Ann Oper Res (2016) 244:545–569
DOI 10.1007/s10479-016-2117-4

ORIGINAL PAPER

Multi-period cardinality constrained portfolio selection models with interval coefficients

Yong-Jun Liu[1] · **Wei-Guo Zhang**[1] · **Jun-Bo Wang**[2]

Published online: 30 August 2016

Abstract In this paper, we discuss a multi-period portfolio selection problem in emerging markets. To provide investors with more choices, we propose four multi-period cardinality constrained portfolio selection models with interval coefficients in both objective functions and constraints. The proposed models can be equivalently represented as the parameter programming problems with interval coefficients in constraints. We utilize the definition of the possibility degree for interval inequality to handle the interval inequality constraints in the proposed models and express investors' different risk attitudes. Then, the proposed models are transformed into deterministic models. After that, we design a new dynamic differential evolution algorithm with self-adapting control parameter to solve the transformed deterministic models. Finally, we provide a numerical example to illustrate the applications of the proposed models and demonstrate the effectiveness of the designed algorithm.

Keywords Multi-period portfolio · Interval coefficient · Order relation · Possibility degree · Differential evolution algorithm

1 Introduction

Portfolio selection aims at selecting a set of securities to optimize a performance measure under some specific constraints, particularly a budget constraint. Investors often want to

✉ Yong-Jun Liu
liuyongjun0202@aliyun.com; yjlgx0202@126.com

✉ Wei-Guo Zhang
wgzhang@scut.edu.cn; zhwg61@263.net

Jun-Bo Wang
jwang2@cityu.edu.hk

1 School of Business Administration, South China University of Technology, Guangzhou 510641, People's Republic of China

2 Department of Economics and Finance, City University of Hong Kong, 88 Tat Chee Avenue, Kowloon, Hong Kong

obtain a most profitable return and avoid risk at the same time. However, how to construct a satisfying portfolio has become a challenging problem owing to the uncertainty associated with the returns of securities.

In real financial markets, there exist many non-random factors that affect portfolio decision-making such as economic, social, political, people's psychological factors and so on. Especially, experts' knowledge and experiences play important roles in portfolio decision-making. So, in many cases, the market data of risky assets are usually imprecise and ambiguous. Fuzzy set theory in Zadeh (1965) has been regarded as a powerful tool for describing an uncertain environment with vagueness, ambiguity or some other type of fuzziness, which appears in many aspects of financial markets, such as portfolio managers' unpredictable behaviors. With the wide use of fuzzy set theory, more and more people have realized that they could use it to investigate the fuzzy uncertainty associated with financial markets. To model the fuzzy uncertainty associated with financial markets, the return of a risky asset is characterized by a fuzzy variable with possibility distribution. A possibility distribution is identified to reflect the returns of risky assets associated with possibility grades offered by portfolio experts. Tanaka et al. (2000) pointed out that fuzzy portfolio models integrated the historical data and human factors to catch variations of stock markets better. By using fuzzy mathematical approaches, quantitative analysis, qualitative analysis, the experts' knowledge and the investors' subjective opinions can be better incorporated into a portfolio selection model as mentioned by Wang and Zhu (2002).

Fuzzy portfolio selection problem has been researched from 1990s. Various researchers have investigated fuzzy portfolio selection problem by using different approaches and proposed kinds of portfolio optimization models. For example, Watada (1997) proposed a fuzzy portfolio selection model by using fuzzy numbers to represent decision makers' aspiration levels for the expected return rate and a certain degree of risk. Inuiguchi and Tanino (2000) introduced a novel possibilistic programming approach to investigate portfolio selection problem based on the minimax regret criterion and yielded a distributive investment solution. By using fuzzy goal programming approach, Parra et al. (2001) presented a multiple criteria model for portfolio selection. León et al. (2002) discussed some fuzzy optimization schemes for managing portfolio selection problems under the framework of risk–return tradeoff. Bilbao-Terol et al. (2006) formulated a fuzzy compromise programming model for portfolio selection. Gupta et al. (2008) applied fuzzy mathematical programming to develop comprehensive models for asset portfolio optimization. Tiryaki and Ahlatcioglu (2009) used fuzzy analytic hierarchy process to investigate fuzzy portfolio selection problem. Calvo et al. (2014) proposed a fuzzy multi-criteria model for portfolio selection with non-financial goals. In Vercher et al. (2007), Zhang et al. (2007) and Zhang et al. (2009), possibility theory was applied to handle uncertainty and solve portfolio optimization problems. Huang (2008), Zhang et al. (2011), Li et al. (2012) and Gupta et al. (2013) used credibility theory to study portfolio selection problems in a fuzzy economic environment.

All the literature mentioned above are formulated under the framework of fuzzy set theory. They assume that investors can accurately predict the possibility distributions of the uncertain parameters on risky assets by historical data. However, in emerging markets or financial incidents, it is impossible for investors to predict the possibility distributions of the uncertain parameters on risky assets due to lack of historical information. In these cases, the investors have no choice but to predict the approximate ranges of the uncertain parameters on risky assets based on experts' experiences and their own market sentiment. Nowadays, some researchers have used interval programming approaches to study fuzzy portfolio selection problems. Lai et al. (2002) developed an interval absolute semideviation model for portfolio selection. Wang and Zhu (2002) employed interval programming for portfolio selection.

Ida (2003, 2004) discussed multi-objective portfolio selection problems with interval coefficients. Giove et al. (2006) formulated a minimax regret portfolio selection problem in which the prices of the securities were considered as interval variables. Based on fuzzy decision theory, Fang et al. (2006) proposed a linear interval programming model for the portfolio rebalancing problem with transaction costs. Li and Xu (2007) dealt with a possibilistic portfolio selection model with interval center values. Bhattacharyya et al. (2011) utilized the concept of interval numbers in fuzzy set theory to extend the classical mean–variance model into a mean–variance–skewness model with transaction cost for fuzzy portfolio selection. Liu (2011) discussed the uncertain portfolio selection problem where the returns of risky assets were represented by interval numbers. Wu et al. (2013) studied an interval portfolio selection model in which both the returns and the risks of risky assets were defined as interval numbers.

Though, there are some researches about portfolio selection problems in emerging markets, most of them are mainly focused on single period portfolio selection. In the real world, an investor's behavior is usually multi-period. Thus, it is necessary to investigate multi-period portfolio selection problems in emerging markets. Recently, Liu et al. (2013) studied a multi-period portfolio selection problem in emerging markets by using interval programming approach. However, they still neglected many real constraints such as floor and ceiling constraints and cardinality constraints. As far as investors are concerned, cardinality constraints enable them to limit the complexity of a portfolio and control transaction costs. Bound constraints enable them to control the amount invested on each asset. Based on the facts mentioned above, we investigate multi-period portfolio selection problem with some real features in emerging markets by using interval programming approach. Our contributions can be summarized as the following three aspects: (1) We propose four interval portfolio optimization models with cardinality constraints for multi-period portfolio selection in emerging markets. To achieve greater flexibility in portfolio selection, we incorporate some decision criteria including return, transaction costs, risk, liquidity, diversification degree, bound constraints and cardinality constraints into the proposed models. (2) We first use the concepts of possibility degree of interval inequalities to express investors' different risk attitudes and transform the proposed interval programming portfolio selection models into deterministic forms. (3) Considering the complexity of the proposed models, we design a new dynamic differential evolution algorithm with a self-adapting control parameter to solve them.

The remainder of this paper is structured as follows. In Sect. 2, we summarize some basic concepts about interval numbers. In Sect. 3, we propose four interval portfolio optimization models with cardinality constraints for multi-period portfolio selection in emerging markets. In Sect. 4, based on the definition of possibility degree for interval inequality, we transform the proposed interval programming models into the corresponding deterministic forms. Then, we design a new dynamic differential evolution algorithm with self-adapting control parameter to solve the deterministic models. In Sect. 5, we give a numerical example to demonstrate the applications of our models and the effectiveness of the designed algorithm. Finally, we present conclusions of our study in Sect. 6.

2 Basic conceptions

Denote the set of all real numbers by $\mathbb{R}$. An ordered pair in a bracket defines an interval number

$$\widetilde{a} = [\underline{a}, \overline{a}] = \{x : \underline{a} \leq x \leq \overline{a}, x \in \mathbb{R}\}, \tag{1}$$

where $\underline{a}$ and $\overline{a}$ are the lower and upper bounds of interval number $\widetilde{a}$. In the special case, when $\underline{a} = \overline{a}$, $\widetilde{a}$ is reduced to a real number. The center and width of interval number $\widetilde{a} = [\underline{a}, \overline{a}]$ are defined as

$$m(\widetilde{a}) = \frac{1}{2}(\underline{a} + \overline{a}) \quad \text{and} \quad w(\widetilde{a}) = \frac{1}{2}(\overline{a} - \underline{a}). \tag{2}$$

Using $m(\widetilde{a})$ and $\omega(\widetilde{a})$, the interval number $\widetilde{a}$ can be rewritten as

$$\begin{aligned}\widetilde{a} &= \langle m(\widetilde{a}), \omega(\widetilde{a})\rangle \\ &= \{x : m(\widetilde{a}) - \omega(\widetilde{a}) \leq x \leq m(\widetilde{a}) + \omega(\widetilde{a}), x \in \mathbb{R}\}.\end{aligned}$$

Notice that $m(\widetilde{a})$ is the crisp value of $\widetilde{a}$, which behaves like the expected value of a uniform distribution. $\omega(\widetilde{a})$ is similar to the spread of a symmetrical fuzzy number. It represents the uncertainty associated with interval number $\widetilde{a}$ and behaves likes the fluctuation range of a random variable, which is determined by historical data and human's subjective judgement in decision making.

Definition 1 (Alefeld and Herzberger (1983)). Let $\circ \in \{+, -, \times, \div\}$ be a binary operation on $\mathbb{R}$. For any given two interval numbers $\widetilde{a}$ and $\widetilde{b}$, the binary operation on them is defined by

$$\widetilde{a} \circ \widetilde{b} = \{x \circ y : x \in \widetilde{a}, y \in \widetilde{b}\},$$

where 0 is not in $\widetilde{b}$ for the case of division.

