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MULTIPLE OBJECTIVE OPTIMIZATION

Portfolio selection problem: a review of deterministic and stochastic multiple objective programming models

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Abstract The literature on portfolio selection mostly concentrates on computational analysis rather than on modelling efforts. In response, this paper provides a comprehensive literature review of multiple objective deterministic and stochastic programming models for the portfolio selection problem. First, we summarize different concepts related to portfolio selection theory, including pricing models and portfolio risk measures. Second, we report the mathematical models that are generally used to solve deterministic and stochastic multiple objective programming problems. Finally, we present how these models can be used to solve the portfolio selection problem.

Keywords Portfolio selection · Multiple objective programming · Multiple objective stochastic programming

1 Introduction

Portfolio management deals with the selection of best portfolios in a context of volatile returns due to random changes in future securities prices (Crundwell 2008). Therefore, investors are always looking for securities that provide a good balance between return opportunities and risk.

We start by reviewing basic definitions related to portfolio selection. We present the capital asset pricing model (Sharpe 1964), quantifiers for portfolio risk and the well-known Markowitz model (Markowitz 1952) for portfolio selection.

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1.1 The return

Return on investment is a measure that evaluates the worth of the investment. In a financial market, the return on investment in security i at time t can be defined as the ratio of the net gain to the cost of the investment, which can be expressed as follows (Rasmussen 2003):

$$r_{i,t} = \frac{p_{i,t} - p_{i,t-1} + d_{i,t}}{p_{i,t}}$$

where $p_{i,t}$ is the price of the security i at time t, $p_{i,t} - p_{i,t-1}$ is the capital gain over the period $[t-1,\ t]$ for security i, and $d_{i,t}$ is the dividend payment during the same period for security i. Other formulas can be used to compute the return over time (Brentani 2004). For example, some authors define returns as "excess returns," which are the amount of investment return above the risk free rate of return (Rasmussen 2003).

Let us denote by $x = (x_1, ..., x_n)$ a vector whose component x_i denotes the weight or proportion of the investor's wealth allocated to the ith security in the portfolio. Obviously, we have $\sum_{i=1}^{n} x_i = 1$ and $0 \le x_i \le 1$, $i = 1, ..., n$. Any portfolio P can be characterized by a vector $x = (x_1, ..., x_n)$.

Mathematically, the total random return R_P of the portfolio P can be written as follows:

$$R_P = \sum_{i=1}^{n} r_i x_i$$

where r_i is the random return of the ith security in portfolio P at a future date.

To be able to estimate the return of portfolio P at a future date, we need to estimate the securities' future prices. These prices cannot be known in advance as they are random. Therefore, several possible prices (or prices intervals) are defined, and a probability is assigned to each of these prices. The portfolio return can be evaluated by its expected return:

$$E(R_P) = \sum_{i=1}^{n} \bar{r}_i x_i$$

where $E(.)$ is the expected value of the random return and $\bar{r}_i = E(r_i)$.

Calculating the security return seems to difficult, as we need to define a probability distribution for a set of plausible returns. In their Capital Asset Pricing Model (CAPM), Sharpe (1964) and Lintner (1965) estimated the return using a simple linear regression model.

1.2 The capital asset pricing model

The main hypothesis in the CAPM is that investors focus only on the market portfolio return R_M when managing their portfolios, where the market portfolio consists of all securities and the proportion invested in each security corresponds to its relative market value (Cohen and Natoli 2003). The Sharpe model includes a set of simplifying assumptions that make the CAPM applicable (Athanasoulis and Shiller 2000):

- The probability distribution of returns is homogenously anticipated by investors;
- There are no commissions, taxes or expenses for markets which are supposed perfect;
- Unlimited sums of money can be lent or borrowed by investors at the same interest rate, which is equal to the risk-free rate;
- The mean variance criterion is maximized by risk-averse investors.

The CAPM proposes that a security's return can be described completely by a linear combination of a market return and its security's co-variation (Rasmussen 2003):

$$r_i = r_f + \beta_i (R_M - r_f) \tag{1}$$

where r_f is the return of the risk free security and β_i is known as Sharpe's Beta. The Beta of a security i is defined as the covariance between the security return and the market return:

$$\beta_i = \frac{Cov\,(r_i,\ R_M)}{Var\,(R_M)}$$

We may observe three possible absolute values of the beta (Lee et al. 2010):

- If the beta is equal to 1, then the security's return moves with the market return,
- If the beta is less than 1, then the security's return is less volatile than the market return,
- If the beta is greater than 1, then the security's return is more volatile than the market return.

From historical values of the stock, the value of Beta can be determined by re-arranging Eq. (1) into the following form known as the market model (Crundwell 2008):

$$r_i = \alpha + \beta_i R_M$$

where the value of the coefficients α and β_i are respectively the intercept and the slope of a linear regression of the historical data for r_i and R_M.

