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A class of nonsmooth fractional multiobjective optimization problems

Thai Doan Chuong[1] · Do Sang Kim[2]

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Abstract This paper focuses on the study of optimality conditions and duality in nonsmooth fractional multiobjective optimization problems. Applying some advanced tools of variational analysis and generalized differentiation, we establish necessary optimality conditions for (weakly) efficient solutions of a fractional multiobjective optimization problem involving inequality and equality constraints. Sufficient optimality conditions for such solutions to the considered problem are also obtained by means of (strictly) generalized convex-affine functions. In addition, we address a dual problem to the primal one and examine duality relations between them.

Keywords Fractional multiobjective programming · Optimality condition · Duality · Limiting/Mordukhovich subdifferential · Generalized convex-affine function

Mathematics Subject Classification 49K99 · 65K10 · 90C29 · 90C46

1 Introduction

Optimality conditions and duality for (weakly) Pareto/efficient solutions in *fractional multi-objective optimization problems* have been investigated intensively by many researchers (see

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✉ Do Sang Kim
dskim@pknu.ac.kr

Thai Doan Chuong
chuongthaidoan@yahoo.com

1 Department of Mathematics and Applications, Saigon University, 273 An Duong Vuong Street, Ward 3, District 5, Ho Chi Minh City, Vietnam

2 Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea

e.g., Antczak 2006, 2008; Bahatia and Garg 1998; Bector et al. 1993; Chen 2002; Chinchuluun and Pardalos 2007; Chinchuluun et al. 2007; Kim et al. 2006; Konno and Kuno 1990; Kuk et al. 2001; Lee and Lai 2005; Lai and Ho 2012; Lalitha et al. 2003; Liu 1996; Liu and Yokoyama 1999; Liu and Feng 2007; Long 2011; Nobakhtian 2008; Niculescu 2007; Soleimani-damaneh 2008; Zalmai 2006 and the references therein). One of the main tools used to examine a fractional multiobjective optimization problem is that one employs the separation theorem of *convex sets* (see e.g., Rockafellar 1970) to provide necessary optimality conditions for (weakly) efficient solutions of the considered problem and exploits various kinds of (generalized) convex/or invex functions to formulate sufficient optimality conditions for such solutions. It should be noted further that since the kinds of (generalized) invex functions mentioned above have been constructed via the convexified/Clarke subdifferential of locally Lipschitz functions, we therefore have to remain using tacitly the separation theorem of convex sets in the proof schemes.

In fact, a characteristic of a fractional multiobjective optimization problem is that its objective function is generally *not* a convex function. Even under more restrictive concavity/convexity assumptions fractional multiobjective optimization problems are generally *nonconvex* ones. Besides, the (approximate) *extremal principle* (Mordukhovich 2006a), which plays a key role in variational analysis and generalized differentiation, has been well-recognized as a variational counterpart of the separation theorem for *nonconvex* sets. Hence using the extremal principle and other advanced techniques of variational analysis and generalized differentiation to establish optimality conditions seems to be suitable for *nonconvex/nonsmooth* fractional multiobjective optimization problems.

In this work, we employ some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, the sum rule and the quotient rule for the limiting/Mordukhovich subdifferential, and the intersection rule for the normal/Mordukhovich cone) to establish necessary optimality conditions for (weakly) Pareto/efficient solutions of a nonsmooth fractional multiobjective optimization problem with inequality and equality constraints. Since the limiting/Mordukhovich subdifferential of a real-valued function at a given point is *contained in* the convexified/Clarke subdifferential of such a function at the corresponding point (cf. Mordukhovich 2006a), the necessary optimality conditions formulated in terms of the limiting subdifferential are *sharper* than the corresponding ones expressed in terms of the convexified subdifferential. Sufficient optimality conditions for such solutions to the considered problem are also provided by means of introducing (strictly) generalized convex-affine functions defined in terms of the limiting subdifferential for locally Lipschitz functions. Along with optimality conditions, we state a dual problem to the primal one and explore weak, strong and converse duality relations under assumptions of (strictly) generalized convexity-affineness. Furthermore, examples are given for analyzing and illustrating the obtained results.

In passing, we wish to point out that besides the (weakly) Pareto/efficient solutions, the notion of super minimality/efficiency introduced by Borwein and Zhuang (1993) and more recently investigated by Bao and Mordukhovich (2009) plays also an important role in multiobjective optimization. Since the latter paper successfully established necessary optimality conditions for such efficiency in a general setting by using the above-mentioned generalized differential constructions, it could be possible to obtain results in this vein for fractional multiobjective optimization problems. We leave this for future study.

The rest of the paper is organized as follows. Section 2 contains some basic definitions from variational analysis and several auxiliary results. In Sect. 3, we first establish necessary optimality conditions for (weakly) efficient solutions of a fractional multiobjective optimization problem. Then we supply sufficient optimality conditions for such solutions. Section 4 is devoted to describing duality relations.

2 Preliminaries

Throughout the paper we use the standard notation of variational analysis (see e.g., Mordukhovich 2006a, b). Unless otherwise specified, all spaces under consideration are assumed to be *Asplund* (i.e., Banach spaces whose separable subspaces have separable duals). The canonical pairing between space X and its topological dual X^* is denoted by $\langle\cdot\,,\cdot\rangle$, while the symbol $\|\cdot\|$ stands for the norm in the considered space. As usual, the *polar cone* of a set $\Omega\subset X$ is defined by

$$\Omega^\circ := \{x^*\in X^*\mid \langle x^*,x\rangle\le 0\quad \forall x\in\Omega\}. \tag{2.1}$$

Also, for each $m\in\mathbb{N}:=\{1,2,\ldots\}$, we denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$.

Given a multifunction $F\colon X\rightrightarrows X^*$, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^*\in X^*\Big|\ \exists\ \text{sequences}\ \ x_n\to\bar{x}\ \ \text{and}\ \ x_n^*\xrightarrow{w^*}x^*\ \ \text{with}\ \ x_n^*\in F(x_n)\ \ \text{for all}\ \ n\in\mathbb{N}\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x\to\bar{x}$, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

A set $\Omega\subset X$ is *locally closed* if for each $\bar{x}\in\Omega$, there is a neighborhood U of $\bar{x}$ such that $\Omega\cap\operatorname{cl}U$ is closed. From now on, we always assume that sets under consideration are locally closed. Given $\Omega\subset X$, the *regular/Fréchet normal cone* to Ω at $\bar{x}\in\Omega$ is defined by

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^*\in X^*\Big|\ \limsup_{x\xrightarrow{\Omega}\bar{x}}\frac{\langle x^*,x-\bar{x}\rangle}{\|x-\bar{x}\|}\le 0\Big\}, \tag{2.2}$$

where $x\xrightarrow{\Omega}\bar{x}$ means that $x\to\bar{x}$ with $x\in\Omega$. If $\bar{x}\notin\Omega$, we put $\widehat{N}(\bar{x};\Omega):=\emptyset$.