Let $\widetilde{a} = [\underline{a}, \overline{a}]$ and $\widetilde{b} = [\underline{b}, \overline{b}]$ be two interval numbers, and let $\lambda \in \mathbb{R}$ be a real number. The binary operations on interval numbers used in this paper are given as follows (see Moore (1966))

$$\widetilde{a} + \widetilde{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{3}$$

$$\lambda\widetilde{a} = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}], & \text{if } \lambda \geq 0, \\ [\lambda\underline{a}, \lambda\overline{a}], & \text{if } \lambda < 0, \end{cases} \tag{4}$$

$$\widetilde{a} \pm \lambda = [\underline{a} \pm \lambda, \overline{a} \pm \lambda], \tag{5}$$

$$\widetilde{a} \times \widetilde{b} = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}]. \tag{6}$$

Let $\widetilde{a}_1 = [\underline{a}_1, \overline{a}_1], \widetilde{a}_2 = [\underline{a}_2, \overline{a}_2], \ldots, \widetilde{a}_n = [\underline{a}_n, \overline{a}_n]$ be n interval numbers, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n nonnegative real numbers. Then, the linear combination of the n interval numbers can be expressed by

$$\lambda_1\widetilde{a}_1 + \lambda_2\widetilde{a}_2 + \ldots + \lambda_n\widetilde{a}_n = \left[\sum_{i=1}^{n} \lambda_i\underline{a}_i, \sum_{i=1}^{n} \lambda_i\overline{a}_i\right]. \tag{7}$$

It follows from Eqs. (2) and (7) that

$$\lambda_1 m(\widetilde{a}_1) + \lambda_2 m(\widetilde{a}_2) + \ldots + \lambda_n m(\widetilde{a}_n) = \sum_{i=1}^{n} \lambda_i m(\widetilde{a}_i),$$

$$\lambda_1 \omega(\widetilde{a}_1) + \lambda_2 \omega(\widetilde{a}_2) + \ldots + \lambda_n \omega(\widetilde{a}_n) = \sum_{i=1}^{n} \lambda_i \omega(\widetilde{a}_i).$$

Definition 2 (Ishibuchi and Tanaka (1990)). Let $\widetilde{a}$ and $\widetilde{b}$ be two interval numbers. Then, the interval inequality $\widetilde{a} \succeq \widetilde{b}$ is defined as follows

$$\widetilde{a} \succeq \widetilde{b} \text{ if and only if } \underline{a} \geq \underline{b} \text{ and } \overline{a} \geq \overline{b}.$$

For describing the interval inequality above in detail, Jiang et al. (2008) gave the concept of possibility degree as follows.

Definition 3 For any given two interval numbers $\widetilde{a} = [\underline{a}, \overline{a}]$ and $\widetilde{b} = [\underline{b}, \overline{b}]$, the possibility degree of $\widetilde{a} \succeq \widetilde{b}$ (denoted by $P(\widetilde{a} \succeq \widetilde{b})$) is defined as

$$P(\widetilde{a} \succeq \widetilde{b}) = \begin{cases} 1, & \text{if } \underline{a} \geq \overline{b}, \\ \frac{\overline{a}-\overline{b}}{\overline{a}-\underline{a}} + \frac{\overline{b}-\underline{a}}{\overline{a}-\underline{a}} \cdot \frac{\underline{a}-\underline{b}}{\overline{b}-\underline{b}} + \frac{1}{2} \cdot \frac{\overline{b}-\underline{a}}{\overline{a}-\underline{a}} \cdot \frac{\overline{b}-\underline{a}}{\overline{b}-\underline{b}}, & \text{if } \underline{b} \leq \underline{a} < \overline{b} \leq \overline{a}, \\ \frac{\overline{a}-\overline{b}}{\overline{a}-\underline{a}} + \frac{1}{2} \cdot \frac{\overline{b}-\underline{b}}{\overline{a}-\underline{a}}, & \text{if } \underline{a} < \underline{b} < \overline{b} \leq \overline{a}, \\ \frac{1}{2} \cdot \frac{\overline{a}-\underline{b}}{\overline{a}-\underline{a}} \cdot \frac{\overline{a}-\underline{b}}{\overline{b}-\underline{b}}, & \text{if } \underline{a} < \underline{b} \leq \overline{a} < \overline{b}, \\ \frac{\underline{a}-\underline{b}}{\overline{b}-\underline{b}} + \frac{1}{2} \cdot \frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}, & \text{if } \underline{b} \leq \underline{a} < \overline{a} < \overline{b}, \\ 0, & \text{if } \overline{a} < \underline{b}. \end{cases}$$

Especially, in Definition 3, if $\widetilde{b}$ is reduced into a real number b, then

$$P(\widetilde{a} \succeq b) = \begin{cases} 0, & \text{if } b > \overline{a}, \\ \frac{\overline{a}-b}{\overline{a}-\underline{a}}, & \text{if } \underline{a} < b \leq \overline{a}, \\ 1, & \text{if } b \leq \underline{a}. \end{cases} \tag{8}$$

If $\widetilde{a}$ is reduced into a real number a in Definition 3, then

$$P(a \succeq \widetilde{b}) = \begin{cases} 0, & \text{if } a < \underline{b}, \\ \frac{a-\underline{b}}{\overline{b}-\underline{b}}, & \text{if } \underline{b} < a \leq \overline{b}, \\ 1, & \text{if } a \geq \overline{b}. \end{cases} \tag{9}$$

Thus, the possibility degree in the concept of an interval inequality represents the degree to which one interval number is larger or smaller than another.

Notice that an interval number is similar to a uniform distribution in form. However, they have obvious difference as follows: The interval number can be viewed as a special fuzzy number whose membership function takes the value 1 over the interval and 0 anywhere else. Correspondingly, a uniform distribution variable is special case of random variable which takes a constant value in an interval, and 0 anywhere else. In financial decision-making, an interval number can be regarded as an alternative for a uniform distribution, in which the un-quantifiable factors such as experts' knowledge and investors' subjective opinions can be easily reflected. What's more, in contrast with the uniform distribution, we need less information to determine the interval number of a parameter than to determine the uniform distribution. It only requires us to determine the lower and upper bounds of the interval by using both historical data and human subjective judgement. Unlike the uniform distribution, it does not require us to determine any distribution function defined over the interval with equal probability.

3 Interval programming models for multi-period cardinality constrained portfolio selection

In this section, we discuss a multi-period portfolio selection problem in emerging markets, in which the expected proceeds, the variances of the return rates and the turnover rates on securities are characterized by interval numbers. We propose four multi-period cardinality constrained portfolio selection models with interval coefficients. In the proposed models, we use the expected proceeds of a portfolio as the investment return, the variance of the return rate on a portfolio as the investment risk and the turnover rate of a portfolio as the portfolio liquidity. Before formulating the proposed models, we first introduce an interval estimation approach to predict the aforementioned three factors of interval values.

3.1 Estimation of the interval-valued expected proceeds, variance and turnover rate

Since the future states of securities cannot be accurately predicted in emerging markets, in this paper, we use the interval estimation approach in Fang et al. (2008) to predict the interval values of the investment return, investment risk and turnover rate of a security. Here, we use the estimation of the investment return interval of the security as an example to introduce the estimation approach.

Traditionally, the arithmetic mean of the historical proceeds of a security is used as the approximate value of its expected proceeds. So the expected proceeds of the security is a crisp value in this way. However, using the traditional estimation approach, we need to solve the following two main problems. Firstly, if the observation period of the historical data of the security is long, it may lead to the result that the influence of both the earlier historical data and the recent data are the same. However, in the real world, the later historical data of the security most often indicate that the performance of a corporation is more important than that of the earlier historical data. Secondly, if the historical data of the security are not enough, one cannot accurately estimate the statistical parameters due to data scarcity.

To take the two problems above into consideration, it is a good idea to characterize the expected proceeds of a security as an interval number rather than a crisp value based on the arithmetic mean of historical data. Investors may make use of a corporation's financial reports and the historical data of the security to determine its expected proceeds interval. To determine the expected proceeds interval of a security, let us consider the following three factors:

(1) *Arithmetic mean proceeds factor* Although the arithmetic mean of the proceeds of a security should not be represented as its expected proceeds directly, it is a good approximation. Generally, the arithmetic mean proceeds factor of security i at period t is calculated by its historical data as follows

$$r_{t,i}^{a} = \frac{1}{t-t_0} \sum_{s=t_0}^{t-1} r_{s,i}, \quad t-1 \geq t_0,$$

where t_0 is the starting period and t the current period and $r_{s,i}$ is the real proceeds of unit capital invested on security i at period s.

(2) *Historical proceeds tendency factor* The historical proceeds tendency is also an important index for predicting the future proceeds of a security. In the real world, if the recent proceeds of the security have been increasing, we believe that its expected proceeds will be greater than the arithmetic mean proceeds based on historical data. Conversely, if the recent proceeds of the security have been declining, we believe that its expected proceeds will be smaller than the arithmetic mean proceeds based on historical data. Denote the historical proceeds tendency factor of security i at period t as $r_{t,i}^{h}$, which reflects the tendency of the proceeds on security i. The value of $r_{t,i}^{h}$ can be regarded as the arithmetic mean proceeds of the recent historical data on security i. The observation period of recent historical data is subjectively determined by experts. In many cases, the value of $r_{t,i}^{h}$ can be directly given by experts.

(3) *Forecast proceeds factor*: The third factor influencing the expected proceeds of a security is its estimated future proceeds. According to the financial reports of a corporation, if the proceeds on the corporation's stocks will increase, then we believe that the expected proceeds of security i at period t is larger than its arithmetic mean proceeds factor $r^a_{t,i}$. Contrarily, if the proceeds of the corporation's stocks will decrease in future, then we believe that the expected proceeds of security i at period t is smaller than its arithmetic mean proceeds factor $r^a_{t,i}$. Denote the forecast proceeds factor of security i at period t by $r^f_{t,i}$. The value of $r^f_{t,i}$ can be determined by some forecasts based on the financial reports and experts' experiences.

Using the three factors above, we can determine lower and upper limits of the expected proceeds of security i at period t. We set the minimum and maximum values of the three factors as the lower and upper limits of its expected proceeds, denoted them by $\underline{r}_{t,i} = \min\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\}$ and $\overline{r}_{t,i} = \max\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\}$. Then, the expected proceeds of security i at period t can be estimated by $\widetilde{r}_{t,i} = [\underline{r}_{t,i}, \overline{r}_{t,i}]$. Similarly, the turnover rate of security i at period t can be estimated by $\widetilde{l}_{t,i} = [\min\{l^a_{t,i}, l^h_{t,i}, l^f_{t,i}\}, \max\{l^a_{t,i}, l^h_{t,i}, l^f_{t,i}\}]$ and the covariance of the return rates of securities i and j can be estimated by $\widetilde{\delta}_{t,ij} = [\min\{\delta^a_{t,ij}, \delta^h_{t,ij}, \delta^f_{t,ij}\}, \max\{\delta^a_{t,ij}, \delta^h_{t,ij}, \delta^f_{t,ij}\}]$. Here, $l^a_{t,i}$, $l^h_{t,i}$ and $l^f_{t,i}$ denote the arithmetic mean, historical tendency and future forecast factors of the turnover rate of security i at period t, respectively; $\delta^a_{t,ij}$, $\delta^h_{t,ij}$ and $\delta^f_{t,ij}$ denote the arithmetic mean, historical tendency and future forecast covariance factors of securities i and j at period t, respectively. For convenience, we denote $\widetilde{l}_{t,i} = [\underline{l}_{t,i}, \overline{l}_{t,i}]$ and $\widetilde{\delta}_{t,ij} = [\underline{\delta}_{t,ij}, \overline{\delta}_{t,ij}]$ in the following sections.