The Beta can be used to measure the security risk, which is the risk that the return will decrease due to moves in the market. In the following, we report some other measures of the volatilities of returns.

1.3 Risk measures

Portfolio selection involves several forms of risk, including (Zenios and Ziemba 2006):

- Credit risk: risk of a nonpayment;
- Liquidity risk: risk related to non-availability of cash to support the investment activities;
- Operational risk: risk of losses due to operational errors, and;
- Business risk: risks due to volatility of security volumes.

For investors, an appropriate measurement of risk should quantify the chances that the actual return of an investment will not be as expected (Huang 2008).

For example, the mean absolute deviation (*MAD*) measures the average deviation in absolute terms around the mean of the distribution as follows (Rachev and Stoyanov 2008):

$$MAD_P = E\,(|R_P - E\,(R_P)|)$$

Another dispersion measure that penalizes symmetrically both negative and positive deviations from the mean is variance. Variance (σ_P) is the widely-used measure to quantify the risk of a portfolio P as follows:

$$\sigma_P^2 = E\left((R_P - E(R_P))^2\right) = \left[\sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{\substack{j=1 \\ i \neq j}}^{n} x_i x_j \sigma_{ij}\right] = x^t V x$$

where $\sigma_{ij} = E\left((r_i - \bar{r}_i)\left(r_j - \bar{r}_j\right)\right)$ is the covariance of security i and security j and V is the $n \times n$ covariance matrix:

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

The higher are the variance and dispersion of the returns, the more uncertain the future returns and therefore the riskier the portfolio.

Markowitz (1959) proposed semi-variance, which considers observations below the mean as a measure of risk (Kandasamy 2008):

$$SV_P = \left(E\left(\min\left\{R_P - E\left(R_P\right), 0\right\}\right)^2\right)$$

While semi-variance is well suited to describe security risk, the most often used risk measure is Morgan's value at risk (*VaR*) (Jorion 2001). The *VaR* represents the predicted maximum loss with a specified probability level ε over a certain period (Giannopoulos et al. 2005):

$$P\left(R_P \geq -VaR_{\varepsilon}\left(R_P\right)\right) = \varepsilon$$

If we take the case of a portfolio which is held with a 1 day 99% *VaR* equal to \$1 million, this means that over 1 day, there is a 1% probability that the portfolio will lose more than \$1 million.

Despite its popularity, the *VaR* has the disadvantage of not being a sub-additive measure and does not take account of severe losses beyond the *VaR* value (Filho 2006).

Risk can be influenced by different factors, such as income uncertainty, interest rates, inflation, exchange rates, tax rates, etc. Markowitz (1952) observed that investors should diversify their investment to reduce risk, and that an efficient portfolio is one where no added diversification can lower the portfolio's risk for a given expected return.

1.4 The Markowitz model

Markowitz's (1952) portfolio selection model aims to select the least risky portfolio for a given level of return:

$$\begin{aligned} Min \quad & \sigma_P^2 = x^t V x \\ s.t. \quad & \sum_{i=1}^{n} \bar{r}_i x_i \geq R \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0 \leq x_i \leq 1, i = 1, \ldots, n \end{aligned} \tag{2}$$

where R is the desired level of expected return.

The Markowitz model (2) is a quadratic program with a unique solution as the covariance matrix V is a positive definite matrix (variances of risky portfolios are strictly positive).

Many studies have followed Markowitz' analysis and used or extended mean variance analysis for portfolio selection (see Liu 2004). The mean variance model (2) can be rewritten differently to maximize the return for a given level of risk σ:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{n} \bar{r}_i x_i \\
s.t. \quad & x^t V x \le \sigma^2 \\
& \sum_{i=1}^{n} x_i = 1 \\
& 0 \le x_i \le 1, i = 1, ..., n
\end{aligned} \tag{3}
$$

For a given level of return *R,* program (1) provides an efficient portfolio.

A portfolio that solves program (2) or (3) is called an efficient portfolio if it provides minimum risk for a given return, or a maximum return for a given risk. The concept of efficient optimal solution results from multiple objective programming and stochastic dominance fields.

2 Multiple objective programming

A multiple objective program can be written in the following form (Ehrgott 2005):

$$
\begin{aligned}
Min \quad & f(x) = \left[f_1(x), \ f_2(x), ..., \ f_p(x) \right] \\
s.t. \quad & x \in X \subset IR^n
\end{aligned} \tag{4}
$$

where X is the set of feasible solutions and $f_1(x), ..., f_p(x)$ are p functions that evaluate the performance of the decision variable x.

For $p = 1$, the problem (4) is a uni-objective mathematical program. With a single objective function and under some convexity assumptions, the problem (4) has a unique optimal solution. In the case of multiple objectives $p > 1$, problem (4) may not have a feasible solution that optimizes all objective functions at the same time because of the usually observed conflict between objective functions (Evans 1984). The concept of Pareto efficient solution is generally proposed for such situations.