The *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ at $\bar{x}\in\Omega$ is obtained from $\widehat{N}(\cdot;\Omega)$ by taking the sequential Painlevé–Kuratowski upper limits as

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\xrightarrow{\Omega}\bar{x}}\widehat{N}(x;\Omega). \tag{2.3}$$

If $\bar{x}\notin\Omega$, we put $N(\bar{x};\Omega):=\emptyset$.

For an extended real-valued function $\varphi: X\to\overline{\mathbb{R}}:=[-\infty,\infty]$, we set

$$\operatorname{gph}\varphi := \{(x,\mu)\in X\times\mathbb{R}\mid \mu=\varphi(x)\},\quad \operatorname{epi}\varphi := \{(x,\mu)\in X\times\mathbb{R}\mid \mu\ge\varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* of φ at $\bar{x}\in X$ with $|\varphi(\bar{x})|<\infty$ is defined by

$$\partial\varphi(\bar{x}) := \{x^*\in X^*\mid (x^*,-1)\in N((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi)\}. \tag{2.4}$$

If $|\varphi(\bar{x})|=\infty$, then one puts $\partial\varphi(\bar{x}):=\emptyset$. It is known (cf. Mordukhovich 2006a) that when φ is a convex function, the above-defined subdifferential coincides with the subdifferential in the sense of convex analysis (Rockafellar 1970).

Considering the indicator function $\delta(\cdot;\Omega)$ defined by $\delta(x;\Omega):=0$ for $x\in\Omega$ and by $\delta(x;\Omega):=\infty$ otherwise, we have a relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows (see Mordukhovich 2006a, Proposition 1.79):

$$N(\bar{x};\Omega)=\partial\delta(\bar{x};\Omega)\quad\forall\bar{x}\in\Omega. \tag{2.5}$$

The nonsmooth version of Fermat's rule (see e.g., Mordukhovich 2006a, Proposition 1.114), which is an important fact for many applications, can be formulated as follows: If $\bar{x}\in X$ is a *local minimizer* for $\varphi: X\to\overline{\mathbb{R}}$, then

$$0\in\partial\varphi(\bar{x}). \tag{2.6}$$

The following limiting subdifferential sum rule is needed for our study.

Lemma 2.1 (See Mordukhovich 2006a, Theorem 3.36) *Let* $\varphi_i : X \to \overline{\mathbb{R}}, i = 1, 2, \ldots, n, n \geq 2$, *be lower semicontinuous around* $\bar{x}\in X$, *and let all these functions except, possibly, one be Lipschitz continuous around* $\bar{x}$. *Then one has*

$$\partial(\varphi_1+\varphi_2+\cdots+\varphi_n)(\bar{x})\subset\partial\varphi_1(\bar{x})+\partial\varphi_2(\bar{x})+\cdots+\partial\varphi_n(\bar{x}). \tag{2.7}$$

Combining this limiting subdifferential sum rule with the quotient rule (cf. Mordukhovich 2006a, Corollary 1.111(ii)), we get an estimate for the limiting subdifferential of quotients.

Lemma 2.2 *Let* $\varphi_i : X \to \overline{\mathbb{R}}, i = 1, 2$, *be Lipschitz continuous around* $\bar{x}$. *Assume that* $\varphi_2(\bar{x})\neq 0$. *Then one has*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x})\subset\frac{\partial\big(\varphi_2(\bar{x})\varphi_1\big)(\bar{x})+\partial\big(-\varphi_1(\bar{x})\varphi_2\big)(\bar{x})}{[\varphi_2(\bar{x})]^2}. \tag{2.8}$$

Recall Mordukhovich (2006a) that a set $\Omega\subset X$ is *sequentially normally compact* (SNC) at $\bar{x}\in\Omega$ if for any sequences

$$x_k\xrightarrow{\Omega}\bar{x}\text{ and }x_k^*\xrightarrow{w^*}0\text{ with }x_k^*\in\widehat{N}(x_k;\Omega),$$

one has $\|x_k^*\|\to 0$ as $k\to\infty$. Obviously, this SNC property is automatically satisfied in finite dimensional spaces. The interested reader is referred to Fabian and Mordukhovich (2003) for a comprehensive comparison of the SNC property and other constructions of this type.

A function $\varphi : X\to\mathbb{R}$ is called *sequentially normally compact* (SNC) at $\bar{x}\in X$ if $\operatorname{gph}\varphi$ is SNC at $(\bar{x},\varphi(\bar{x}))$. According to Mordukhovich (2006a, Corollary 1.69(i)), φ is SNC at $\bar{x}\in X$ if it is Lipschitz continuous around $\bar{x}$.

In what follows, we also need the intersection rule for the normal cones under the fulfillment of the SNC condition.

Lemma 2.3 (See Mordukhovich 2006a, Corollary 3.5) *Assume that* $\Omega_1,\Omega_2\subset X$ *are closed around* $\bar{x}\in\Omega_1\cap\Omega_2$ *and that at least one of* $\{\Omega_1,\Omega_2\}$ *is* SNC *at this point. If*

$$N(\bar{x};\Omega_1)\cap\big(-N(\bar{x};\Omega_2)\big)=\{0\},$$

then

$$N(\bar{x};\Omega_1\cap\Omega_2)\subset N(\bar{x};\Omega_1)+N(\bar{x};\Omega_2).$$

3 Optimality conditions in fractional multiobjective optimization

This section is devoted to studying optimality conditions for fractional multiobjective optimization problems. More precisely, by using the nonsmooth version of Fermat's rule, the sum rule and the quotient rule for the limiting subdifferentials, and the intersection rule for the Mordukhovich cones, we first establish necessary optimality conditions for (weakly) efficient solutions of a fractional multiobjective optimization problem. Then by imposing assumptions of (strictly) generalized convexity-affineness, we give sufficient optimality conditions for such solutions.