3.2 Assumptions and notations

Suppose that an investor with initial wealth W_0 selects n securities from emerging markets for constructing T consecutive time periods investment. At the beginning of each of the following $T - 1$ periods, the investor can reallocate his wealth. Owing to lack of historical data, the proceeds, the turnover rates and the variances of the return rates on the n securities in the T investment periods are assumed to be interval numbers. To aid the description, we introduce the following notations:

$\widetilde{r}_{t,i}$	the expected proceeds of per unit capital invested on security i at period t, where $\widetilde{r}_{t,i} = [\underline{r}_{t,i}, \overline{r}_{t,i}]$;
$\widetilde{\delta}_{t,ij}$	the covariance of the return rates on securities i and j at period t, where $\widetilde{\delta}_{t,ij} = [\underline{\delta}_{t,ij}, \overline{\delta}_{t,ij}]$;
$c_{t,i}$	the transaction cost rate of security i at period t, where $c_{t,i}$ is a real number;
$x_{t,i}$	the investment proportion of security i at period t;
x_t	the portfolio at period t, where $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,n})$;
$\widetilde{\delta}_t$	the variance of the return rate on the portfolio at period t;
$\widetilde{l}_{t,i}$	the turnover rate of security i $(i = 1, 2, \ldots, n)$, where $\widetilde{l}_{t,i} = [\underline{l}_{t,i}, \overline{l}_{t,i}]$;
W_t	the available wealth at the end of period t for all $t = 1, 2, \ldots, T$.

3.3 Decision objective and investment constraints

With the same consideration as Markowitz (1987), we assume that the transaction costs at period t is a V-shaped function of the differences between the given portfolio at period $t - 1$ and the new portfolio at period t. Hence, the transaction costs of the portfolio $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,n})$ at period t can be expressed as

$$C_t = \sum_{i=1}^{n} c_{t,i} \left| x_{t,i} - x_{t-1,i} \right|, \quad t = 1, 2, \ldots, T. \tag{10}$$

Using the interval estimation approach in Sect. 3.1, the expected proceeds of per unit capital invested on security i at period t can be given by

$$\widetilde{r}_{t,i} = [\underline{r}_{t,i}, \overline{r}_{t,i}] = \left[\min\left\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\right\}, \max\left\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\right\}\right]. \tag{11}$$

Considering the fact that short selling may increase investment risk and incur transaction costs, as is usual in most of the existing literature, we assume that short selling is not allowed over the whole investment horizon, that is, $x_{t,i} \geq 0$ $(i = 1, 2, \ldots, n; t = 1, 2, \ldots, T)$. From Eq. (7), the expected proceeds of per unit capital invested on the portfolio at period t can be represented as the following interval number

$$\widetilde{R}_t = \sum_{i=1}^{n} x_{t,i} \widetilde{r}_{t,i} = \left[\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i}, \sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i}\right]. \tag{12}$$

Derived from Eqs. (5), (10) and (12), the expected net proceeds of per unit capital invested on the portfolio at period t is

$$\widetilde{R}_{N,t} = \sum_{i=1}^{n} x_{t,i} \widetilde{r}_{t,i} - C_t = \left[\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t, \sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t\right]. \tag{13}$$

According to Eqs. (6) and (13), the wealth obtained at the end of period t is

$$W_t = W_{t-1} \widetilde{R}_{N,t} = W_{t-1} \left[\left(\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t\right), \left(\sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t\right)\right]. \tag{14}$$

It follows from Eqs. (6) and (14) that the terminal wealth obtained at the end of period T is

$$W_T = W_0 \prod_{t=1}^{T} \widetilde{R}_{N,t} = W_0 \left[\prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t\right), \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t\right)\right]. \tag{15}$$

Then, by Eqs. (2) and (15), the crisp value of the terminal wealth W_T is

$$m(W_T) = \frac{1}{2} W_0 \left[\prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t\right) + \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t\right)\right]. \tag{16}$$

According to the assumption above, the covariance of the return rates on securities i and j $(i, j = 1, 2, \ldots, n)$ at each period are characterized by interval numbers. Similar to the estimation of the expected proceeds above, the arithmetic mean covariance factor of securities i and j at period t can be computed by $\delta^a_{t,ij} = \frac{1}{t-t_0} \sum_{s=t_0}^{t-1} \delta_{s,ij}$. Assume that the historical covariance tendency factor $\delta^h_{t,ij}$ and the forecast covariance factor $\delta^f_{t,ij}$ are directly given by experts. Then, the covariance of the return rates of securities i and j can be expressed by

$$\widetilde{\delta}_{t,ij} = [\underline{\delta}_{t,ij}, \overline{\delta}_{t,ij}] = \left[\min\left\{\delta^a_{t,ij}, \delta^h_{t,ij}, \delta^f_{t,ij}\right\}, \max\left\{\delta^a_{t,ij}, \delta^h_{t,ij}, \delta^f_{t,ij}\right\}\right].$$

By Eqs. (3) and (4), the variance of the return rate on the portfolio at period t is

$$\widetilde{\delta}_t = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{t,i} x_{t,j} \widetilde{\delta}_{t,ij} = \left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_{t,i} x_{t,j} \underline{\delta}_{t,ij}, \sum_{i=1}^{n}\sum_{j=1}^{n} x_{t,i} x_{t,j} \overline{\delta}_{t,ij}\right]. \tag{17}$$

Liquidity is another one of the main concerns for portfolio managers. It measures the degree of probability of being able to convert an investment into cash without any significant loss in value. The liquidity of a security is commonly measured by its turnover rate, which is the proportion of turnover volume to the tradable volume. Generally, investors prefer greater liquidity. Especially, in a bull market, the returns of securities with high liquidity tend to increase with time. According to the assumption above, the turnover rates of securities at each period are characterized by interval numbers. By using the estimation approach of Sect. 3.1, the arithmetic mean turnover rate factor of security i at period t can be estimated by $l^a_{t,i} = \frac{1}{t-t_0}\sum_{s=t_0}^{t-1} l_{s,i}$. Assume that the historical turnover rate tendency factor $l^h_{t,i}$ and the forecast turnover rate factor $l^f_{t,i}$ are directly given by experts. Then, the turnover rate of security i at period t can be given as an interval number with the following form

$$\widetilde{l}_{t,i} = [\underline{r}_{t,i}, \overline{r}_{t,i}] = \left[\min\left\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\right\}, \max\left\{r^a_{t,i}, r^h_{t,i}, r^f_{t,i}\right\}\right].$$

Then, derived from Eq. (7), the turnover rate of the portfolio at period t can be represented by

$$\widetilde{l}_t = \sum_{i=1}^{n} x_{t,i}\widetilde{l}_{t,i} = \left[\sum_{i=1}^{n} x_{t,i}\underline{l}_{t,i}, \sum_{i=1}^{n} x_{t,i}\overline{l}_{t,i}\right]. \tag{18}$$

To construct a diversified portfolio, investors usually wish to control the maximum and minimum fractions of the capital invested on each security. Mathematically, the lower and upper bound constraints on $x_{t,i}$ can be expressed as

$$m_{t,i}z_{t,i} \le x_{t,i} \le M_{t,i}z_{t,i}, \quad i = 1, 2, \ldots, n; t = 1, 2, \ldots, T,$$

where $m_{t,i}$ and $M_{t,i}$ are the maximum and minimum fractions of the capital invested on security i at period t; $z_{t,i}$ is a binary variable which will be 1 if security i at period t is held and 0 otherwise.

There is a widely accepted policy for reducing the investment risk of a portfolio by not allocating the whole investment in just a few securities. This idea is reflected in a well-known saying that “don’t put all your eggs in one basket”. However, how to construct a well diversified portfolio has become a problem to researchers. So far, some researchers have investigated the problem and used proportion entropy to measure the diversification degree of a portfolio, see for example Fang et al. (1997). Similarly, we also use proportion entropy to measure the diversification degree of the portfolio at each period. Mathematically, the diversification degree of the portfolio $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,n})$ at period t can be represented by

$$h(x_t) = -\sum_{i=1}^{n} x_{t,i}\ln(x_{t,i}), \quad t = 1, 2, \ldots, T. \tag{19}$$

In the real world, to control transaction costs and reduce the complexity of a portfolio, investor may wish to restrict the maximum number of securities in the portfolio at each period. In this paper, we assume that the investor intends to hold not more than K securities in the portfolio at period t. Then, the cardinality constraints about the portfolio at period t can be expressed by

$$\sum_{i=1}^{n} z_{t,i} \le K, \quad t = 1, 2, \ldots, T,$$

where $z_{t,i} \in \{0, 1\}$ is a binary variable that controls whether security i at period t should be selected in the portfolio or not.

3.4 Model formulation

Based on the discussion in the previous section, we investigate a multi-period portfolio problem in emerging markets by taking several criteria into consideration including transaction costs, expected proceeds, variance, turnover rate, diversification degree, bound constraints and cardinality constraints. To demonstrate the effects of these decision criteria on portfolio selection and provide investors with more choices, we propose four multi-period cardinality constrained portfolio optimization models with interval coefficients, viz., the return–risk (RR) model, the return–risk–liquidity (RRL) model, the return–risk–entropy (RRE) model and the return–risk–liquidity–entropy (RRLE) model.