2.1 Efficiency and non-dominance

The concept of efficiency can be defined as follows:

Definition (*Chankong and Haimes*): A feasible solution $x^* \in X$ is called efficient or Pareto optimal, if there is no other $x \in X$ such that $f(x) \le f(x^*)$. If x^* is efficient, $f(x^*)$ is called a non-dominated point. If $x_1, x_2 \in X$ and $f(x_1) \le f(x_2)$, we say that x_1 dominates x_2 and $f(x_1)$ dominates $f(x_2)$. The set of all efficient solutions $x^* \in X$ is denoted X_E and called the efficient set. The set of all non-dominated points $y^* = f(x^*)$, where $x^* \in X_E$, is denoted Y_N and called the non-dominated set.

The literature gives some mathematical properties of the efficient set X_E and the non-dominated set Y_N (for more details see Chankong and Haimes 1983; Figueira et al. 2005). These sets may be empty or defined via isolated points.

The literature also defines other forms of efficiency, such as weak efficiency, strict efficiency and proper efficiency.

Definition (*Chankong and Haimes*): A feasible solution $x^* \in X$ is called weakly efficient (weakly Pareto optimal) if there is no $x \in X$ such that $f(x) < f(x^*)$, i.e. $f_k(x) < f_k(x^*)$ for all $k = 1, ..., p$. The point $y^* = f(x^*)$ is then called weakly non-dominated.

A feasible solution $x^* \in X$ is called strictly efficient (strictly Pareto optimal) if there is no $x \in X, x \neq x^*$ such that $f(x) < f(x^*)$. The weakly (strictly) efficient and non-dominated sets are denoted, $X_{wE}(X_{sE})$ and Y_{wE} , respectively

A feasible solution $x^* \in X$ is called properly efficient, if it is efficient and if there is a real number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(x^*)$ there exists an index j such that $f_j(x^*) < f_j(x)$ such that

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M$$

The corresponding point $y^* = f(x^*)$ is called properly non-dominated.

We note that $Y_N \subset Y_{wN}$ and that $X_{sE} \subset X_E \subset X_{wE}$ (Figueira et al. 2005).

It is difficult to explicitly generate all these sets correctly. The decision maker is usually interested in one or few solutions. In the next subsections, we discuss solution strategies used to solve the multiple objective program (4) to obtain a reasonable sample of efficient solutions.

2.2 The weighted sum method

The most commonly used approach to solve the multiple objective program (4) is the weighted sum method:

$$\begin{aligned} &Min \quad \sum_{i=1}^{p} w_i \ f_i\ (x) \\ &s.t. \quad x \in X \end{aligned} \tag{5}$$

where w_i is the weight for the objective function f_i.

The problem (5) is easy to solve and it has been demonstrated that it can be used to generate efficient solutions:

Proposition (Chankong and Haimes): *Suppose that x^* is an optimal solution of the weighted sum optimization problem (5) then one of the following statements hold:*

1. *If $w \geq 0$ then $x^* \in X_{wE}$.*
2. *If $w > 0$ then $x^* \in X_E$.*
3. *If $w \geq 0$ and x^* is a unique optimal solution of (5) then $x^* \in X_{sE}$.*
4. *If $w > 0$ and $\sum_{i=1}^{p} w_i = 1$ then x^* is a properly efficient solution.*

Let X be a convex set and f_k, $k = 1, ..., p$; be convex functions. Then the following statements hold:

1. *If x^* is a properly efficient solution of (4) then there is some $w > 0$ such that x^* is an optimal solution of (5).*
2. *If x^* is a weak efficient solution of (4) then there is some $w \geq 0$ such that x^* is an optimal solution of (5).*

2.3 The ε -constraint model

The ε-constraint method consists in transforming *p-1* objective functions into constraints by fixing a threshold to the transformed functions as follows:

$$
\begin{aligned}
Min \quad & f_j(x) \\
s.t. \quad & f_k(x) \geq \varepsilon_k,\ k = 1, \ldots,\ p \text{ and } k \neq j \\
& x \in X
\end{aligned} \tag{6}
$$

where ε_k is the threshold for the objective function f_k, $k = 1, \ldots,\ p$ and $k \neq j$. We note that a solution x^* is efficient if, and only if, there exist $\varepsilon \in \Re_p$ such that this solution x^* is optimal for the problem (6) for all $j = 1, \ldots,\ p$.

While it is computationally difficult to obtain an efficient solution for p programs (6) for a given threshold $\varepsilon \in \Re_p$, in the following proposition we report results for other forms of efficiency.

