_p^{'}(X)) \geq \lambda, \\
& \mu(Z_p^{'}(X)) \leq 1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \ \ \geq \ \ b_i^{'}, \ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \ \ \leq \ \ b_i^{'}, \ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \ \ = \ \ b_i^{'}, \ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& x_j \ \ \geq \ \ 0, j = 1, 2, 3, ..., n; \lambda \geq 0.
\end{aligned}
\tag{4.2}
$$

From Theorem 2, Lemma 1 and properties of hyperbolic membership function, we conclude the following corollary.

Corollary 1 *The sets* $\{X : \mu_L(Z_p^{'}(X)) \geq \lambda\}$, $\{X : \mu_H(Z_p^{'}(X)) \geq \lambda, Z_p^{'}(X) \leq \frac{1}{2}(U_p + L_p)\}$ *and* $\{X : \mu_P(Z_p^{'}(X)) \geq \lambda\}$ *are convex sets.*

Zimmermann (1978) proved that if the convex model (4.2) has unique optimal solution at some point X^*, then X^* is an efficient solution for (3.3). Now, using various membership functions, Problem (4.2) is transformed to the following crisp convex programming problems.

- Using linear membership function:

$$
\begin{aligned}
& Max \quad \lambda \\
s.t. \quad & U_p - Z_p^{'} \geq \lambda(U_p - L_p), \quad p = 1, 2, 3, ..., K, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \geq \quad b_i^{'}, \ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \leq \quad b_i^{'}, \ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad = \quad b_i^{'}, \ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& Z_p^{'} \geq L_p, \lambda \geq 0, \\
& x_j \quad \geq \quad 0, j = 1, 2, 3, ..., n.
\end{aligned}
\tag{4.3}
$$

- Using hyperbolic membership function:

$$
\begin{aligned}
& Max \quad \lambda \\
s.t. \quad & Z_p^{'} \alpha_p + \beta = \frac{U_p + L_p}{2} \alpha_p, \quad p = 1, 2, 3, ..., K, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \geq \quad b_i^{'}, \ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \leq \quad b_i^{'}, \ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad = \quad b_i^{'}, \ i = m_2 + 1, m_2 + 2, m_2 + 3, ..., m, \\
& tanh\beta \geq 2\lambda - 1, tanh\beta \leq 1, \\
& Z_p^{'} \leq 1/2(U_p + L_p), \\
& x_j \quad \geq \quad 0, j = 1, 2, 3, ..., n; \lambda \geq 0.
\end{aligned}
\tag{4.4}
$$

- Using parabolic membership function:

$$
\begin{aligned}
& Max \quad \lambda \\
s.t. \quad & \left(Z_p^{'} - L_p\right)^2 + \lambda \left(\left(U_p - L_p\right)^2\right) \leq \left(U_p - L_p\right)^2, \quad p = 1, 2, 3, ..., K, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \geq \quad b_i^{'}, \ i = 1, 2, 3, ..., m_1, \\
& \sum_{j=1}^{n} a_{ij}^{'} x_j \quad \leq \quad b_i^{'}, \ i = m_1 + 1, m_1 + 2, m_1 + 3, ..., m_2,
\end{aligned}
\tag{4.5}
$$

$$\sum_{j=1}^{n} a_{ij}^{'} x_j = b_i^{'}, \; i = m_2+1, m_2+2, m_2+3, ..., m,$$
$$\left(Z_p^{'}\right)^2 \geq \left(L_p\right)^2, \lambda \geq 0,$$
$$x_j \geq 0, j = 1, 2, 3, ..., n.$$

Models (4.3)–(4.5) are convex programming problems, which can be solved by suitable algorithms or software packages.

5 Algorithm

The whole solution procedure developed in Sect. 4 has been summarized as an algorithm in this section.