Let Ω be a nonempty locally closed subset of X, and let $K := \{1, \ldots, m\}, I := \{1, \ldots, n\} \cup \emptyset$ and $J := \{1, \ldots, l\} \cup \emptyset$ be index sets. In what follows, Ω is always assumed to be SNC at the point under consideration. This assumption is automatically fulfilled when X is a finite dimensional space.

We consider the following *fractional multiobjective optimization problem*:

$$\min_{\mathbb{R}^m_+} \left\{ f(x) := \left(\frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)} \right) \,\Big|\, x \in C \right\}, \tag{P}$$

where the constraint set C is defined by

$$C := \big\{ x \in \Omega \mid g_i(x) \leq 0,\ i \in I, h_j(x) = 0,\ j \in J \big\}, \tag{3.1}$$

and the functions $p_k, q_k, k \in K, g_i, i \in I$, and $h_j, j \in J$ are locally Lipschitz on X. For the sake of convenience, we further assume that $q_k(x) > 0, k \in K$ for all $x \in \Omega$, and that $p_k(\bar{x}) \leq 0, k \in K$ for the reference point $\bar{x} \in \Omega$. Also, we use hereafter the notation $g := (g_1, \ldots, g_n), h := (h_1, \ldots, h_l)$ and $f := (f_1, \ldots, f_m)$, where $f_k := \frac{p_k}{q_k}, k \in K$.

Definition 3.1 (i) We say that $\bar{x} \in C$ is an *efficient solution* of problem (P), and write $\bar{x} \in \mathcal{S}(P)$, iff

$$\forall x \in C, \quad f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \backslash \{0\}.$$

(ii) A point $\bar{x} \in C$ is called a *weakly efficient solution* of problem (P), and write $\bar{x} \in \mathcal{S}^w(P)$, iff

$$\forall x \in C, \quad f(x) - f(\bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+.$$

For $\bar{x} \in \Omega$, let us put

$$I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}, \quad J(\bar{x}) := \{j \in J \mid h_j(\bar{x}) = 0\}.$$

Definition 3.2 We say that condition (CQ) is satisfied at $\bar{x} \in \Omega$ if there do not exist $\beta_i \geq 0, i \in I(\bar{x})$ and $\gamma_j \geq 0, j \in J(\bar{x})$, such that $\sum_{i \in I(\bar{x})} \beta_i + \sum_{j \in J(\bar{x})} \gamma_j \neq 0$ and

$$0 \in \sum_{i \in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J(\bar{x})} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \big) + N(\bar{x}; \Omega).$$

It is worth to mention here that when considering $\bar{x} \in C$ defined in (3.1) with $\Omega = X$ in the smooth setting, the above-defined (CQ) is guaranteed by the Mangasarian-Fromovitz constraint qualification (see e.g., Mordukhovich 2006a for more details).

The following theorem gives a Karush–Kuhn–Tucker (KKT) type necessary optimality condition for (weakly) efficient solutions of problem (P).

Theorem 3.3 *Let the* (CQ) *be satisfied at* $\bar{x} \in \Omega$. *If* $\bar{x} \in \mathcal{S}^w(P)$, *then there exist* $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \backslash \{0\}$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$, *and* $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ *such that*

$$0 \in \sum_{k \in K} \lambda_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \big) + N(\bar{x}; \Omega), \quad \beta_i g_i(\bar{x}) = 0, \quad i \in I. \tag{3.2}$$

Proof Let $\bar{x} \in \mathcal{S}^w(P)$ and let

$$\varphi(x) := \max_{k \in K} \left\{ \frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right\}.$$

We are going to show that

$$\varphi(\bar{x}) \leq \varphi(x) \quad \forall x \in C. \tag{3.3}$$

Indeed, if this is not the case, then there exists $x_0 \in C$ such that $\varphi(x_0) < \varphi(\bar{x})$. Since $\varphi(\bar{x}) = 0$, it holds that $\max_{k \in K} \left\{ \frac{p_k(x_0)}{q_k(x_0)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right\} < 0$. Thus,

$$f(x_0) - f(\bar{x}) \in -\text{int}\, \mathbb{R}^m_+,$$

which contradicts the fact that $\bar{x} \in \mathcal{S}^w(P)$.

We assert by (3.3) that $\bar{x}$ is a minimizer of the following scalar problem

$$\min_{x \in C} \varphi(x).$$

Thus, $\bar{x}$ is a minimizer of the following unconstrained scalar optimization problem

$$\min_{x \in X} \varphi(x) + \delta(x; C). \tag{3.4}$$

Applying the nonsmooth version of Fermat's rule (2.6) to problem (3.4), we have

$$0 \in \partial \big(\varphi + \delta(\cdot; C) \big)(\bar{x}). \tag{3.5}$$

Since the function φ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; C)$ is l.s.c around this point, it follows from the sum rule (2.7) applied to (3.5) and from the relation in (2.5) that

$$0 \in \partial \varphi(\bar{x}) + N(\bar{x}; C). \tag{3.6}$$

On the one side, employing the formula for the basic subdifferential of maximum functions (see Mordukhovich 2006a, Theorem 3.46(ii)) and the sum rule (2.7) we obtain

$$\partial \varphi(\bar{x}) = \partial \left(\max_{k \in K} \left\{ \frac{p_k}{q_k}(\cdot) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right\} \right)(\bar{x}) \subset \left\{ \sum_{k \in K} \alpha_k \partial \left(\frac{p_k}{q_k} \right)(\bar{x}) \,\middle|\, \alpha_k \geq 0, k \in K, \sum_{k \in K} \alpha_k = 1 \right\}.$$

Taking (2.8) into account, we arrive at

$$\partial\varphi(\bar{x}) \subset \left\{ \sum_{k\in K} \alpha_k \frac{\partial\big(q_k(\bar{x})p_k\big)(\bar{x}) + \partial\big(-p_k(\bar{x})q_k\big)(\bar{x})}{[q_k(\bar{x})]^2} \,\Big|\, \alpha_k \geq 0, k \in K, \sum_{k\in K}\alpha_k = 1 \right\}$$
$$= \left\{ \sum_{k\in K} \alpha_k \frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{[q_k(\bar{x})]^2} \,\Big|\, \alpha_k \geq 0, k \in K, \sum_{k\in K}\alpha_k = 1 \right\}, \qquad (3.7)$$

where the equality holds due to the fact that $-p_k(\bar{x}) \geq 0$, $q_k(\bar{x}) > 0$ for all $k \in K$.