(1) In the RR model, we assume that an investor intends to seek an investment strategy with the objective of maximizing terminal wealth. Meanwhile, as shown in Constraints (20)–(24), the investor requires that the expected proceeds of the portfolio on unit capital investment at period t must achieve or exceed the preset minimum proceeds level R_t, the variance of the return rate on the portfolio at period t must not exceed the preset maximum tolerable variance level Var_t, the maximum number of securities in the portfolio at period t must be not more than K, the investment proportion of at period t must sum to one and the investment proportion on each selected security at period t must satisfy the lower and upper bound constraints. Then, the RR model for the multi-period portfolio selection in emerging markets can be formulated as the following interval programming problem (P_1):

$$(P_1)\begin{cases} \max \quad W_T = \left[W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t \right), W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t \right) \right] \\ \text{s.t.} \quad \left[\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t, \sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t \right] \succeq R_t, & (20) \\ \quad \left[\sum_{i=1}^{n} \sum_{k=1}^{n} x_{t,i} x_{t,k} \underline{\delta}_{i,k}^{t}, \sum_{i=1}^{n} \sum_{k=1}^{n} x_{t,i} x_{t,k} \overline{\delta}_{i,k}^{t} \right] \preceq \mathrm{Var}_t, & (21) \\ \quad \sum_{i=1}^{n} z_{t,i} \le K, \ z_{t,i} \in \{0,1\}, & (22) \\ \quad \sum_{i=1}^{n} x_{t,i} = 1, \ x_{t,i} \ge 0, & (23) \\ \quad m_{t,i} z_{t,i} \le x_{t,i} \le M_{t,i} z_{t,i}, \ i = 1, 2, \ldots, n; t = 1, 2, \ldots, T. & (24) \end{cases}$$

From Constraint (23), we have $x_{t,i} \ge 0$ for all $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$. It indicates that short selling is not allowed over the whole investment horizon.

(2) In the RRL model, we assume that the investor not only considers the criteria in the model (P_1) but also takes the effect of the turnover rate of the portfolio at each period into consideration. As shown in Constraint (25), he demands that the turnover rate of the portfolio at period t must be larger than or equal to the given turnover rate level l_t. Then, on the basis of (P_1), the RRL model for the multi-period portfolio selection in emerging markets can be formulated as the following interval programming problem (P_2):

$$(P_2)\begin{cases} \max \quad W_T = \left[W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i} \underline{r}_{t,i} - C_t \right), W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i} \overline{r}_{t,i} - C_t \right) \right] \\ \text{s.t.} \quad \text{Constraints (20)–(24)}, \\ \quad \left[\sum_{i=1}^{n} x_{t,i} \underline{l}_{t,i}, \sum_{i=1}^{n} x_{t,i} \overline{l}_{t,i} \right] \succeq l_t. \end{cases} \quad (25)$$

(3) In the RRE model, we assume that the investor considers all the criteria in the RR model (P_1). Meanwhile, he requires that the diversification degree of the portfolio at period

t must be no less than the preset minimum diversification level e_t as shown in Constraint (26). Then, on the basis of (P_1), the RRE model can be formulated as the following interval programming problem (P_3):

$$(P_3)\begin{cases} \max \quad {}_T = \left[W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\underline{r}_{t,i} - C_t \right), W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\bar{r}_{t,i} - C_t \right) \right] \\ \text{s.t.} \quad \text{Constraints (20)–(24)}, \\ \qquad -\sum_{i=1}^{n} x_{t,i} \ln(x_{t,i}) \geq e_t. \end{cases} \tag{26}$$

(4) In the RRLE model, we assume that the investor takes all the decision criteria in the aforementioned three models into consideration. Then, the RRLE model can be formulated as the following interval programming problem (P_4):

$$(P_4)\begin{cases} \max \quad W_T = \left[W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\underline{r}_{t,i} - C_t \right), W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\bar{r}_{t,i} - C_t \right) \right] \\ \text{s.t.} \quad \text{Constraints (20)–(26)}. \end{cases}$$

3.5 Deterministic multi-period portfolio optimization models

Notice that the proposed four models in previous section are interval programming problems with interval coefficients in both objectives and inequality constraints. To solve the proposed models, it is necessary to transform them into deterministic programming models. In this section, we first transform the proposed models into parameter programming problems with interval coefficients in inequality constraints. Then, based on the definitions of the possibility degree for the interval inequalities, we can convert the interval inequality constraints in the transformed models into deterministic inequality constraints.

3.5.1 Treatment of the uncertain objective functions

To handle the uncertain objective functions in the proposed models, we apply a commonly-used variable transformation method to transform them into the corresponding parameter programming problems with interval coefficients in constraints. Here, we take the RR model (P_1) as an example to introduce the treatment of its uncertain objective function. By using the variable transformation method, the RR model (P_1) can be equivalently converted into the following parameter programming model $(P_1^{'})$:

$$(P_1^{'})\begin{cases} \max \quad f \\ \text{s.t.} \quad \text{Constraints (20)–(24)}, \\ \qquad \left[W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\underline{r}_{t,i} - C_t \right), W_0 \prod_{t=1}^{T} \left(\sum_{i=1}^{n} x_{t,i}\bar{r}_{t,i} - C_t \right) \right] \succeq f. \end{cases} \tag{27}$$

Similarly, models (P_2), (P_3) and (P_4) can be converted into the corresponding parameter programming models. Denote them by $(P_2^{'})$, $(P_3^{'})$ and $(P_4^{'})$, respectively.

3.5.2 Treatment of the uncertain inequality constraints

In stochastic optimization problems (see Liu et al. (2003)), we often make an uncertain inequality constraint satisfied with a confidence level and transform it into deterministic inequality constraint. Analogously, in fuzzy optimization problems, we can make an interval inequality constraint satisfied with a possibility degree level and convert it into deterministic inequality constraint.

In this paper, for the uncertainty inequality constraints in Constraints (20), (21), (25) and (27), we use the definitions of the possibility degree for interval inequalities in Eqs. (8) and (9) to transform them into the corresponding deterministic forms. Let $\theta_{\widetilde{R}_t}$, $\theta_{\widetilde{l}_t}$ and θ_{W_T} be the possibility degree levels of Constraints (20), (25) and (27), respectively. Then, by Eq. (8), the three interval inequality constraints can be, respectively, converted into deterministic inequality constraints as follows

$$\sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t - R_t \geq \theta_{\widetilde{R}_t} \sum_{i=1}^{n} x_{t,i}(\overline{r}_{t,i} - \underline{r}_{t,i}), \tag{28}$$

$$\sum_{i=1}^{n} x_{t,i}\overline{l}_{t,i} - l_t \geq \theta_{\widetilde{l}_t} \sum_{i=1}^{n} (x_{t,i}\overline{l}_{t,i} - \underline{l}_{t,i}), \tag{29}$$

$$W_0 \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t\right) - f \geq \theta_{W_T} W_0 \left[\prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t\right) - \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\underline{r}_{t,i} - C_t\right)\right]. \tag{30}$$

For the uncertainty inequality constraint in Constraint (21), we denote its possibility degree level by $\theta_{\widetilde{\delta}_t}$. Then, by Eq. (9), it can be transformed into the following deterministic inequality constraint

$$\mathrm{Var}_t - \sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i}x_{t,k}\underline{\delta}^t_{i,k} \geq \theta_{\widetilde{\delta}_t} \sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i}x_{t,k}(\overline{\delta}^t_{i,k} - \underline{\delta}^t_{i,k}). \tag{31}$$

Notice that, with the increasing of $\theta_{\widetilde{R}_t}$, $\theta_{\widetilde{l}_t}$, θ_{W_T} and $\theta_{\widetilde{\delta}_t}$, the investor will become more and more conservative to demand that Constraints (28)–(31) should be held. Thus, we can freely express investor's different risk attitudes by varying the values of the possibility degree levels on the aforementioned uncertain inequality constraints.

Then, by Eqs. (28), (30) and (31), the model $(P_1^{'})$ can be transformed into the following deterministic parameter programming problem $(P_1^{''})$:

$$(P_1^{''})\begin{cases} \max \quad f \\ s.t. \quad W_0 \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t\right) - f \geq \theta_{W_T} W_0 \left[\prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t\right) - \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}\underline{r}_{t,i} - C_t\right)\right], \\ \qquad \sum_{i=1}^{n} x_{t,i}\overline{r}_{t,i} - C_t - R_t \geq \theta_{\widetilde{R}_t} \sum_{i=1}^{n} x_{t,i}(\overline{r}_{t,i} - \underline{r}_{t,i}), \\ \qquad \mathrm{Var}_t - \sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i}x_{t,k}\underline{\delta}^t_{i,k} \geq \theta_{\widetilde{\delta}_t} \sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i}x_{t,k}(\overline{\delta}^t_{i,k} - \underline{\delta}^t_{i,k}), \\ \qquad \sum_{i=1}^{n} x_{t,i} = 1, \ x_{t,i} \geq 0, \\ \qquad \sum_{i=1}^{n} z_{t,i} \leq K, \quad z_{t,i} \in \{0, 1\}, \\ \qquad m_{t,i}z_{t,i} \leq x_{t,i} \leq M_{t,i}z_{t,i}, \ i = 1, 2, \ldots, n, t = 1, 2, \ldots, T. \end{cases}$$

By substituting Eqs. (28)–(31) into the models $(P_2^{'})$, $(P_3^{'})$ and $(P_4^{'})$, the corresponding deterministic parameter programming models can also be obtained. Denote them by $(P_2^{''})$, $(P_3^{''})$ and $(P_4^{''})$, respectively.

It is well known that portfolio selection is affected by many realistic investment constraints including cardinality constraints, bound constraints and so on. To highlight the effects of both the cardinality constraints and the bound constraints on portfolio selection, on the basis of the model $(P_1^{''})$, we formulate a contrast model without considering Constraints (22) and (24) as follows

$$(P_5)\begin{cases}\max & f\\ s.t. & W_0\prod_{t=1}^{T}\left(\sum_{i=1}^{n}x_{t,i}\overline{r}_{t,i}-C_t\right)-f\geq\theta_{W_T}W_0\left[\prod_{t=1}^{T}\left(\sum_{i=1}^{n}x_{t,i}\overline{r}_{t,i}-C_t\right)-\prod_{t=1}^{T}\left(\sum_{i=1}^{n}x_{t,i}\underline{r}_{t,i}-C_t\right)\right],\\ & \sum_{i=1}^{n}x_{t,i}\overline{r}_{t,i}-C_t-R_t\geq\theta_{\widetilde{R}_t}\sum_{i=1}^{n}x_{t,i}(\overline{r}_{t,i}-\underline{r}_{t,i}),\\ & \mathrm{Var}_t-\sum_{i=1}^{n}\sum_{k=1}^{n}x_{t,i}x_{t,k}\underline{\delta}_{i,k}^{t}\geq\theta_{\widetilde{\delta}_t}\sum_{i=1}^{n}\sum_{k=1}^{n}x_{t,i}x_{t,k}(\overline{\delta}_{i,k}^{t}-\underline{\delta}_{i,k}^{t}),\\ & \sum_{i=1}^{n}x_{t,i}=1,\ x_{t,i}\geq 0,\ t=1,2,\ldots,T.\end{cases}$$

4 Solution algorithm

Because Constraint (22) is a 0–1 integer constraint and the first derivatives of Constraints (28) and (30) are discontinuous, traditional optimization approaches usually fail to find the optimal solutions of the models mentioned in previous section. In this section, we design a new dynamic differential evolution algorithm with self-adapting control parameter to solve them.