- *Step 1* Model the uncertain MOLPP using TIFNs.
- *Step 2* Transform the IFMOLPP into MOLPP by using the accuracy function.
- *Step 3* Find optimal solution of each SOLPP. Let the solution set be $\mathbb{S}$.
- *Step 4* Find the values of all objective functions at all points of $\mathbb{S}$.
- lit Step 5 Choose the optimum value L_p of the pth objective function as the goal value for the pth objective function.
- *Step 6* Find the maximum value U_p for the pth objective function over $\mathbb{S}$, i.e., $U_p = max\{Z_p^{'}(X); X \in \mathbb{S}, p = 1, 2, ..., K\}$.
- *Step 7* Determine the goal programming model as in (4.1).
- *Step 8* Use suitable membership function and transform the FGP model to crisp programming model for various membership functions as in (4.3)–(4.5) accordingly.
- *Step 9* Solve the crisp convex programming problem using suitable techniques or software packages.

6 Numerical example

Let us consider the following IFMOLPP.

$$\begin{aligned} & Min\ \tilde{Z_1}^I = \tilde{5}^I x_1 \oplus \tilde{3}^I x_2 \\ & Min\ \tilde{Z_2}^I = \tilde{2_a}^I x_1 \oplus \tilde{7}^I x_2 \\ s.t.\ & \tilde{2_b}^I x_1 \oplus \tilde{4}^I x_2 \geq \tilde{25}^I, \\ & \tilde{1_a}^I x_1 \oplus \tilde{1_b}^I x_2 \geq \tilde{10}^I, \\ & \tilde{4}^I x_1 \oplus \tilde{5}^I x_2 \leq \tilde{50}^I, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{6.1}$$

Here the estimated parameters by the DM are as follows:
$\tilde{25}^I = (22, 25, 25; 18, 25, 25)$, $\tilde{10}^I = (9, 10, 10; 8, 10, 10)$, $\tilde{50}^I = (50, 50, 55; 50, 50, 60)$, $\tilde{5}^I = (4, 5, 6; 4, 5, 7)$, $\tilde{3}^I = (3, 3, 4; 3, 3, 4.5)$, $\tilde{2_a}^I = (2, 2, 3; 2, 2, 4)$, $\tilde{7}^I = (7, 7, 7.5; 6, 7, 8)$, $\tilde{2_b}^I = (1.5, 2, 2; 1, 2, 2)$, $\tilde{1_a}^I = (0.5, 1, 1; 0.2, 1, 1.5)$, $\tilde{1_b}^I = (1, 1, 1; 0.5, 1, 2)$, $\tilde{4}^I = (3, 4, 4; 2, 4, 4)$.

Using accuracy function (Definition 4), Problem (6.1) is equivalent to the following crisp MOLPP:

$$\begin{aligned} & Min\ Z_1^{'} = 5.125x_1 + 3.33x_2 \\ & Min\ Z_2^{'} = 2.37x_1 + 6.2x_2 \\ s.t.\quad & 1.8x_1 + 3.62x_2 \ \geq 23.75, \\ & 0.9x_1 + 1.06x_2 \ \geq 9.6, \\ & 3.62x_1 + 5.125x_2 \ \leq 51.8, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{6.2}$$

Solving Problem (6.2) as an SOLPPs, we have the following solutions:
$X_1 = (0, 9.056)$, $X_2 = (13.19, 0)$, $L_1 = 30.15$, $U_1 = 67.59$, $L_2 = 31.26$ and $U_2 = 56.14$. Problem (6.1) is now equivalent to the following FGP model:

$$\begin{aligned} & Find\ \{x_j : j = 1, 2\} \\ s.t.\quad & 5.125x_1 + 3.33x_2 \sim 30.15, \\ & 2.37x_1 + 6.2x_2 \sim 31.26, \\ & 1.8x_1 + 3.62x_2 \ \geq 23.75, \\ & 0.9x_1 + 1.06x_2 \ \geq 9.6, \\ & 3.62x_1 + 5.125x_2 \ \leq 51.8, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{6.3}$$

Applying models (4.3)–(4.5) and solving by LINGO, the solution of (6.1) is summarized in Table 1.