Proposition (Chankong and Haimes)*: Let x^* be an optimal solution of (6) for some j, $j = 1, \ldots,\ p$, then x^* is weakly efficient. If x^* is the unique optimal solution of (6) for some j, $j = 1, \ldots,\ p$, then x^* is a strictly efficient (and therefore efficient).*

2.4 The goal programming model

The goal-programming model is similar to the ε-constraint method as objective functions are transformed into constraints by setting up target values for these functions, called goals. The goal programming model can be formulated as follows (Jones and Tamiz 2002):

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{p} \alpha_j\ d_j^+ + \beta_j\ d_j^- \\
s.t. \quad & f_j(x) - d_j^+ + d_j^- = g_j,\quad j = 1, \ldots,\ p \\
& x \in X
\end{aligned} \tag{7}
$$

where g_j is the goal for the objective function f_j; d_j^+ and d_j^- are deviation variables representing the under and over achievement of the jth goal respectively and α_j and β_j are weights for positive and negative deviations of the jth objective functions f_j, $j = 1, \ldots,\ p$.

Goal programming was introduced by Charnes et al. (1955) in an application of a single-objective linear programming problem to estimate executive compensation. Nowadays, the goal programming technique is commonly used for a long list of applications (Romero 1991).

The above program (7) is called the weighted goal programming model. Other goal programming models exist, either related to the type of goal criteria (less than or equal to, greater than or equal to, equal to, within a range) or to the aggregating approach between deviations (weighted, lexicographic, minimax, etc.). For more details, see Jones and Tamiz (2002). In all variants of the goal programming approach, the principle is to minimize deviations from a target value. One of the goal programming variants is the compromise model.

2.5 The compromise programming model

The compromise program aims to minimize the deviation from the target values, which are the ideal or the nadir point.

Definition (Ehrgott 2005): The point $y^I = (y_1^I, \ldots,\ y_p^I)$ given by $y_k^I = \underset{x \in X}{Min}\ f_k(x)$ is called the ideal of the multiple objective optimization problem (4). The point $y^N = (y_1^N, \ldots,\ y_p^N)$ given by $y_k^N = \underset{x \in X}{Max}\ f_k(x)$ is called the nadir of the multiple objective optimization problem (4).

The compromise programming approach consists in finding the nearest solution to the Nadir as follows (Aouni et al. 2005):

$$\begin{aligned} & Min \quad d(f(x),\, y^N) \\ & s.t. \quad x \in X \end{aligned} \tag{8}$$

where $d(.,.)$ is a distance in IR^p. When we choose the Euclidian L_p distance, program (4) is equivalent to the following program i (Ehrgott 2005):

$$\begin{aligned} & Min \quad \left(\sum_{i=1}^{p} w_i \, \left(f_i\,(x) - y_i^N \right)^p \right)^{\frac{1}{p}} \\ & s.t. \quad x \in X \end{aligned}$$

where w_i is the weight for the ith deviation.

3 The multiple objective portfolio selection models

The Markowitz model (2) can be viewed as an epsilon constrained transformation of a bi-objective model where the investor maximizes the expected return value (the first objective) and minimizes the return variance (the second objective). These two objective functions represent the first and second moment of the random return. In some situations, the investor may be concerned with higher moments, for example skewness (Prakash et al. 2003).

Anagnostopoulos and Mamanis (2010) presented a portfolio selection model with three objective functions: (i) to minimize the risk, (ii) to maximize the return and (iii) to minimize the number of securities included in the portfolio. Xidonas et al. (2009) selected a portfolio from the Athens stock exchange taking into account the following criteria: profitability (return on security and return on equity), management performance (Asset turnover and Inventory turnover) and capital structure (assets to liabilities and liabilities to equity). Ehrgott et al. (2004) presented a multiple objective portfolio selection model where they replaced the risk and return objectives by five objective functions: 12-month performance, 3-year performance, the annual dividend or revenue, the Standard and Poor's star ranking, and 12-month volatility. Ida (2003) presented a multiple objective problem for the portfolio selection problem in which some of the model coefficients are defined by intervals. The main idea was to consider that the security return and variance are defined on an interval. The obtained model is a mean variance model with an interval coefficient:

$$\begin{aligned} & Max \quad \sum_{j=1}^{n} \left[r\,\inf{}_i,\, r\,\sup{}_i \right] x_i \\ & \qquad\quad \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sigma\,\inf{}_{ij},\, \sigma\,\sup{}_{ij} \right] x_i x_j \\ & s.t. \quad \sum_{i=1}^{n} x_i = 1 \\ & \qquad\quad x_i \geq 0,\; i = 1, \ldots, n \end{aligned}$$

where the return r_i and the covariance σ_{ij} are restricted by intervals $\left[r\inf_i, r\sup_i\right]$ and $\left[\sigma\inf_{ij}, \sigma\sup_{ij}\right]$ respectively.