Differential evolution (DE) is a stochastic population-based search method, which was originally proposed by Storn and Price (1995). Like other evolutionary algorithms (EAs), it encodes each decision variable by a real number and utilizes the three important operators (i.e., mutation, crossover and selection) to evolve from randomly generated initial population to final individual solution. The key idea of DE is a scheme for generating trial vectors. Mutation and crossover operators are used to generate trial vectors, and the selection operator determines which of the vectors will survive into the next generation (see Brest et al. (2006)).

DE is of simple concept, few control parameters, easy implementation and high convergence characteristics (see Storn and Price (1997)), and has attracted much attention and wide applications for solving the unconstrained optimization problems. But, DE lacks an explicit mechanism to guide the search process towards the feasible region, which limits its application. As we know, many real-world optimization problems are with constraints. The main challenge in solving constrained optimization is how to balance the search between feasible and infeasible regions effectively. Namely, how to design an efficient constraint-handling mechanism to locate the global optimum in the feasible region. Numerous researchers have concentrated on constraint-handling mechanisms for DEs and presented a series of approaches for handling constraints (see e.g., Mezura-Montes et al. (2010) and Wang and Li (2010)). To solve constrained optimization problems with mixed-type variables, Mohamed and Sabry (2012) designed a novel dynamic differential evolution algorithm. On the basis of Alia and Törn (2004) and Mohamed and Sabry (2012), we design a new dynamic differential evolution with self-adapting control parameter to solve the proposed models. Here, we introduce its fitness function, representation and coding, constraint-handling, mutation, crossover, selection and stopping criterion.

4.1 Fitness function

The fitness function is a factor for measuring the quality of a solution. In an optimization problem, fitness is often determined by the value of its objective function. Thus, for the maximization problems, the individual with higher fitness will have more chance to generate offspring. For example, in the maximization problem $(P_1^{''})$, we regard its objective function as the fitness function of the designed algorithm.

4.2 Representation and coding

In this algorithm, we use the hybrid representation to define a portfolio, in which two vectors are expressed as the following forms

$$\triangle = \{z_{1,1}, z_{1,2}, \ldots, z_{1,n}; \ldots; z_{T,1}, z_{T,2}, \ldots, z_{T,n}\}, \tag{32}$$

$$X = \{x_{1,1}, x_{1,2}, \ldots, x_{1,n}; \ldots; x_{T,1}, x_{T,2}, \ldots, x_{T,n}\}. \tag{33}$$

To simplify the description, we denote $\triangle$ and X by $X = (x_1, x_2, \ldots, x_T)$ with $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,n})$ and $\triangle = \{z_1, z_2, \ldots, z_T\}$ with $z_t = (z_{t,1}, z_{t,2}, \ldots, z_{t,n})$ for all $t = 1, 2, \ldots, T$. Here, z_t is a binary vector that specifies whether a particular security participates in the portfolio at period t, where $z_{t,i} \in \{0, 1\}$ $(i = 1, 2, \ldots, n; t = 1, 2, \ldots, T)$. x_t is a real-valued vector used to compute the investment proportions of the budget invested on the portfolio at period t, where $x_{t,i} \in [0, 1]$ $(i = 1, 2, \ldots, n; t = 1, 2, \ldots, T)$.

4.3 Constraint-handling

Similar to Chang et al. (2000) and Mishra et al. (2014), we perform the following repair mechanism to find the portfolio at period t associated with z_t and x_t. First, if the number of securities in the portfolio at period t (i.e., the number of 1's in z_t) exceeds the maximum allowable number K, then we delete (by changing its value from 1 to 0 in z_t) those securities with the $(n - K)$–smallest weights in x_t. In this way, we can keep the portfolio at each period satisfy the cardinality constraints (i.e., Constraint (22)).

To meet the budget constraint (i.e., Constraint (23)), we perform the following normalization operation

$$x'_{t,i} = \frac{x_{t,i} z_{t,i}}{\sum_{j=1}^{n} x_{t,j} z_{t,j}}, \quad i = 1, 2, \ldots, n; t = 1, 2, \ldots, T. \tag{34}$$

Since the normalized investment proportion in Eq. (34) may not satisfy the bound constraints, we need to discuss the following three different cases.

Case 1: If both the lower and upper bound constraints are presented, then the adjusted investment proportions are computed by

$$x'_{t,i} = y_{t,i} + \frac{x_{t,i} z_{t,i}}{\sum_{j=1}^{n} x_{t,j} z_{t,j}} \left(1 - \sum_{j=1}^{n} y_{t,j}\right), \quad t = 1, 2, \ldots, T, \tag{35}$$

where $y_{t,j} = m_{t,j} z_{t,j} + M_{t,j}(1 - z_{t,j})$ means that the investment proportion $x_{t,i}$ cannot escape either the lower or upper bound.

Case 2: If the investment proportion has to be adjusted only for the lower bound constraint, and there is no restriction on the upper bound constraint, then the adjusted investment proportions are calculated by

$$x'_{t,i} = m_{t,i} z_{t,i} + \frac{x_{t,i} z_{t,i}}{\sum_{j=1}^{n} x_{t,j} z_{t,j}} \left(1 - \sum_{j=1}^{n} m_{t,j} z_{t,j}\right), \quad t = 1, 2, \ldots, T. \tag{36}$$

Case 3: If the investment proportion has to be adjusted for the upper bound constraint, and there is no restriction on the lower bound constraint, then the adjusted investment proportions are computed by

$$x'_{t,i} = M_{t,i} z_{t,i} - \frac{x_{t,i} z_{t,i}}{\sum_{j=1}^{n} x_{t,j} z_{t,j}} \left(\sum_{i=1}^{n} M_{t,i} z_{t,i} - 1\right), t = 1, 2, \ldots, T. \tag{37}$$

For the remaining inequality constraints in the maximization problem, we rewrite them into the common forms $g_j(x) \leq 0 \, (j = 1, 2, \ldots, q)$. Then, the violation value of the solution X is defined as

$$\text{Viol}(X) = \sum_{j=1}^{q} \frac{g_j(x)}{g_{\max,j}(x)}, \tag{38}$$

where $g_j(x)$ is the jth constraint of the problem and $g_{\max,j}(x)$ is the largest violation of the constraint $g_j(x)$ found so far.

According to the feasibility-based rule in Deb (2000), we compare each trial vector with its corresponding target vector in current population by taking both fitness values and violation values into consideration. We replace the target vector by the trial vector as an individual of next generation by the following criteria:

(i) The trial vector is feasible and the target vector is infeasible;
(ii) Both the trial vector and the target vector are feasible, but the fitness value of the trial vector is no less than the corresponding target vector;
(iii) Both the trial vector and the target vector are infeasible, but the violation value of the trial vector is smaller the corresponding target vector.

After repeating this operation N times, we can obtain a population with N initialized individuals.

4.4 Mutation operation

Since the basic mutation strategy DE/rand/1/bin slows down the convergence of DE algorithms, we design a new mutation strategy with self-adapting control parameter to perform mutation operation as follows: For each target vector X_i^G, a mutant vector V is generated as

$$\text{if } (\text{rand}[0, 1] \leq 0.5) \tag{39}$$

$$\text{then } V_i^{G+1} = X_r^G + F(X_b^G - X_w^G) \tag{40}$$

$$\text{else } V_i^{G+1} = X_{r_1}^G + F(X_{r_2}^G - X_{r_3}^G) \tag{41}$$

where G is the current generation number; X_r^G is a randomly chosen vector at generation G; X_b^G and X_w^G are the best and the worst vectors in the entire population at current generation, respectively; rand[0, 1] is a random number in [0,1]; $r_1, r_2, r_3 \in \{1, 2, \ldots, N\}$ are randomly chosen indices; F is a self-adapting control parameter with the following form (see Alia and

Törn (2004))

$$F = \begin{cases} \max\left(F_{\min}, 1 - \left|\frac{f_{\max}}{f_{\min}}\right|\right), & \text{if } |\frac{f_{\max}}{f_{\min}}| < 1, \\ \max\left(F_{\min}, 1 - \left|\frac{f_{\min}}{f_{\max}}\right|\right), & \text{otherwise,} \end{cases}$$

where $f_{\min}$ and $f_{\max}$ are the minimum and maximum objective values of vector X at current generation, $F_{\min}$ is the minimum value of F. In Alia and Törn (2004), the value of $F_{\min}$ is set as 0.4.

4.5 Crossover operation

To balance between global exploration ability and local exploitation tendency in constrained optimization, a dynamic non-linear increased crossover probability scheme is performed as follows

$$u_{i,j}^{G+1} = \begin{cases} x_{i,j}^{G+1}, & \text{if } \mathrm{rand}(j) \leq \mathrm{CR} \ \text{ or } j = \mathrm{randn}(i), \\ v_{i,j}^{G}, & \text{if } \mathrm{rand}(j) > \mathrm{CR} \text{ and } j \neq \mathrm{randn}(i), \end{cases} \tag{42}$$

where $j = 1, 2, \ldots, nT$; $\mathrm{rand}(j)$ is the jth evaluation of a uniform random generator number; $\mathrm{randn}(i) \in \{1, 2, \ldots, nT\}$ is a randomly chosen index and it ensures that u_i^{G+1} gets at least one element from v_i^{G+1}; the crossover probability CR is defined as

$$\mathrm{CR} = \mathrm{CR}_{\max} + (\mathrm{CR}_{\min} - \mathrm{CR}_{\max})\left(1 - \frac{G}{G_{\max}}\right)^p.$$

Here, $G_{\max}$ is the maximum generation number; $\mathrm{CR}_{\min}$ and $\mathrm{CR}_{\max}$ are the minimum and maximum values of CR, respectively; p is a positive number. In this paper, the values of $\mathrm{CR}_{\min}$, $\mathrm{CR}_{\max}$ and p are set as 0.5, 0.95 and 4, respectively.

4.6 Selection operation

In this algorithm, the selection operation is based on a greedy selection strategy. Namely, if the trial vector u_i^{G+1} yields a better fitness function value than X_i^G, then we set u_i^{G+1} as X_i^G. Otherwise, we retain X_i^G. Thus, the selection operation can be expressed as

$$X_i^{G+1} = \begin{cases} u_i^{G+1}, & \text{if } f(u_i^{G+1}) > f(X_i^{G+1}), \\ X_i^G, & \text{if } f(u_i^{G+1}) \leq f(X_i^{G+1}). \end{cases} \tag{43}$$

4.7 Stopping criterion

We terminate the algorithm when the maximum number of generations $G_{\max}$ is reached or there is no improvement after a certain number of generations.