On the other side, by letting

$$\tilde{\Omega} := \big\{x \in X \mid g_i(x) \leq 0,\ i \in I,\ h_j(x) = 0,\ j \in J\big\},$$

we have $C = \tilde{\Omega} \cap \Omega$. The (CQ) being satisfied at $\bar{x}$ entails that there do not exist $\beta_i \geq 0, i \in I(\bar{x})$, and $\gamma_j \geq 0$, $j \in J(\bar{x}) = J$ such that $\sum_{i\in I(\bar{x})} \beta_j + \sum_{j\in J} \gamma_j \neq 0$ and

$$0 \in \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big).$$

Hence, we get by Mordukhovich (2006a, Corollary 4.36) that

$$N(\bar{x}; \tilde{\Omega}) \subset \Big\{ \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) \,\Big|\, \beta_i \geq 0, i \in I(\bar{x}), \gamma_j \geq 0,\ j \in J \Big\}. \qquad (3.8)$$

As the (CQ) is satisfied at $\bar{x}$ and Ω is assumed to be SNC at this point, we apply Lemma 2.3 to obtain the following

$$N(\bar{x}; C) = N(\bar{x}; \tilde{\Omega} \cap \Omega) \subset N(\bar{x}; \tilde{\Omega}) + N(\bar{x}; \Omega). \qquad (3.9)$$

It follows from (3.6)–(3.9) that

$$0 \in \left\{ \sum_{k\in K} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) \,\Big|\, \alpha_k \geq 0, k \in K, \sum_{k\in K}\alpha_k = 1 \right\}$$
$$+ \left\{ \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) \,\Big|\, \beta_i \geq 0, i \in I(\bar{x}), \gamma_j \geq 0,\ j \in J \right\}$$
$$+ N(\bar{x}; \Omega). \qquad (3.10)$$

Now, put $\beta_i := 0$ for $i \in I\backslash I(\bar{x})$, and let $\lambda_k := \frac{\alpha_k}{q_k(\bar{x})}$ for $k \in K$. It is clear that (3.10) implies (3.2) and so, the proof is complete. □

A simple example below shows that the conclusion of Theorem 3.3 may fail if the (CQ) is not satisfied at the point in question.

Example 3.4 Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right),$$

where $p_1(x) = p_2(x) := x, q_1(x) = q_2(x) := x^2 + 1, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := x^2, h(x) := 0, x \in \mathbb{R}$. We consider problem (P) with $m := 2$ and

$\Omega := (-\infty, 0] \subset \mathbb{R}$. Then $C = \{0\}$ and thus, $\bar{x} := 0 \in \mathcal{S}^w(P)(= \mathcal{S}(P))$. In this setting, we have $N(\bar{x}; \Omega) = [0, +\infty)$. Now, we can check that condition (CQ) is not satisfied at $\bar{x}$. Meantime, $\bar{x}$ does not satisfy (3.2) either.

We refer the reader to a result (Bao et al. 2007) about necessary optimality conditions for a more general multiobjective fractional program with *equilibrium constraints* by way of a different approach. More concretely, the paper (Bao et al. 2007) considers the problem (P) with an additional equilibrium constraint:

$$0 \in G(x) + Q(x),$$

where $G, Q : X \rightrightarrows Y$ are multifunctions between Banach spaces. Their approach is to compute the *Mordukhovich/limiting coderivative* of the epigraphical multifunction of the component functions $\varphi_k := \frac{p_k}{q_k}, k \in K$ for deriving the optimality conditions. Under a constraint qualification condition (Bao et al. 2007, Theorem 4.2) obtains a KKT type necessary optimality condition for a (local) weakly efficient solution $\bar{x}$ involving the values of the limiting coderivatives of $D^*G(\bar{x}, \bar{y})$ and $D^*Q(\bar{x}, \bar{y})$, where $\bar{y} \in G(\bar{x}) \cap (-Q(\bar{x}))$, and of the subdifferentials $\frac{\partial\left(q_k(\bar{x})p_k\right)(\bar{x})+\partial\left(-p_k(\bar{x})q_k\right)(\bar{x})}{[q_k(\bar{x})]^2}$, $k \in K$. Hence, Bao et al. (2007, Theorem 4.2) is more general than Theorem 3.3 due to the former contains the data of equilibrium constraint $D^*G(\bar{x}, \bar{y})$ and $D^*Q(\bar{x}, \bar{y})$. However, by exploiting the exclusive structure of our problem, we can elaborate separately the subdifferentials of the functions p_k and $q_k, k \in K$ at the referenced point and turn the necessary optimality criterion into a traditional representation of the KKT condition (cf. 3.2), which allows us to be able to provide sufficient optimality conditions and explore duality relations in the sequel.

The next example illustrates that Theorem 3.3 works better in comparison with some of the existing results about optimality conditions for fractional multiobjective optimization problems, for instance (Kim et al. 2006).

Example 3.5 Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}\right),$$

where $p_1(x) = p_2(x) := |x|, q_1(x) = q_2(x) := -|x| + 1, x \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := -x - 1, x \in \mathbb{R}$. Let us consider problem (P) with $K := \{1, 2\}$, $I := \{1\}$, $J := \emptyset$, and $\Omega := (-1, 1) \subset \mathbb{R}$. It is easy to check that $\bar{x} := 0 \in \mathcal{S}^w(P)$ and the (CQ) is satisfied at this point. So, in this setting we can apply Theorem 3.3 to conclude that $\bar{x}$ satisfies condition (3.2). Meanwhile, since the functions q_1, q_2 are not differentiable at $\bar{x}$ (Kim et al. 2006, Theorem 2.2) is not applicable to this problem.

It should be noted further that, in general, a feasible point of problem (P) satisfying condition (3.2) is not necessarily to be a weakly efficient solution even in the smooth case. This will be illustrated by the following example.

Example 3.6 Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}\right),$$

where $p_1(x) = p_2(x) := x^3 - 1, q_1(x) = q_2(x) := x^2 + 1, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := -x^2, h(x) := 0, x \in \mathbb{R}$. Let us consider problem (P) with $m := 2$ and $\Omega := (-\infty, 1] \subset \mathbb{R}$. Then $C = \Omega$ and thus, $\bar{x} := 0 \in C$. In this setting, we have $N(\bar{x}; \Omega) = \{0\}$. Observe that $\bar{x}$ satisfies condition (3.2). However, $\bar{x} \notin \mathcal{S}^w(P)$.

By virtue of Example 3.6, obtaining sufficient optimality conditions for (weakly) efficient solutions of problem (P) requires concepts of (generalized) convexity-affineness-type for locally Lipschitz functions.