Steuer et al. (2005) reported the following set of objective functions that were considered in the literature for portfolio selection:

$$\begin{aligned}
&\max\{z_1 = \textit{portfolio return}\}\\
&\max\{z_2 = \textit{dividends}\}\\
&\max\{z_3 = \textit{amount invested in R\&D}\}\\
&\max\{z_4 = \textit{social responsibility}\}\\
&\max\{z_5 = \textit{liquidity}\}\\
&\max\{z_6 = \textit{portfolio return over that of a benchmark}\}\\
&\max\{z_7 = -\textit{deviation from asset allocation percentage}\}\\
&\max\{z_8 = -\textit{number of } \text{securities} \textit{ in portfolio}\}\\
&\max\{z_9 = -\textit{turnover } (\textit{i.e., costs of adjustment})\}\\
&\max\{z_{10} = -\max \textit{imum investment proportion weight}\}\\
&\max\{z_{11} = -\textit{amount of short selling}\}\\
&\max\{z_{12} = -\textit{number of } \sec \textit{urities sold short}\}
\end{aligned}$$

Some of the objective functions listed above are stochastic, and we discuss these stochastic objective functions in a following section.

3.1 The weighted sum model

Xia et al. (2000) presented a model to improve the performance of the mean variance model by considering the expected return of securities as decision variables:

$$\begin{aligned}
Max \quad & (1-w)\sum_{i=1}^{n}\bar{r}_i x_i - w\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\\
s.t. \quad & \sum_{i=1}^{n} x_i = 1\\
& \bar{r}_i \geq \bar{r}_{i+1},\ i = 1, \ldots, n-1\\
& a_i \leq r_i \leq b_i,\ i = 1, \ldots, n\\
& x_i \geq 0,\ i = 1, \ldots, n
\end{aligned} \tag{9}$$

where $[a_i,\ b_i]$ is the range in which the expected return of security i can vary. They solved program (9) using genetic algorithms and reported a comparison with the traditional Markowitz model to show the importance of the above model.

3.2 The goal programming model

Usually, objective functions in the portfolio selection problem (return, liquidity, risk, etc.) have a target value or goal that we need to achieve (Aouni and Tarre 2010). This leads to the construction of goal programming models for the portfolio selection problem (Azmi and Tamiz 2010)

Below, we report a case study from the Tunisian security market, in which Mansour et al. (2007) applied an interval goal programming model that considered three objective functions:

return, risk, in terms of the portfolio Beta and exchange flow ratio. For each of these objective functions lower and upper target values are given (goals intervals) and a satisfaction function is used to measure the difference between the actual value of the objective function and the goal. The proposed model was written as follows:

$$\begin{aligned}
Max \quad & Z = \sum_{j=1}^{3}\left(w_j^+ F_j^+\left(\delta_j^+\right) + w_j^- F_j^-\left(\delta_j^-\right)\right) \\
s.t. \quad & \sum_{i=1}^{n}\bar{r}_i x_i - \delta_1^+ + \delta_1^- = \zeta_1 \\
& \sum_{i=1}^{n}\beta_i x_i - \delta_2^+ + \delta_2^- = \zeta_2 \\
& \sum_{i=1}^{n}L_i x_i - \delta_3^+ + \delta_3^- = \zeta_3 \\
& \sum_{i=1}^{n}x_i = 1 \\
& 0 \le x_i \le 0.1 \\
& \zeta_j \in \left[g_j^l, g_j^u\right] \ (for\ j = 1, 2, 3) \\
& \delta_j^+, \delta_j^- \ge 0 \ (for\ j = 1, 2, 3)
\end{aligned} \tag{10}$$

where F_j^+ (.) and F_j^- (.) are the satisfaction functions associated with respectively positive δ_j^+ and negative deviations δ_j^-; w_i^+ and w_i^- are weights associated respectively with positive and negative deviations and g_j^l, g_j^u are respectively the lower and upper values for goal ζ_j.

3.3 The compromise programming model

Ballestero and Romero (1996) presented one of the first applications of compromise programming model for portfolio selection. They proposed a compromise mean variance portfolio selection model based on L_p distance and defined as follows:

$$\begin{aligned}
Min \quad & L_p = \left[w_1^p \left| R - \sum_{i=1}^{n}\bar{r}_i x_i \right|^p + w_2^p \left| \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij} x_i x_j - V \right|^p \right]^{\frac{1}{p}} \\
s.t. \quad & \sum_{i=1}^{n} x_i = 1 \\
& 0 \le x_i \le 1, \ i = 1, 2, ..., n
\end{aligned} \tag{11}$$

where R and V are the ideal values for expected return and variance, respectively. Ballestero and Romero (1996) studied the case where $p = 1$ and $p = \infty$. Ballestero and PlaSantamaria (2003) applied the compromise model (11) to a set of securities from the Madrid security market.