7 Conclusion

This research proposed IFMOLPP and a method for its solution has been developed. The IFMOLPP is transformed to MOLPP by using the accuracy function and by applying the scalarization technique it is transformed to FGP problem. After that, we have introduced various membership functions to solve the FGP model. Introduction of various membership functions provides the flexibility to the DM for choosing the membership function which fits better for the problem. We observed from Table 1 that for the given numerical problem, solutions are better in case of hyperbolic membership function. Thus the efficiency of the models in terms of satisfaction level or achievement level of the DM can be ordered as Hyperbolic > Parabolic > Linear. Here, the other flexibility is that the solution can be chosen according to the priority of the objectives. That is, the DM can adopt the membership function, which is giving better solution for the objective function having higher priority.

Table 1 Solutions

Membership functions	Solutions	Objective values	Deviations from (L_1, L_2)	λ
Linear	$x_1 = 4.94, x_2 = 4.1$	(38.97, 37.12)	(8.82, 5.86)	0.76
Hyperbolic	$x_1 = 4.97, x_2 = 4.31$	(39.82, 38.5)	(9.67, 7.24)	1
Parabolic	$x_1 = 5.62, x_2 = 4.27$	(43.02, 39.79)	(12.87, 8.53)	0.88

Jana and Roy (2005) decomposed the problem into eight sub-objectives, which increases the dimension of the problem eight times and that increases the complexity of the problem a lot. In Gupta and Mehlawat (2009), Jana and Roy (2007) and Zangiabadi and Maleki (2013), authors have applied hyperbolic and exponential membership functions, which are not concave in whole domain. So, the resulting problem need not be convex and hence there is no guarantee for the obtained solution to be global. The proposed approach overcomes these deficiencies and provides the global solution.

Our research opens many possible avenues for future research. First, from a practical point of view, there is value to implement the approach developed in this paper on a complete real life problem from industry such as in manufacturing, scheduling, planning, transportation etc., which can be a challenging work. Second, from a methodological viewpoint, there is value to develop faster heuristic methods to solve large scale problems.

Acknowledgements The authors gratefully acknowledge the financial support given by the Ministry of Human Resource and Development (MHRD), Govt. of India, India. Also the authors would like to thank the anonymous reviewers for making critical and conceptual comments, which helped to improve the manuscript in the present form.

References

Asuncin, M. D. L., Castillo, L., Olivares, J. F., Prez, O. G., Gonzlez, A., & Palao, F. (2007). Handling fuzzy temporal constraints in a planning environment. *Annals of Operations Research*, *155*, 391–415.

Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*, 87–96.

Bector, C. R., & Chandra, S. (2002). On duality in linear programming under fuzzy environment. *Fuzzy Sets and Systems, 125*(3), 317–325.

Bellman, R. E., & Zadeh, L. A. (1970). Decision making in a fuzzy environment. *Management Science*, *17*, 141–164.

Bit, A. K., Biswal, M. P., & Alam, S. S. (1992). Fuzzy programming approach to multicriteria decision making transportation problem. *Fuzzy Sets and Systems*, *50*, 135–141.

Bit, A. K., Biswal, M. P., & Alam, S. S. (1993). Fuzzy programming approach to multi-objective solid transportation problem. *Fuzzy Sets and Systems*, *57*, 183–194.

Cascetta, E., Gallo, M., & Montella, B. (2006). Models and algorithms for the optimization of signal settings on urban networks with stochastic assignment models. *Annals of Operations Research*, *144*, 301–328.

Chanas, D. (1989). Fuzzy programming in multi-objective linear programming—a parametric approach. *Fuzzy Set and System*, *29*, 303–313.

Das, S. K., Goswami, A., & Alam, S. S. (1999). Multi-objective transportation problem with interval cost, source and destination parameters. *European Journal of Operational Research*, *117*, 100–112.

De, S. K., & Sana, S. S. (2015). Backlogging EOQ model for promotional effort and selling price sensitive demand-an intuitionistic fuzzy approach. *Annals of Operations Research*, *233*(1), 57–76.