Definition 3.7 (i) We say that $(f, g; h)$ is *generalized convex-affine* on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x}), i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})$, $j \in J$ there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &\geq \langle u_k^*, \nu\rangle, k \in K, \\ q_k(x) - q_k(\bar{x}) &\geq \langle v_k^*, \nu\rangle, k \in K, \\ g_i(x) - g_i(\bar{x}) &\geq \langle x_i^*, \nu\rangle, i \in I, \\ h_j(x) - h_j(\bar{x}) &= \omega_j \langle y_j^*, \nu\rangle, j \in J, \end{aligned}$$

where $\omega_j = 1$ (respectively, $\omega_j = -1$) whenever $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$).

(ii) We say that $(f, g; h)$ is *strictly generalized convex-affine* on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega\backslash\{\bar{x}\}$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x}), i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})$, $j \in J$ there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &> \langle u_k^*, \nu\rangle, k \in K, \\ q_k(x) - q_k(\bar{x}) &\geq \langle v_k^*, \nu\rangle, k \in K, \\ g_i(x) - g_i(\bar{x}) &\geq \langle x_i^*, \nu\rangle, i \in I, \\ h_j(x) - h_j(\bar{x}) &= \omega_j \langle y_j^*, \nu\rangle, j \in J, \end{aligned}$$

where $\omega_j = 1$ (respectively, $\omega_j = -1$) whenever $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$).

It is clear that if Ω is convex, $p_k, q_k, k \in K, g_i, i \in I$ are convex, and $h_j, j \in J$ are affine, then $(f, g; h)$ is generalized convex-affine on Ω at $\bar{x} \in \Omega$ with $\nu := x - \bar{x}$ for each $x \in \Omega$. And besides, when $q_k(x) \equiv 1$ for $k \in K$ (i.e., $f := (p_1, \ldots, p_m)$), the above-defined notions reduce respectively to L-(strictly) invex-infine functions given in Chuong (2012), Chuong and Kim (2014). Hence, the class of generalized convex-affine functions surely contains some *nonconvex* functions (see e.g., Chuong 2012, Example 3.3). The reader is referred to Chuong (2012, 2013), Chuong and Kim (2014), Sach et al. (2003) for some properties and applications of (L-) invex-infine functions.

We are now in a position to provide sufficient conditions for a feasible point of problem (P) to be a *weakly efficient* (or *efficient*) solution.

Theorem 3.8 *Assume that $\bar{x} \in C$ satisfies condition (3.2).*

(i) If $(f, g; h)$ is generalized convex-affine on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w(P)$.

(ii) If $(f, g; h)$ is strictly generalized convex-affine on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{S}(P)$.

Proof Since $\bar{x}$ satisfies condition (3.2), there exist $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m\backslash\{0\}, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}_+^n, \gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}_+^l, u_k^* \in \partial p_k(\bar{x}), v_k^* \in \partial q_k(\bar{x}), k \in K, x_i^* \in \partial g_i(\bar{x}), i \in I$ with $\mu_i g_i(\bar{x}) = 0$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})$, $j \in J$ such that

$$-\left[\sum_{k\in K} \lambda_k \left(u_k^* - \frac{p_k(\bar{x})}{q_k(\bar{x})} v_k^*\right) + \sum_{i\in I} \mu_i x_i^* + \sum_{j\in J} \gamma_j y_j^*\right] \in N(\bar{x}; \Omega). \tag{3.11}$$

We first justify (i). Assume on the contrary that $\bar{x} \notin \mathcal{S}^w(P)$. This means that there is $\hat{x} \in C$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\mathrm{int}\,\mathbb{R}^m_+. \tag{3.12}$$

By the generalized convex-affine property of $(f, g; h)$ on Ω at $\bar{x}$, for $\hat{x}$ above, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned}
&\sum_{k\in K} \lambda_k \left(\langle u_k^*, \nu\rangle - \frac{p_k(\bar{x})}{q_k(\bar{x})}\langle v_k^*, \nu\rangle\right) + \sum_{i\in I}\mu_i\langle x_i^*, \nu\rangle + \sum_{j\in J}\gamma_j\langle y_j^*, \nu\rangle \\
&\le \sum_{k\in K}\lambda_k \left[p_k(\hat{x}) - p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}\big(q_k(\hat{x}) - q_k(\bar{x})\big)\right] + \sum_{i\in I}\mu_i\big(g_i(\hat{x}) - g_i(\bar{x})\big) \\
&\quad + \sum_{j\in J}\frac{1}{\omega_j}\gamma_j\big(h_j(\hat{x}) - h_j(\bar{x})\big) \\
&= \sum_{k\in K}\lambda_k\left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}q_k(\hat{x})\right) + \sum_{i\in I}\mu_i\big(g_i(\hat{x}) - g_i(\bar{x})\big) \\
&\quad + \sum_{j\in J}\frac{1}{\omega_j}\gamma_j\big(h_j(\hat{x}) - h_j(\bar{x})\big),
\end{aligned}$$

where $\omega_j \in \{-1, 1\}$, $j \in J$. Due to the definition of polar cone (2.1), it follows from (3.11) and the relation $\nu \in N(\bar{x}; \Omega)^\circ$ that

$$0 \le \sum_{k\in K}\lambda_k\left(\langle u_k^*, \nu\rangle - \frac{p_k(\bar{x})}{q_k(\bar{x})}\langle v_k^*, \nu\rangle\right) + \sum_{i\in I}\mu_i\langle x_i^*, \nu\rangle + \sum_{j\in J}\gamma_j\langle y_j^*, \nu\rangle. \tag{3.13}$$

Hence,

$$\begin{aligned}
0 \le &\sum_{k\in K}\lambda_k\left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}q_k(\hat{x})\right) + \sum_{i\in I}\mu_i\big(g_i(\hat{x}) - g_i(\bar{x})\big) \\
&+ \sum_{j\in J}\sigma_j\big(h_j(\hat{x}) - h_j(\bar{x})\big),
\end{aligned} \tag{3.14}$$

where $\sigma_j := \frac{\gamma_j}{\omega_j} \in \mathbb{R}$, $j \in J$. In addition, it holds that

$$\sum_{i\in I}\mu_i\big(g_i(\hat{x}) - g_i(\bar{x})\big) + \sum_{j\in J}\sigma_j\big(h_j(\hat{x}) - h_j(\bar{x})\big) \le 0$$

due to the fact that $\mu_i g_i(\bar{x}) = 0$, $\mu_i g_i(\hat{x}) \le 0$, $i \in I$, and $h_j(\bar{x}) = 0$, $h_j(\hat{x}) = 0$, $j \in J$. So, we get by (3.14) that

$$0 \le \sum_{k\in K}\lambda_k\left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}q_k(\hat{x})\right).$$

This entails that there is $k_0 \in K$ such that

$$0 \le p_{k_0}(\hat{x}) - \frac{p_{k_0}(\bar{x})}{q_{k_0}(\bar{x})}q_{k_0}(\hat{x}) \tag{3.15}$$

due to $\lambda \in \mathbb{R}^m_+\backslash\{0\}$. The inequality in (3.15) is equivalent to the following one

$$f_{k_0}(\bar{x}) \le f_{k_0}(\hat{x}),$$

which contradicts (3.12) and, therefore, the proof of (i) is complete.