3.4 Perspective and limitations of multiple objective portfolio selection models

According to Aouni et al. (2014), goal programming models are popular among researchers for the portfolio selection problem because of the flexibility they offer concerning decision maker preferences in terms of goals and importance of the criteria. Xidonas et al. (2012) noted that these mathematical models are inefficient unless supported by algorithmic tools and mechanisms. The decision maker is looking for a solution that can consider his/her preferences and that takes account of the risk related to the return on the security, the nature of the security and the state of the market (Al-Shammari and Masri 2015).

4 The multiple objective stochastic portfolio selection models

The return is primarily defined as a stochastic parameter, and the Markowitz model optimizes the first and the second moment of this random return. In addition, almost all objective functions reported by Steuer et al. (2005) are stochastic. In this section, we review stochastic programming approaches and we review some of the single and multiple objective stochastic models for portfolio selection.

4.1 Single and multiple objective stochastic programs

A stochastic program is a mathematical program where the objective function and/or the constraints parameters are random variables (Kall and Wallace 1995):

$$\begin{aligned} Min \quad & f(x, w) \\ s.t. \quad & g_j(x, w) \geq 0, \quad j = 1, ..., m \\ & x \in X \end{aligned} \tag{12}$$

where $f(x, w)$, $g_j(x, w) \geq 0$, $j = 1, ..., m$ are functions defined using the random parameter w, X is the set of feasible solutions defined by deterministic constraints and w is a random event whose probability distribution P is defined on a set $\Omega \subset \mathbf{R^d}$.

A stochastic linear program can model problems under risk where explicit knowledge of the probability distribution is the main hypothesis. An intuitive approach to solve the stochastic program (12) is to replace the random parameters by their expected value and then obtain the following equivalent mathematical program:

$$\begin{aligned} Min \quad & E(f(x, w)) \\ s.t. \quad & E\left(g_j(x, w)\right) \geq 0, \quad j = 1, ..., m \\ & x \in X \end{aligned}$$

This approach is easy to implement but it may lead to an unfeasible decision (Sen and Higle 1999).

To solve the stochastic program (12), we need to transform it into an equivalent mathematical program. A first strategy, called the wait and see approach, is to solve program (12) for each realization of the random variable w and deliver the final solution after we observe the value of the random parameter w (Bereanu and Peeters 1970). A second strategy, the most used in practice and known as the here and now approach, supposes that we cannot wait for the occurrence of the random event and we have to find a solution here and now. Of the different here and now approaches, we describe in more detail below the chance constrained approach and the recourse approach.

The chance constrained approach was first proposed by Charnes and Cooper (1963) where a solution does not have to satisfy random constraints for all scenarios, but only for a percentage α of these scenarios. In a chance constrained approach, the equivalent of the stochastic program (12) can be written as follows:

$$\begin{aligned} Min \quad & E\,(f\,(x,\ w)) \\ s.t. \quad & P\left[g_j(x,\ w) \geq 0\right] \geq \alpha, \quad j = 1, ..., m \\ & x \in X \end{aligned} \tag{13}$$

where $\alpha \in [0,\ 1]$ is the probability (or reliability) level that characterizes the minimum degree of satisfaction of the uncertain constraints and $P\,[.]$ denotes the probability distribution.

Program (13) is called the joint chance constrained program because the probability level α is the same for all random constraints. If the probability level is different from one constraint to another, the resulting program, called the joint chance constrained program and is defined as follows:

$$\begin{aligned} Min \quad & E\,(f\,(x,\ w)) \\ s.t. \quad & P\left[g_j(x,\ w) \geq 0\right] \geq \alpha_j,\ j = 1, ..., m \\ & x \in X \end{aligned}$$

where $\alpha_j \in [0, 1]$ is the probability level related to the jth random constraints ($j = 1,...,m$).

Under some predefined condition for probability distribution, the chance constrained program (13) has a convex set of feasible solutions (Prékopa 1995). In the case of normal probability distribution, the set of feasible solutions in program (13) can be rewritten using linear constraints (Ben Abdelaziz et al. 2007).

The recourse approach considers that all solutions in X are feasible solutions and if a solution $x \in X$ does not verify some of the uncertain constraints $g_j(x,\ w) \geq 0$, then we penalize such a solution by introducing into the objective function an additional cost, called the recourse function $Q(x)$ (Birge and Louveaux 1997). The resulting certainty equivalent program under a recourse approach for the stochastic program (12) can be written as follows (Kall and Wallace 1995):

$$\begin{aligned} Min \quad & E\,(f\,(x,\ w)) + Q(x) \\ s.t. \quad & x \in X \end{aligned} \tag{14}$$

where $Q(x) = E\,(Q(x,\ w))$ and

$$\begin{aligned} Q(x,\ w) = \quad Min \quad & q(w)^t y \\ s.t \quad & W(w)y = g^-(x,\ w) \\ & y \geq 0 \end{aligned}$$

where $q(w)$ is the recourse cost, $W(w)$ is the recourse matrix, y is the recourse decision and $g^-(x,\ w) = g(x,\ w)$ if $g(x,\ w) < 0$ otherwise 0.