The concrete procedures of the designed algorithm are summarized as follows:

Step 1: Input population size N, crossover rate parameters ($\mathrm{CR}_{\min}$, $\mathrm{CR}_{\max}$ and p) and maximum generation number $G_{\max}$;

Step 2: Randomly generate N initial individuals and convert them into corresponding feasible ones by using the constraint-handing mechanisms;

Step 3: Calculate the fitness value of each individual;

Step 4: Perform mutation operation by Eqs. (39)–(41) and crossover operation by Eq. (42);

Step 5: Compare the fitness values of the offspring u_i^{G+1} and its parent X_i^G, and then select out the better one as an individual for next generation by Eq. (43);

Table 1 The interval-valued proceeds and turnover rates of the 8 stocks at each period

St. i	Proceeds			Turnover rates		
	$t = 1$	$t = 2$	$t = 3$	$t = 1$	$t = 2$	$t = 3$
St. 1	[1.0463, 1.0825]	[1.0527, 1.0963]	[1.0469, 1.0670]	[0.0143, 0.0255]	[0.0103, 0.0294]	[0.0168, 0.0438]
St. 2	[0.9910, 1.0419]	[0.9901, 1.0486]	[1.0446, 1.0943]	[0.0331, 0.0470]	[0.0245, 0.0461]	[0.0170, 0.0422]
St. 3	[1.0265, 1.0536]	[1.0324, 1.0584]	[1.0260, 1.0485]	[0.0322, 0.0467]	[0.0263, 0.0392]	[0.0223, 0.0406]
St. 4	[1.0352, 1.0679]	[1.0383, 1.0602]	[1.0078, 1.0442]	[0.0252, 0.0374]	[0.0181, 0.0377]	[0.0313, 0.0473]
St. 5	[1.0492, 1.0929]	[1.0512, 1.0887]	[1.0674, 1.0918]	[0.0124, 0.0265]	[0.0232, 0.0412]	[0.0221, 0.0341]
St. 6	[1.0452, 1.0602]	[1.0299, 1.0741]	[1.0395, 1.0688]	[0.0221, 0.0343]	[0.0161, 0.0301]	[0.0160, 0.0451]
St. 7	[1.0441, 1.0839]	[1.0354, 1.0614]	[1.0371, 1.0504]	[0.0167, 0.0340]	[0.0232, 0.0412]	[0.0121, 0.0493]
St. 8	[1.0307, 1.0503]	[1.0388, 1.0869]	[1.0423, 1.0819]	[0.0322, 0.0441]	[0.0121, 0.0311]	[0.0243, 0.0350]

Step 6: Check the stopping criterion. If the stopping criterion is satisfied, then quit the iteration operation and report the best individual as the optimal solution. Otherwise, go back to Step 3.

5 Numerical example

In this section, we give a numerical example to illustrate the applications of the proposed models and demonstrate the validity of the designed algorithm.

Assume that an investor chooses 8 stocks from the Shanghai Stock Exchange for his investment. He intends to make three consecutive periods investment among the 8 stocks with initial wealth 10,000 RMB. To simulate the real transaction, we collect the historical data about weekly return rates, variances and turnover rates of the 8 stocks from January 2010 to January 2013. We set each year as an investment period to handle these historical data. In this example, the values of $r^a_{t,i}$, $\delta^a_{t,i}$ and $l^a_{t,i}$ are calculated by their arithmetic means of the historical data in the recent a year for $i = 1, 2, \ldots, 8$ and $t = 1, 2, 3$. Namely, the value of t_0 is set as 1. The values of $r^h_{t,i}$, $\delta^h_{t,i}$ and $l^h_{t,i}$ are subjectively given by experts. Here, we assume that they are calculated by their arithmetic means of the historical data in the first half of the recent a year. The values of $r^f_{t,i}$, $\delta^f_{t,i}$ and $l^f_{t,i}$ are roughly computed by their arithmetic means of the historical data in the second half of the recent a year. According to the interval estimation approach in Fang et al. (2008), the proceeds obtained by per unit capital investment, the turnover rates and the covariances of the return rates on the 8 stocks at each period are characterized by interval numbers as shown in Tables 1 and 2.

In this example, the maximum holding number of stocks in the portfolio at each period is set as 6, that is, $K = 6$. The transaction costs of per unit capital adjustment are set as 0.003 for all stocks over the whole investment horizon. The minimum expected return level of the portfolio on unit capital investment at each period is set as 1.0475. The maximum risk tolerance levels for the portfolios in the three investment periods are set as 0.012, 0.015 and 0.018, respectively. The minimum expected turnover rate level of the portfolio at each period is set as 0.21. The diversification degree of the portfolio at each period is set as 1.6. The lower and upper bounds of the investment proportion of each selected stock over the whole investment horizon are set as 0.005 and 0.3, respectively. The parameters of the designed algorithm are set as follows: The population size is 200 and the maximum generation

Table 2 The interval-valued covariances of the return rates on the 8 stocks at each period

t	St. i	St. 1	St. 2	St. 3	St. 4	St. 5	St. 6	St. 7	St. 8
$t=1$	St. 1	[0.005,0.021]	[0.005, 0.012]	[0.006, 0.012]	[0.003, 0.017]	[0.007, 0.019]	[0.003, 0.012]	[0.005, 0.016]	[0.005, 0.015]
	St. 2	[0.005, 0.012]	[0.006, 0.007]	[0.006, 0.010]	[0.001, 0.010]	[0.008, 0.011]	[0.003, 0.007]	[0.005, 0.010]	[0.006, 0.008]
	St. 3	[0.006, 0.012]	[0.006, 0.007]	[0.007, 0.012]	[0.001, 0.013]	[0.009, 0.014]	[0.004, 0.009]	[0.005, 0.013]	[0.006, 0.011]
	St. 4	[0.003, 0.017]	[0.001, 0.010]	[0.001, 0.013]	[0.002, 0.015]	[0.002, 0.016]	[0.001, 0.010]	[0.001, 0.014]	[0.001, 0.012]
	St. 5	[0.007, 0.019]	[0.008, 0.011]	[0.009, 0.014]	[0.002, 0.016]	[0.010, 0.017]	[0.004, 0.010]	[0.007, 0.015]	[0.008, 0.013]
	St. 6	[0.003, 0.012]	[0.003, 0.007]	[0.004, 0.009]	[0.001, 0.010]	[0.004, 0.010]	[0.002, 0.006]	[0.002, 0.010]	[0.003, 0.008]
	St. 7	[0.005, 0.016]	[0.005, 0.010]	[0.005, 0.013]	[0.001, 0.014]	[0.007, 0.015]	[0.002, 0.010]	[0.004, 0.013]	[0.005, 0.012]
	St. 8	[0.005, 0.015]	[0.006, 0.008]	[0.006, 0.011]	[0.001, 0.012]	[0.002, 0.010]	[0.003, 0.008]	[0.005, 0.012]	[0.006, 0.011]
$t=2$	St. 1	[0.000, 0.031]	[0.001, 0.014]	[0.001, 0.020]	[0.000, 0.010]	[0.001, 0.021]	[0.001, 0.013]	[0.001, 0.009]	[0.001, 0.019]
	St. 2	[0.001, 0.014]	[0.002, 0.006]	[0.003, 0.009]	[0.002, 0.005]	[0.003, 0.009]	[0.004, 0.006]	[0.003, 0.004]	[0.005, 0.008]
	St. 3	[0.001, 0.020]	[0.003, 0.009]	[0.003, 0.013]	[0.002, 0.007]	[0.003, 0.014]	[0.003, 0.009]	[0.003, 0.006]	[0.005, 0.012]
	St. 4	[0.000, 0.010]	[0.002, 0.005]	[0.002, 0.007]	[0.001, 0.003]	[0.002, 0.007]	[0.002, 0.004]	[0.002, 0.003]	[0.003, 0.006]
	St. 5	[0.001, 0.021]	[0.003, 0.009]	[0.003, 0.014]	[0.002, 0.007]	[0.003, 0.015]	[0.002, 0.009]	[0.003, 0.006]	[0.005, 0.013]
	St. 6	[0.001, 0.013]	[0.004, 0.006]	[0.003, 0.009]	[0.002, 0.004]	[0.002, 0.009]	[0.002, 0.006]	[0.002, 0.004]	[0.004, 0.008]
	St. 7	[0.001, 0.009]	[0.003, 0.004]	[0.003, 0.006]	[0.002, 0.003]	[0.003, 0.006]	[0.002, 0.004]	[0.002, 0.003]	[0.005, 0.006]
	St. 8	[0.001, 0.019]	[0.005, 0.008]	[0.005, 0.012]	[0.003, 0.006]	[0.005, 0.013]	[0.004, 0.008]	[0.005, 0.006]	[0.008, 0.011]
$t=3$	St. 1	[0.002, 0.005]	[0.002, 0.014]	[0.002, 0.009]	[0.001, 0.004]	[0.004, 0.025]	[0.001, 0.007]	[0.003, 0.006]	[0.004, 0.012]
	St. 2	[0.002, 0.014]	[0.002, 0.038]	[0.003, 0.025]	[0.001, 0.011]	[0.003, 0.070]	[0.001, 0.018]	[0.004, 0.018]	[0.005, 0.034]
	St. 3	[0.002, 0.009]	[0.003, 0.025]	[0.003, 0.017]	[0.001, 0.007]	[0.003, 0.046]	[0.001, 0.012]	[0.004, 0.012]	[0.005, 0.023]
	St. 4	[0.001, 0.004]	[0.001, 0.011]	[0.001, 0.007]	[0.001, 0.003]	[0.001, 0.019]	[0.000, 0.005]	[0.001, 0.005]	[0.003, 0.010]
	St. 5	[0.004, 0.025]	[0.003, 0.070]	[0.003, 0.046]	[0.001, 0.019]	[0.003, 0.127]	[0.001, 0.034]	[0.004, 0.033]	[0.005, 0.062]
	St. 6	[0.001, 0.007]	[0.001, 0.018]	[0.001, 0.012]	[0.000, 0.005]	[0.001, 0.034]	[0.001, 0.009]	[0.001, 0.009]	[0.001, 0.016]
	St. 7	[0.003, 0.006]	[0.004, 0.018]	[0.004, 0.012]	[0.001, 0.005]	[0.004, 0.033]	[0.001, 0.009]	[0.005, 0.008]	[0.007, 0.016]
	St. 8	[0.004, 0.012]	[0.005, 0.034]	[0.005, 0.023]	[0.003, 0.010]	[0.005, 0.062]	[0.001, 0.016]	[0.007, 0.016]	[0.003, 0.009]

number is 1000. The values of CR_{min} and CR_{max} are set as 0.5 and 0.95. To demonstrate the performances of the proposed models, we assume that the possibility degree level for each interval inequality constraint is 0.8. After running the designed algorithm 1000 iterations on each model, we record the corresponding investment strategies as shown in Table 3.