Now, we prove (ii). Suppose to the contrary that $\bar{x} \notin \mathcal{S}(P)$. Then there is $\hat{x} \in C$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\mathbb{R}^m_+ \setminus \{0\}. \tag{3.16}$$

By the strictly generalized convex-affine property of $(f, g; h)$ on Ω at $\bar{x}$, for $\hat{x}$ above, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\sum_{k\in K} \lambda_k \left(\langle u_k^*, \nu\rangle - \frac{p_k(\bar{x})}{q_k(\bar{x})} \langle v_k^*, \nu\rangle \right) + \sum_{i\in I} \mu_i \langle x_i^*, \nu\rangle + \sum_{j\in J} \gamma_j \langle y_j^*, \nu\rangle$$
$$< \sum_{k\in K} \lambda_k \left[p_k(\hat{x}) - p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \big(q_k(\hat{x}) - q_k(\bar{x})\big) \right] + \sum_{i\in I} \mu_i \big(g_i(\hat{x}) - g_i(\bar{x})\big)$$
$$+ \sum_{j\in J} \frac{1}{\omega_j} \gamma_j \big(h_j(\hat{x}) - h_j(\bar{x})\big),$$

where $\omega_j \in \{-1, 1\}$, $j \in J$. Similar to the proof of (i), we obtain (3.13) and then arrive at

$$0 < \sum_{k\in K} \lambda_k \left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} q_k(\hat{x}) \right).$$

This entails that there is $k_0 \in K$ such that

$$0 < p_{k_0}(\hat{x}) - \frac{p_{k_0}(\bar{x})}{q_{k_0}(\bar{x})} q_{k_0}(\hat{x}).$$

Equivalently,

$$f_{k_0}(\bar{x}) < f_{k_0}(\hat{x}).$$

It together with (3.16) gives a contradiction, which completes the proof. □

4 Duality in fractional multiobjective optimization

In this section we propose a dual problem to the primal one in the sense of Mond and Weir (1981) and examine weak, strong, and converse duality relations between them. Note further that another dual problem formulated in the sense of Wolfe (1961) can be similarly treated.

Let $z \in X, \lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$. In connection with the fractional multiobjective optimization problem (P), we consider a *fractional multiobjective dual problem* of the form:

$$\max_{\mathbb{R}^m_+} \left\{ \bar{f}(z, \lambda, \mu, \gamma) := \left(\frac{p_1(z)}{q_1(z)}, \cdots, \frac{p_m(z)}{q_m(z)} \right) \,\Big|\, (z, \lambda, \mu, \gamma) \in C_D \right\}. \tag{D}$$

Here the constraint set C_D is defined by

$$C_D := \Big\{ (z, \lambda, \mu, \gamma) \in \Omega \times (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^n_+ \times \mathbb{R}^l_+ \mid 0 \in \sum_{k\in K} \lambda_k \left(\partial p_k(z) - \frac{p_k(z)}{q_k(z)} \partial q_k(z) \right)$$
$$+ \sum_{i\in I} \mu_i \partial g_i(z) + \sum_{j\in J} \gamma_j \big(\partial h_j(z) \cup \partial(-h_j)(z)\big) + N(z; \Omega),$$
$$\langle \mu, g(z)\rangle + \langle \sigma, h(z)\rangle \geq 0 \quad \forall \sigma \in \mathbb{S}(0, ||\gamma||) \Big\},$$

where $\mathbb{S}(0, ||\gamma||) := \{\sigma \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma||\}$.

We need to address here that an *efficient solution* (resp., a *weakly efficient solution*) of problem (D) is similarly defined as in Definition 3.1 by replacing $-\mathbb{R}^m_+$ (resp., $\operatorname{int}\mathbb{R}^m_+$) by $\mathbb{R}^m_+$ (resp., $-\operatorname{int}\mathbb{R}^m_+$). Also, we denote the set of efficient solutions (resp., weakly efficient solutions) of problem (D) by $\mathcal{S}(D)$ (resp., $\mathcal{S}^w(D)$).

In what follows, we use the following notation for convenience.

$$u \prec v \Leftrightarrow u - v \in -\operatorname{int}\mathbb{R}^m_+,\ \ u \not\prec v \text{ is the negation of } u \prec v,$$
$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+\setminus\{0\},\ \ u \not\preceq v \text{ is the negation of } u \preceq v.$$

The first theorem in this section describes weak duality relations between the primal problem (P) and the dual problem (D).

Theorem 4.1 (Weak Duality) *Let $x \in C$ and let $(z, \lambda, \mu, \gamma) \in C_D$.*

(i) If $(f, g; h)$ is generalized convex-affine on Ω at z, then

$$f(x) \not\prec \bar{f}(z, \lambda, \mu, \gamma).$$

(ii) If $(f, g; h)$ is strictly generalized convex-affine on Ω at z, then

$$f(x) \not\preceq \bar{f}(z, \lambda, \mu, \gamma).$$

Proof Since $(z, \lambda, \mu, \gamma) \in C_D$, there exist $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+\setminus\{0\}, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+, \gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+, u^*_k \in \partial p_k(z), v^*_k \in \partial q_k(z), k \in K, x^*_i \in \partial g_i(z), i \in I$, and $y^*_j \in \partial h_j(z) \cup \partial(-h_j)(z), j \in J$ such that

$$-\left[\sum_{k\in K}\lambda_k\left(u^*_k - \frac{p_k(z)}{q_k(z)}v^*_k\right) + \sum_{i\in I}\mu_i x^*_i + \sum_{j\in J}\gamma_j y^*_j\right] \in N(z;\Omega), \tag{4.1}$$

$$\langle\mu, g(z)\rangle + \langle\sigma, h(z)\rangle \geq 0 \ \text{ for all } \ \sigma \in \mathbb{R}^l \text{ with } ||\sigma|| = ||\gamma||. \tag{4.2}$$