Ganesan, K., & Veeramani, P. (2006). Fuzzy linear programs with trapezoidal fuzzy numbers. *Annals of Operations Research*, *143*, 305–315.

Gasimov, R. N., & Yenilmez, K. (2002). Soving fuzzy linear programming with linear membership funtions. *Turkish Journal of Mathematics*, *26*, 375–396.

Gupta, P., & Mehlawat, M. K. (2009). Bector–Chandra type duality in fuzzy linear programming with exponential membership functions. *Fuzzy Sets and Systems*, *160*, 3290–3308.

Jana, B., & Roy, T. K. (2005). Multi-objective fuzzy linear programming and its application in transportation model. *Tamsui Oxford Journal of Mathematical Sciences*, *21*(2), 243–268.

Jana, B., & Roy, T. K. (2007). Multi-objective intuitionistic fuzzy linear programming and its application in transportation model. *Notes on Intuitionistic Fuzzy Sets*, *13*(1), 34–51.

Lai, Y. J., & Hawng, C. L. (1992). *Fuzzy mathematical programming, lecture notes in economics and mathematical systems* (p. 394). New York: Springer.

Sakawa, M., & Yano, H. (1985). Interactive decision making for multi-objective linear fractional programming problems with fuzzy parameters. *Cybernetics Systems*, *16*, 377–394.

Singh, S. K., & Yadav, S. P. (2015a). Intuitionistic fuzzy non linear programming problem: Modeling and optimization in manufacturing systems. *Journal of Intelligent and Fuzzy Systems*, *28*, 1421–1433.

Singh, S. K., & Yadav, S. P. (2015b). Modeling and optimization of multi objective non-linear programming problem in intuitionistic fuzzy environment. *Applied Mathematical Modelling*, *39*, 4617–4629.

Singh, S. K., & Yadav, S. P. (2015c). Efficient approach for solving type-1 intuitionistic fuzzy transportation problem. *International Journal of System Assurance Engineering and Managment*, *6*(3), 259–267.

Singh, S. K., & Yadav, S. P. (2016). A new approach for solving intuitionistic fuzzy transportation problem of type-2. *Annals of Operations Research*, *243*, 349–363.

Tanaka, H., & Asai, K. (1984). Fuzzy linear programming problems with fuzzy numbers. *Fuzzy Sets and Systems*, *13*, 1–10.

Tiwari, A. K., Tiwari, A., Smuel, C., & Pandey, S. K. (2013). Flexibility in assisgnment problem using fuzzy numbers with non-linear membership functions. *International Journal of Industrial Engineering and Technology*, *3*(2), 1–10.

Tong, S. (1994). Interval number and fuzzy number linear programming. *Fuzzy Sets and Systems*, *66*, 301–306.

Wiecek, M. M., Ehrgott, M., & Engau, A. (2016). Continuous multi-objective programming. doi:10.1007/978-1-4939-3094-4_18.

Wu, Z., Xu, J., & Xu, Z. (2016). A multiple attribute group decision making framework for the evaluation of lean practices at logistics distribution centers. *Annals of Operations Research*, *247*, 735–757.

Xidonas, P., Doukas, H., Mavrotas, G., & Pechak, O. (2016). Environmental corporate responsibility for investments evaluation: An alternative multi-objective programming model. *Annals of Operations Research*, *247*, 395–413.

Xu, L. D. (1988). A fuzzy multi-objective programming algorithm in decision support systems. *Annals of Operations Research*, *12*, 315–320.

Yager, R. R. (1981). A procedure for ordering fuzzy subsets of the unit interval. *Information Sciences*, *24*(2), 143–161.

Zangiabadi, M., & Maleki, H. R. (2013). Fuzzy goal programming technique to solve multi-objective transportation problems with some non-linear membership functions. *Iranian Journal of Fuzzy Systems*, *10*(1), 61–74.

Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets and System*, *1*, 45–55.