Table 3, representing the investment strategies obtained by different models, shows that different portfolio selection models reflect investors' different investment intentions. From Table 3, we can find that most of the investor's wealth is allocated among Stocks 1, 5, 6, 7 and 8. If the investor uses the model (P_1) to make his portfolio decision, he should follow the investment strategies listed in lines 2, 3 and 4 of Table 3 to adjust his wealth at the beginning of each period. Namely, at the beginning of period 1, the investor needs to assign his initial wealth among Stocks 1, 4, 5, 6 and 7 by the investment proportions of 0.2747, 0.0053, 0.2993, 0.2975 and 0.1231, respectively. At the beginning of period 2, the investor needs to adjust his wealth again. After adjustment, he holds Stocks 1, 4, 5, 6, 7 and 8 by the investment proportions of 0.2998, 0.0052, 0.2999, 0.0935, 0.0050 and 0.2965, respectively. Subsequently, at the beginning of period 3, the investor needs to adjust his wealth again. In this investment period, he constructs a portfolio among Stocks 1, 2, 5, 6, 7 and 8 by the investment proportions of 0.2988, 0.0092, 0.1830, 0.2109, 0.0064 and 0.2917, respectively. In this case, the crisp value of terminal wealth is 11,945.76 RMB. If the investor uses the model (P_2) to make his portfolio decision. He should allocate his wealth by the investment strategies listed in lines 5, 6 and 7 of Table 3 at the beginning of each period. The corresponding crisp value of terminal wealth is 11,879.92 RMB. If the investor uses the model (P_3) to make his portfolio decision, he should follow the investment strategies listed in lines 8, 9 and 10 of Table 3 to allocate his wealth at the beginning of each period. In this case, the crisp value of terminal wealth is 11,909.30 RMB. If the investor selects the model (P_4) to make his portfolio decision, he should follow the investment strategies listed in lines 11, 12 and 13 of Table 3. In this case, the crisp value of terminal wealth is 11,875.29 RMB. If the investor selects all stocks in equal investment proportions and hold them over the whole investment horizon, the corresponding crisp value of terminal wealth obtained at the end of period 3 is 11,616.94 RMB. After comparison, we can find out that the crisp value of the terminal wealth obtained by any one of the proposed models is larger than the one obtained by the equal-weighted investment. It follows that the portfolios generated by the proposed models perform better than the equal-weighted one. To highlight the effects of both the cardinality constraints and the bound constraints on portfolio selection, we also use the designed algorithm to solve the model (P_5) and record the corresponding numerical results in lines 14, 15 and 16 of Table 3. It can be observed that, in the sense of terminal wealth, the result of the model (P_1) is preferable to the result of the contrast model (P_5). Thus, we can conclude that both the cardinality constraints and the bound constraints do affect the optimal portfolio composition. In addition, we also calculate the expected proceeds obtained by per unit capital, the turnover rate and the variance of the return rate on the portfolio at each period obtained by different models as shown in Table 4.

To demonstrate the fact that the proposed models can express the investor's different investment preference freely, we take the model (P_1) as an example and vary the possibility degree level on each interval inequality constraint from 0.60 to 0.80. Under each risk attitude case, a run of the designed algorithm with 1000 generations shows that the investor should assign his wealth according to the investment strategies listed in Table 5. From Table 5, we can find that different possibility degree levels on interval inequality constraints lead to different investment strategies. With the increasing of the possibility degree level, the investor becomes more conservative and the obtained terminal wealth becomes smaller, which is consistent with the real world investment case. If the investor is not satisfied with any one of these

Table 3 Comparative results about the investment strategies obtained by different portfolio selection models

Model (P_i)	t	St. 1	St. 2	St. 3	St. 4	St. 5	St. 6	St. 7	St. 8	W_3	$m(W_3)$
Model (P_1)	$t=1$	0.2747	0.0000	0.0000	0.0053	0.2993	0.2975	0.1231	0.0000	[11,404.07, 12,487.44]	11,945.76
	$t=2$	0.2998	0.0000	0.0000	0.0052	0.2999	0.0935	0.0050	0.2965		
	$t=3$	0.2988	0.0092	0.0000	0.0000	0.1830	0.2109	0.0064	0.2917		
Model (P_2)	$t=1$	0.1671	0.0000	0.0000	0.0077	0.2946	0.1728	0.2128	0.1451	[11,385.33, 12,374.51]	11,879.92
	$t=2$	0.2994	0.0000	0.0000	0.0091	0.2989	0.0293	0.2086	0.1548		
	$t=3$	0.2980	0.0128	0.0000	0.0000	0.2028	0.1454	0.1835	0.1575		
Model (P_3)	$t=1$	0.2150	0.0000	0.0000	0.0473	0.2998	0.2384	0.1587	0.0408	[11,392.49, 12,426.10]	11,909.30
	$t=2$	0.2999	0.0000	0.0000	0.0463	0.2991	0.1028	0.0773	0.1745		
	$t=3$	0.3000	0.0214	0.0000	0.0000	0.1950	0.2239	0.0780	0.1818		
Model (P_4)	$t=1$	0.1746	0.0000	0.0000	0.0561	0.2988	0.1825	0.1673	0.1208	[11,384.75, 12,365.82]	11,875.29
	$t=2$	0.2969	0.0000	0.0000	0.0542	0.2998	0.0603	0.1704	0.1184		
	$t=3$	0.2994	0.0092	0.0000	0.0000	0.2108	0.1901	0.1643	0.1263		
Model (P_5)	$t=1$	0.0423	0.0051	0.0005	0.0634	0.4149	0.3728	0.0615	0.0394	[11,406.87, 12,386.33]	11,896.60
	$t=2$	0.4058	0.0149	0.0041	0.0096	0.3967	0.0840	0.0651	0.0198		
	$t=3$	0.3475	0.0154	0.0021	0.0080	0.2314	0.2567	0.0923	0.0465		

Table 4 The expected proceeds $\widetilde{R}_t$, variance $\widetilde{\delta}_t$ and turnover rate $\widetilde{l}_t$ of the portfolio at period t obtained by different models

t	$\widetilde{R}_t$	$\widetilde{\delta}_t$	$\widetilde{l}_t$	$\widetilde{R}_t$	$\widetilde{\delta}_t$	$\widetilde{l}_t$
	Model (P_1)			Model (P_2)		
$t=1$	[1.0465, 1.0791]	[0.0049, 0.0138]	[0.0049, 0.0138]	[1.0441, 1.0772]	[0.0054, 0.0136]	[0.0183, 0.0319]
$t=2$	[1.0458, 1.0888]	[0.0027, 0.0164]	[0.0027, 0.0164]	[1.0457, 1.0843]	[0.0022, 0.0141]	[0.0174, 0.0357]
$t=3$	[1.0477, 1.0764]	[0.0027, 0.0218]	[0.0027, 0.0218]	[1.0474, 1.0719]	[0.0031, 0.0217]	[0.0181, 0.0416]
	Model (P_3)			Model (P_4)		
$t=1$	[1.0454, 1.0785]	[0.0049, 0.0138]	[0.0049, 0.0138]	[1.0441, 1.0771]	[0.0051, 0.0137]	[0.0051, 0.0137]
$t=2$	[1.0455, 1.0857]	[0.0022, 0.0149]	[0.0022, 0.0149]	[1.0455, 1.0837]	[0.0020, 0.0138]	[0.0020, 0.0138]
$t=3$	[1.0476, 1.0742]	[0.0026, 0.0218]	[0.0026, 0.0218]	[1.0476, 1.0720]	[0.0028, 0.0218]	[0.0028, 0.0218]

obtained portfolios, he can obtain more portfolios by varying the possibility degree levels on these interval inequality constraints. Thus, we can conclude that investors' investment intentions can be freely expressed by the proposed models.

To illustrate the effectiveness of the designed algorithm, we also use the dynamic differential evolution (DDE) algorithm in Mohamed and Sabry (2012) to solve the proposed four models. The parameters of the DDE algorithm are set the same as the designed algorithm. We vary the number of generations from 1000 to 5000 to illustrate the stability of the designed algorithm and record the comparative computational results as shown in Table 6. From Table 6, we can find that the objective value of each model obtained by our algorithm is larger than the one obtained by the DDE algorithm. In addition, we also calculate the maximum deviation of objective for each model with respect to different generations. The maximum deviation generated by the designed algorithm for each model is smaller than the one generated by the DDE algorithm. Thus, we can conclude that the designed algorithm is more suitable for solving the proposed models.

6 Conclusions

Due to serious lack of historical information in emerging markets, the returns, risks and turnover rates of securities usually cannot be predicted accurately. In this paper, we assume that the investor can estimate their approximate ranges. Namely, they are characterized by interval numbers. We propose four multi-period cardinality constrained portfolio selection models with interval coefficients. We use the definitions of the possibility degree for interval inequalities to handle the uncertainty inequality constraints in the proposed models and express investors' different risk attitudes. Then, the proposed models are transformed into the corresponding deterministic forms. After that, we design a new dynamic differential evolution algorithm with self-adapting control parameter to solve them. Finally, a numerical example with real data from the Shanghai Stock Exchange is given to illustrate the ideas of our models and the validity of the designed algorithm. Comparative analysis is also given to demonstrate the effects of real constraints on portfolio decision. Computational results indicate that the proposed models can freely express investors' different investment preferences by adjusting the possibility degree levels on the uncertainty constraints and the designed algorithm is suitable for solving our models.