In order to justify (i), we assume to the contrary that

$$f(x) \prec \bar{f}(z, \lambda, \mu, \gamma).$$

This means that

$$f(x) - f(z) \in -\operatorname{int}\mathbb{R}^m_+. \tag{4.3}$$

By the generalized convex-affine property of $(f, g; h)$ on Ω at z, for such x, there exists $\nu \in N(z;\Omega)^\circ$ such that

$$\begin{aligned}
&\sum_{k\in K}\lambda_k\left(\langle u^*_k, \nu\rangle - \frac{p_k(z)}{q_k(z)}\langle v^*_k, \nu\rangle\right) + \sum_{i\in I}\mu_i\langle x^*_i, \nu\rangle + \sum_{j\in J}\gamma_j\langle y^*_j, \nu\rangle \\
&\quad\leq \sum_{k\in K}\lambda_k\left[p_k(x) - p_k(z) - \frac{p_k(z)}{q_k(z)}\big(q_k(x) - q_k(z)\big)\right] + \sum_{i\in I}\mu_i\big(g_i(x) - g_i(z)\big) \\
&\qquad + \sum_{j\in J}\frac{1}{\omega_j}\gamma_j\big(h_j(x) - h_j(z)\big) \\
&\quad= \sum_{k\in K}\lambda_k\left(p_k(x) - \frac{p_k(z)}{q_k(z)}q_k(x)\right) + \sum_{i\in I}\mu_i\big(g_i(x) - g_i(z)\big) \\
&\qquad + \sum_{j\in J}\frac{1}{\omega_j}\gamma_j\big(h_j(x) - h_j(z)\big),
\end{aligned}$$

where $\omega_j \in \{-1, 1\}$, $j \in J$. Due to the definition of polar cone (2.1), it follows from (4.1) and the relation $\nu \in N(z; \Omega)^\circ$ that

$$0 \leq \sum_{k\in K} \lambda_k \left(\langle u_k^*, \nu\rangle - \frac{p_k(z)}{q_k(z)} \langle v_k^*, \nu\rangle \right) + \sum_{i\in I} \mu_i \langle x_i^*, \nu\rangle + \sum_{j\in J} \gamma_j \langle y_j^*, \nu\rangle.$$

Thus,

$$\begin{aligned} 0 \leq &\sum_{k\in K} \lambda_k \left(p_k(x) - \frac{p_k(z)}{q_k(z)} q_k(x) \right) + \sum_{i\in I} \mu_i \big(g_i(x) - g_i(z)\big) \\ &+ \sum_{j\in J} \sigma_j \big(h_j(x) - h_j(z)\big), \end{aligned} \tag{4.4}$$

where $\sigma_j := \frac{\gamma_j}{\omega_j} \in \mathbb{R}$, $j \in J$. In addition, due to $x \in C$, $\sum_{i\in I} \mu_i g_i(x) \leq 0$ and $\sum_{j\in J} \sigma_j h_j(x) = 0$. So, (4.4) entails that

$$\begin{aligned} 0 &\leq \sum_{k\in K} \lambda_k \left(p_k(x) - \frac{p_k(z)}{q_k(z)} q_k(x) \right) - \sum_{i\in I} \mu_i g_i(z) - \sum_{j\in J} \sigma_j h_j(z) \\ &= \sum_{k\in K} \lambda_k \left(p_k(x) - \frac{p_k(z)}{q_k(z)} q_k(x) \right) - (\langle \mu, g(z)\rangle + \langle \sigma, h(z)\rangle), \end{aligned}$$

where $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_l) \in \mathbb{R}^l$. Moreover, since $||\sigma|| = ||\gamma||$, (4.2) is valid and, thus, we obtain

$$0 \leq \sum_{k\in K} \lambda_k \left(p_k(x) - \frac{p_k(z)}{q_k(z)} q_k(x) \right).$$

This entails that there is $k_0 \in K$ such that

$$0 \leq p_{k_0}(x) - \frac{p_{k_0}(z)}{q_{k_0}(z)} q_{k_0}(x) \tag{4.5}$$

due to $\lambda \in \mathbb{R}^m_+ \backslash \{0\}$. The inequality in (4.5) is equivalent to the following one

$$f_{k_0}(z) \leq f_{k_0}(x),$$

which contradicts (4.3). The proof of (i) is complete.

Let us now prove (ii). Assume to the contrary that

$$f(x) \preceq \bar{f}(z, \lambda, \mu, \gamma),$$

or equivalently,

$$f(x) - f(z) \in -\mathbb{R}^m_+ \backslash \{0\}. \tag{4.6}$$

By the strictly generalized convex-affine property of $(f, g; h)$ on Ω at z, for such x, there is $\nu \in N(z; \Omega)^\circ$ such that

$$\sum_{k\in K}\lambda_k\left(\langle u_k^*,v\rangle-\frac{p_k(z)}{q_k(z)}\langle v_k^*,v\rangle\right)+\sum_{i\in I}\mu_i\langle x_i^*,v\rangle+\sum_{j\in J}\gamma_j\langle y_j^*,v\rangle$$
$$<\sum_{k\in K}\lambda_k\left[p_k(x)-p_k(z)-\frac{p_k(z)}{q_k(z)}\big(q_k(x)-q_k(z)\big)\right]+\sum_{i\in I}\mu_i\big(g_i(x)-g_i(z)\big)$$
$$+\sum_{j\in J}\frac{1}{\omega_j}\gamma_j\big(h_j(x)-h_j(z)\big),$$

where $\omega_j\in\{-1,1\}$, $j\in J$. Proceeding similarly as in the proof of (i), we arrive at

$$0<\sum_{k\in K}\lambda_k\left(p_k(x)-\frac{p_k(z)}{q_k(z)}q_k(x)\right).$$

This entails that there is $k_0\in K$ such that

$$0<p_{k_0}(x)-\frac{p_{k_0}(z)}{q_{k_0}(z)}q_{k_0}(x). \tag{4.7}$$

Equivalently,

$$f_{k_0}(z)<f_{k_0}(x),$$

which contradicts (4.6) and therefore completes the proof. □

The next example asserts the *importance* of the generalized convex-affine property of $(f,g;h)$ imposed in Theorem 4.1. Namely, the conclusion of the theorem may go awry if this property has been violated.