A multiple objective stochastic program is a stochastic program with more than two objective functions. To solve a multiple objective stochastic program, we can follow one of the following strategies (Ben Abdelaziz 2012):

- Transform the multiple objective stochastic program using one of the stochastic programming approaches and then solve the resulting multiple objective program using one of the multiple objective programming approaches;

- Transform the multiple objective stochastic program using one of the multiple objective programming approaches and then solve the resulting stochastic program using one of the stochastic programming approaches.

Both strategies combine two kinds of transformations, a stochastic transformation and a multiple objective transformation, to propose an equivalent mathematical program to the multiple objective stochastic program.

4.2 Multiple objective stochastic portfolio selection models

Few scholars have discussed multiple objective stochastic models for portfolio selection. In this section, we review some single and multiple objective stochastic portfolio selection models.

Early on, Roy (1952) proposed the chance constrained approach to minimize the probability of the portfolio return being less than a predetermined disaster level R_*:

$$Min\, \mathrm{P}\left[\sum_{i=1}^{n} r_i x_i \leq R_*\right]$$

Xu et al. (2011) proposed to maximize the level of return R, such that the probability that the portfolio return exceeds R is equal to α

$$\begin{aligned} & Max \quad R \\ & s.t. \quad \Pr\left\{\sum_{i=1}^{n} r_i x_i \geq R\right\} \geq \alpha \end{aligned} \tag{15}$$

This is a chance constrained program (15), where the portfolio risk is considered through the chance constraint.

Ben Tal (1991) proposed a two-stage recourse approach for the single objective stochastic portfolio selection problem, where we first decide on the amount of money to invest in the non-risky security x_0, and then invest the remaining wealth in risky securities. The model was presented as follows:

$$\begin{aligned} & Max \quad E\left(r_0 x_0 + \sum_{i=1}^{n} r_i x_i\right) \\ & s.t \quad x_0 + \sum_{i=1}^{n} x_i = 1, \\ & \qquad x_0 \geq 0,\ x_i \geq 0,\ i = 1, \ldots, n \end{aligned}$$

Ben Abdelaziz et al. (2007) presented a chance constrained compromise programming approach (CCCP) for the following bi-objective portfolio selection problem:

$$\begin{aligned} & Max \quad \sum_{i=1}^{n} r_i x_i \\ & Opt \quad \sum_{i=1}^{n} \beta_i x_i \end{aligned}$$

$$
\begin{aligned}
s.t.\quad & \sum_{i=1}^{n} x_i = 1 \\
& 0 \leq x_i \leq u_i,\ i = 1, ..., n
\end{aligned} \tag{16}
$$

where u_i is the upper bound for the proportion to be invested in the security i, $i = 1, ..., n$. The "*Opt*" in program (16) refers to the systematic portfolio risk that should be equal to a predefined value. Ben Abdelaziz et al. (2007) assumed that their optimal portfolio should be neither more nor less risky than the market, and therefore the target value for the portfolio beta should be equal to 1.

The CCCP approach is a mix between the compromise programming approach (multiple objective transformation) and the chance constrained programming approach (stochastic transformation). The model was used to select a portfolio among 45 securities listed in the Tunisian stock exchange market using the following three objective functions: rate of return, exchange flow ratio (EF_i) and portfolio beta. The resulting deterministic equivalent mathematical program is written as follows:

$$
\begin{aligned}
Min\quad & Z = \varepsilon + \delta_1^- + \delta_2^- + \delta_3^- + \delta_3^+ \\
s.t.\quad & R^* - \sum_{i=1}^{45} \bar{r}_i x_i + \phi^{-1}(1-\zeta)\sigma\left(R^* - \sum_{i=1}^{45} r_i x_i\right) - \varepsilon + \delta_1^- = 0 \\
& \sum_{i=1}^{45} EF_i x_i + \delta_2^- = EF^* \\
& \sum_{i=1}^{45} \beta_i x_i + \delta_3^- - \delta_3^+ = 1 \\
& \sum_{i=1}^{45} x_i = 1 \\
& 0 \leq x_i \leq 0.1,\ i = 1, ..., 45 \\
& \delta_1^-, \delta_2^-, \delta_3^-, \delta_3^+, \varepsilon \geq 0
\end{aligned}
$$

where R^* and EF^* are the ideal values of the rate of return objective function and the exchange flow ratio objective function, respectively and ϕ is the probability distribution function of the standard normal distribution.