Example 4.2 Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by

$$f(x):=\left(\frac{p_1(x)}{q_1(x)},\frac{p_2(x)}{q_2(x)}\right),$$

where $p_1(x)=p_2(x):=x^3, q_1(x)=q_2(x):=x^2+1, x\in\mathbb{R}$, and let $g,h:\mathbb{R}\to\mathbb{R}$ be given by $g(x):=-|x|$ and $h(x):=x^2+x$ for $x\in\mathbb{R}$. Consider the problem (P) with $m:=2$ and $\Omega:=(-\infty,0]\subset\mathbb{R}$. Then $C=\{-1,0\}$ and let us select $\bar{x}:=-1\in C$. Now, consider the dual problem (D). By choosing $\bar{z}:=0\in\Omega, \bar{\lambda}:=(\frac{1}{2},\frac{1}{2}), \bar{\mu}:=1$, and $\bar{\gamma}:=1$, it holds that $(\bar{z},\bar{\lambda},\bar{\mu},\bar{\gamma})\in C_D$ and that

$$f(\bar{x})=\left(-\frac{1}{2},-\frac{1}{2}\right)\prec(0,0)=\bar{f}(\bar{z},\bar{\lambda},\bar{\mu},\bar{\gamma}),$$

showing that the conclusion of Theorem 4.1 fails to hold. The reason is that $(f,g;h)$ is not generalized convex-affine on Ω at $\bar{z}$.

Strong duality relations between the primal problem (P) and the dual problem (D) read as follows.

Theorem 4.3 (Strong Duality) *Let $\bar{x}\in\mathcal{S}^w(P)$ be such that the* (CQ) *is satisfied at this point. Then there exists $(\bar{\lambda},\bar{\mu},\bar{\gamma})\in(\mathbb{R}^m_+\backslash\{0\})\times\mathbb{R}^n_+\times\mathbb{R}^l_+$ such that $(\bar{x},\bar{\lambda},\bar{\mu},\bar{\gamma})\in C_D$ and $f(\bar{x})=\bar{f}(\bar{x},\bar{\lambda},\bar{\mu},\bar{\gamma})$.*

(i) If in addition $(f,g;h)$ is generalized convex-affine on Ω at any $z\in\Omega$, then $(\bar{x},\bar{\lambda},\bar{\mu},\bar{\gamma})\in\mathcal{S}^w(D)$.

(ii) If in addition $(f, g; h)$ is strictly generalized convex-affine on Ω at any $z \in \Omega$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in \mathcal{S}(D)$.

Proof Thanks to Theorem 3.3, we find $\bar{\lambda} := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \backslash \{0\}$, $\bar{\mu} := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\bar{\gamma} := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that

$$0 \in \sum_{k \in K} \lambda_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I} \mu_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \big)$$
$$+ N(\bar{x}; \Omega), \quad \mu_i g_i(\bar{x}) = 0, \quad i \in I.$$

In addition, due to $\bar{x} \in C$, $h_j(\bar{x}) = 0$ for all $j \in J$, and thus, $\langle \bar{\mu}, g(\bar{x}) \rangle + \langle \sigma, h(\bar{x}) \rangle = 0$ for all $\sigma \in \mathbb{S}(0, ||\bar{\gamma}||)$. So, we conclude that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in C_D$. Obviously,

$$f(\bar{x}) = \bar{f}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}).$$

(i) If $(f, g; h)$ is generalized convex-affine on Ω at any $z \in \Omega$, then by invoking (i) of Theorem 4.1, we obtain

$$\bar{f}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) = f(\bar{x}) \nprec \bar{f}(z, \lambda, \mu, \gamma)$$

for any $(z, \lambda, \mu, \gamma) \in C_D$. It means that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in \mathcal{S}^w(D)$.

(ii) If $(f, g; h)$ is strictly generalized convex-affine on Ω at any $z \in \Omega$, then by invoking (ii) of Theorem 4.1, we assert that

$$\bar{f}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \npreceq \bar{f}(z, \lambda, \mu, \gamma)$$

for any $(z, \lambda, \mu, \gamma) \in C_D$. Hence, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in \mathcal{S}(D)$. □

Remark 4.4 Observe that the (CQ) imposed in Theorem 4.3 plays an *important* role in establishing the strong duality results. More precisely, if $\bar{x}$ is a weakly efficient solution of the primal problem at which the (CQ) is not satisfied, then we might not find out a triplet $(\bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ belongs to the feasible/constraint set of the dual problem. In this case, of course, we do not have strong dual relations. Let us look back at Example 3.4.

We close this section by presenting converse-like duality relations between the primal problem (P) and the dual problem (D).

Theorem 4.5 (Converse Duality) *Let* $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in C_D$.

(i) If $\bar{x} \in C$ and $(f, g; h)$ is generalized convex-affine on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w(P)$.
(ii) If $\bar{x} \in C$ and $(f, g; h)$ is strictly generalized convex-affine on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{S}(P)$.

Proof Since $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in C_D$, there exist $\bar{\lambda} := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \backslash \{0\}$, $\bar{\mu} := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$, $\bar{\gamma} := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x})$, $i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})$, $j \in J$ such that

$$-\left[\sum_{k \in K} \lambda_k \left(u_k^* - \frac{p_k(\bar{x})}{q_k(\bar{x})} v_k^* \right) + \sum_{i \in I} \mu_i x_i^* + \sum_{j \in J} \gamma_j y_j^* \right] \in N(\bar{x}; \Omega), \tag{4.8}$$

$$\langle \bar{\mu}, g(\bar{x}) \rangle + \langle \sigma, h(\bar{x}) \rangle \geq 0 \ \text{ for all } \ \sigma \in \mathbb{R}^l \text{ with } ||\sigma|| = ||\bar{\gamma}||. \tag{4.9}$$

It follows by (4.9) that

$$\langle \bar{\mu}, g(\bar{x}) \rangle \geq |\langle \bar{\gamma}, h(\bar{x}) \rangle| \geq 0. \tag{4.10}$$

Let $\bar{x} \in C$. We have $g_i(\bar{x}) \leq 0$ for all $i \in I$ and thus, $\langle \bar{\mu}, g(\bar{x}) \rangle \leq 0$. This together with (4.10) yields $\langle \bar{\mu}, g(\bar{x}) \rangle = 0$. Then $\mu_i g_i(\bar{x}) = 0$ for all $i \in I$. So, we assert by virtue of (4.8) that $\bar{x}$ satisfies condition (3.2). To finish the proof, it remains to apply Theorem 3.8. □

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