Masmoudi and Ben Abdelaziz (2012) presented a recourse goal programming approach to solve program (16), where the difference between the portfolio and the minimum acceptable rate of return generates a penalty in the objective function. The resulting certainty equivalent model for program (16) is as follows:

$$
\begin{aligned}
Min\quad & \delta^+ + \delta^- + \sum_{s=1}^{S} p_s q(w_s)\, y(w_s) \\
s.t.\quad & \sum_{i=1}^{n} r_i(w_s)\, x_i + y(w_s) = R(w_s),\ s = 1, ..., S \\
& \sum_{i=1}^{n} \beta_i x_i - \delta^+ + \delta^- = 1
\end{aligned}
$$

$$\sum_{i=1}^{n} x_i = 1$$
$$0 \leq x_i \leq u_i,\ i = 1, ..., n$$
$$\delta^+ \geq 0, \delta^- \geq 0,\ y(w_s) \geq 0,\ s = 1, ..., S \quad (17)$$

where the security random return $r_i(w_s)$ is defined over the discrete set of states of the market $\Omega = \{w_1, ..., w_S\}$ and p_s is the probability of occurrence of the event w_s and $q(w_s)$ is the recourse penalty for the event w_s.

Recently, Masmoudi and Ben Abdelaziz (2015) extended their recourse goal programming approach by penalizing the infeasible solution for uncertain constraints with the most probable highest recourse cost rather than with the expected recourse cost. The proposed approach is called the chance constrained recourse (CCR) approach, and the certainty equivalent to program (16) is as follows:

$$\begin{aligned}
Min\quad & \delta^- + \delta^+ + \varepsilon \\
s.t.\quad & R - \sum_{i=1}^{n} \bar{r}_i x_i - \frac{\varepsilon}{q} + \phi^{-1}(\alpha) \sum_{i=1}^{n} \sigma_i x_i \leq 0 \\
& \sum_{i=1}^{n} \beta_i x_i + \delta^- - \delta^+ = 1,\ i = 1,\ 2, ...,\ n \\
& \sum_{i=1}^{n} x_i = 1 \\
& 0 \leq x_i \leq u_i,\ i = 1,\ 2, ...,\ n \\
& \delta^- \geq 0, \delta^+ \geq 0, \varepsilon \geq 0
\end{aligned}$$

Masri (2015) dealt with program (16) for cases where multiple stochastic goals are given for the return objective function. The author proposed a chance constrained approach to address the investors' minimum acceptable rate of return and a recourse approach for the investors' ideal rate of return.

The literature proposes few other models to build a certainty equivalent to the multiple objective stochastic portfolio selection problem. Ben Abdelaziz et al. (2009) used the stochastic goal programming approach to select a portfolio in the United Arab Emirates equity market considering the following five objective functions: capital appreciation; current income; the price earnings ratio; the market to book value ratio; and risk.

Masri et al. (2010) extended the CCCP approach for the following bi-objective stochastic portfolio selection problem:

$$\begin{aligned}
Max\quad & \sum_{i=1}^{n} r_i x_i \\
Max\quad & \sum_{i=1}^{n} l_i x_i \\
s.t.\quad & \sum_{i \in S_h} x_i \leq H_r \\
& \sum_{i \in S_l} x_i \geq L_r
\end{aligned}$$

$$\sum_{i=1}^{n} x_i = 1$$
$$0 \leq x_i \leq u_i, \; i = 1, \; 2, \ldots, \; n$$

where l_i is the liquidity of security i, H_r and L_r are the percentage to be invested in the set S_h of high risk securities and the set S_h of low risk securities, respectively.

4.3 Perspective and limitations of multiple objective stochastic portfolio selection models

The above multiple objective stochastic portfolio selection models may offer investors a powerful tool for managing risky portfolios. Knowledge of the probability distribution was an important assumption of the models presented. Such information does not hold in many situations. Ben Abdelaziz and Masri (2010) proposed a compromise approach for a multiple objective stochastic linear program in which the probability distribution is described by partial linear equations. Investors are aware of the importance of risk and uncertainty in today's rapidly changing environments; they want to avoid making investments based on models that disregard their ignorance of risk assessment.

Future research on the subject should explore the group decision aspect of multiple objective portfolio optimization. It should also address other forms of uncertainty while considering the dynamic aspect of the investment process.

5 Conclusion

The Portfolio selection problem can be intuitively modeled as a multiple objective stochastic program. This paper reviews the models currently proposed on the subject, and compares the different assumptions and proposed solutions.

Today, the risk of not achieving a desired or planned portfolio return is the most challenging issue in portfolio selection. Researchers should develop models that reassess the risk in a way that takes into account the sensitivity of the environment and the inherent ignorance of market trends. New trends in portfolio selection models should consider uncertain probability distribution (Masri and Abdelaziz 2010), uncertain risk measures (Ben Abdelaziz and Masmoudi 2014), socially responsible investment (Al-Shammari and Masri 2016), strategic behaviors, group strategic behaviors and dynamic aspects.

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