Table 5 The investment strategies obtained by the model (P_1) under different possibility degree levels (PDL)

PDL	Period t	St. 1	St. 2	St. 3	St. 4	St. 5	St. 6	St. 7	St. 8	W_3	$m(W_3)$
0.60	$t=1$	0.2994	0.0000	0.0000	0.0073	0.2981	0.1009	0.2943	0.0000	[11,432.38, 12,567.56]	11,999.97
	$t=2$	0.2988	0.0000	0.0000	0.0066	0.2985	0.0911	0.0067	0.2984		
	$t=3$	0.2990	0.0054	0.0000	0.0000	0.2612	0.1323	0.0068	0.2952		
0.65	$t=1$	0.2983	0.0000	0.0000	0.0052	0.2999	0.1157	0.2809	0.0000	[11,424.63, 12,559.88]	11,992.26
	$t=2$	0.3000	0.0000	0.0000	0.0059	0.2998	0.0985	0.0000	0.2958		
	$t=3$	0.2983	0.0070	0.0000	0.0000	0.2383	0.1594	0.0000	0.2969		
0.70	$t=1$	0.3000	0.0000	0.0000	0.0000	0.2998	0.1884	0.2119	0.0000	[11,419.31, 12,533.89]	11,976.60
	$t=2$	0.2999	0.0000	0.0000	0.0000	0.2999	0.0954	0.0056	0.2992		
	$t=3$	0.2999	0.0053	0.0000	0.0000	0.2188	0.1714	0.0054	0.2992		
0.75	$t=1$	0.2996	0.0000	0.0000	0.0000	0.2999	0.2549	0.1456	0.0000	[11,413.97, 12,508.86]	11,961.42
	$t=2$	0.3000	0.0000	0.0000	0.0000	0.3000	0.0954	0.00511	0.2995		
	$t=3$	0.3000	0.0000	0.0000	0.0000	0.2029	0.1925	0.0052	0.2994		
0.80	$t=1$	0.2747	0.0000	0.0000	0.0053	0.2993	0.2975	0.1231	0.0000	[11,404.07, 12,487.44]	11,945.76
	$t=2$	0.2998	0.0000	0.0000	0.0052	0.2999	0.0935	0.0050	0.2965		
	$t=3$	0.2988	0.0092	0.0000	0.0000	0.1830	0.2109	0.0064	0.2917		

Table 6 Comparative computational results obtained by different algorithms under different generations

	DDE	Our algorithm	Difference		DDE	Our algorithm	Difference
Objective value (λ) of the model (P_1)				Objective value (λ) of the model (P_2)			
$G = 1000$	11,615.6348	11,620.7449	5.1101	$G = 1000$	11,581.0191	11,583.1637	2.1446
$G = 2000$	11,619.3762	11,623.1345	3.7583	$G = 2000$	11,584.9937	11,585.2709	0.2772
$G = 3000$	11,622.8731	11,623.4151	0.5420	$G = 3000$	11,585.2938	11,585.3777	0.0839
$G = 4000$	11,622.8971	11,623.4151	0.5180	$G = 4000$	11,585.0403	11,585.3940	0.3537
$G = 5000$	11,623.1780	11,623.4151	0.2370	$G = 5000$	11,585.1750	11,585.4292	0.2542
Maximum deviation	7.5432	2.6702		Maximum deviation	8.5126	1.6793	
Objective value (λ) of the model (P_3)				Objective value (λ of the model (P_4))			
$G = 1000$	11,597.8740	11,599.2164	1.8470	$G = 1000$	11,573.4465	11,580.9628	0.1124
$G = 2000$	11,599.3715	11,599.6860	0.3145	$G = 2000$	11,581.7589	11,581.9348	0.1759
$G = 3000$	11,599.6777	11,599.7243	0.0466	$G = 3000$	11,581.8812	11,582.0795	0.1983
$G = 4000$	11,599.7258	11,599.7277	0.0185	$G = 4000$	11,582.0591	11,582.0862	0.0271
$G = 5000$	11,599.7267	11,599.7279	0.0011	$G = 5000$	11,581.9667	11,582.0868	0.1201
Maximum deviation	2.3405	0.5052		Maximum deviation	8.6125	0.0073	

Acknowledgments This research was supported by the National Natural Science Foundation of China (No. 71501076), the Natural Science Foundation of Guangdong Province of China (No. 2014A030310454), the Fundamental Research Funds for the Central Universities (2015ZM084, 2014ZP0005) and Guangzhou Financial Services Innovation and Risk Management Research Base.

References

Alefeld, G., & Herzberger, J. (1983). *Introduction to interval computations*. New York: Academic Press.
Alia, M. M., & Törn, A. (2004). Population set-based global optimization algorithms: Some modifications and numerical studies. *Computers and Operations Research*, *31*(10), 1703–1725.
Bhattacharyya, R., Kar, S., & Majumder, D. D. (2011). Fuzzy mean-variance-skewness portfolio selection models by interval analysis. *Computers and Mathematics with Applications*, *61*(1), 126–137.
Bilbao-Terol, A., Pérez-Gladish, B., Arenas-Parra, M., & Rodrłguez-Uría, M. V. (2006). Fuzzy compromise programming for portfolio selection. *Applied Mathematics and Computation*, *173*(1), 251–264.
Brest, J., Greiner, S., Boskovic, B., Mernik, M., & Zumer, V. (2006). Self-adapting control parameters in differential evolution: A comparative study on numerical benchmark problems. *IEEE Transactions on Evolutionary Computation*, *10*(6), 646–657.
Calvo, C., Ivorra, C., & Liern, V. (2014). Fuzzy portfolio selection with non-financial goals: exploring the efficient frontier. *Annals of Operations Research*,. doi:10.1007/s10479-014-1561-2.
Chang, T. J., Meade, N., Beasley, J. E., & Sharaiha, Y. M. (2000). Heuristics for cardinality constrained portfolio optimisation. *Computers and Operations Research*, *27*(13), 1271–1302.
Deb, K. (2000). An efficient constraint handling method for genetic algorithms. *Computer Methods in Applied Mechanics and Engineering*, *186*(2/4), 311–338.
Fang, Y., Lai, K. K., & Wang, S. Y. (2008). Fuzzy portfolio optimization theory and methods, Lecture Notes in Economics and Mathematical Systems, vol. 609, Springer.
Fang, Y., Lai, K. K., & Wang, S. Y. (2006). Portfolio rebalancing model with transaction costs based on fuzzy decision theory. *European Journal of Operational Research*, *175*(2), 879–893.
Fang, S. C., Rajasekera, J. R., & Tsao, H. S. J. (1997). *Entropy Optimization and Mathematical programming*. The Netherlands: Academic Publishers: Kluwer.
Giove, S., Funari, S., & Nardelli, C. (2006). An interval portfolio selection problem based on regret function. *European Journal of Operational Research*, *170*(1), 253–264.
Gupta, P., Inuiguchi, M., Mehlawat, M. K., & Mittal, G. (2013). Multiobjective credibilistic portfolio selection model with fuzzy chance-constraints. *Information Sciences*, *229*, 1–17.
Gupta, P., Mehlawat, M. K., & Saxena, A. (2008). Asset portfolio optimization using fuzzy mathematical programming. *Information Sciences*, *178*(6), 1734–1755.
Huang, X. (2008). Mean-entropy models for fuzzy portfolio selection. *IEEE Transactions on Fuzzy Systems*, *16*(4), 1096–1101.
Ida, M. (2003). Portfolio selection problem with interval coefficients. *Applied Mathematics Letters*, *16*(5), 709–713.
Ida, M. (2004). Solutions for the portfolio selection problem with interval and fuzzy coefficients. *Reliable Computing*, *10*(5), 389–400.
Inuiguchi, M., & Tanino, T. (2000). Portfolio selection under independent possibilistic information. *Fuzzy Sets and Systems*, *115*(1), 83–92.
Ishibuchi, H., & Tanaka, H. (1990). Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research*, *48*(2), 219–225.
Jiang, C., Han, X., Liu, G. R., & Liu, G. P. (2008). A nonlinear interval number programming method for uncertain optimization problems. *European Journal of Operational Research*, *188*(1), 1–13.
Lai, K. K., Wang, S. Y., Xu, J. P., & Zhu, S. S. (2002). A class of linear interval programming problems and its application to portfolio selection. *IEEE Transactions on Fuzzy Systems*, *10*(6), 689–704.
León, T., Liern, V., & Vercher, E. (2002). Viability of infeasible portfolio selection problems: A fuzzy approach. *European Journal of Operational Research*, *139*(1), 178–189.
Li, X., Shou, B., & Qin, Z. (2012). An expected regret minimization portfolio selection model. *European Journal of Operational Research*, *218*(2), 484–492.
Liu, S. T. (2011). The mean-absolute deviation portfolio selection problem with interval-valued returns. *Journal of Computational and Applied Mathematics*, *235*(14), 4149–4157.
Liu, Y. J., Zhang, W. G., & Zhang, P. (2013). A multi-period portfolio selection optimization model by using interval analysis. *Economic Modelling*, *33*, 113–119.

Liu, B. D., Zhao, R. Q., & Wang, G. (2003). *Uncertain Programming with Applications*. Beijing: Tsinghua University Press.

Li, J., & Xu, J. (2007). A class of possibilistic portfolio selection model with interval coefficients and its application. *Fuzzy Optimization and Decision Making*, *6*(2), 123–137.

Markowitz, H. (1987). *Mean-variance analysis in portfolio choice and capital markets*. New York: Basil Blackwell.

Mezura-Montes, E., Miranda-Varela, M. E., & del Carmen Gómez-Ramón, R. (2010). Differential evolution in constrained numerical optimization: An empirical study. *Information Sciences*, *180*(22), 4223–4262.

Mishra, S. K., Panda, G., & Majhi, R. (2014). A comparative performance assessment of a set of multiobjective algorithms for constrained portfolio assets selection. *Swarm and Evolutionary Computation*, *16*, 38–51.

Mohamed, A. W., & Sabry, H. Z. (2012). Constrained optimization based on modified differential evolution algorithm. *Information Sciences*, *194*, 171–208.

Moore, R. E. (1966). *Interval analysis*. Englewood Cliffs, New Jersey: Prentice-Hall.

Parra, M. A., Terol, A. B., & Uíıa, M. V. R. (2001). A fuzzy goal programming approach to portfolio selection. *European Journal of Operational Research*, *133*(2), 287–297.

Storn, R., & Price, K. (1995). *Differential evolution-a simple and efficient adaptive scheme for global optimization over continuous spaces*. Berkley: International Computer Science Institute.

Storn, R., & Price, K. (1997). Differential evolution-a simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization*, *11*(4), 341–359.

Tanaka, H., Guo, P., & Türksen, I. B. (2000). Portfolio selection based on fuzzy probabilities and possibility distributions. *Fuzzy Sets and Systems*, *111*(3), 387–397.

Tiryaki, F., & Ahlatcioglu, B. (2009). Fuzzy portfolio selection using fuzzy analytic hierarchy process. *Information Sciences*, *179*(1/2), 53–69.

Vercher, E., Bermúdez, J. D., & Segura, J. V. (2007). Fuzzy portfolio optimization under downside risk measures. *Fuzzy Sets and Systems*, *158*(7), 769–782.

Wang, L., & Li, L. (2010). An effective differential evolution with level comparison for constrained engineering design. *Structural Multidisciplinary Optimization*, *41*(6), 947–963.

Wang, S., & Zhu, S. (2002). On fuzzy portfolio selection problems. *Fuzzy Optimization and Decision Making*, *1*(4), 361–377.

Watada, J. (1997). Fuzzy portfolio selection and its applications to decision making. *Tatra Mountains Mathematical Publication*, *13*, 219–248.

Wu, M., Kong, D., Xu, J., & Huang, N. (2013). On interval portfolio selection problem. *Fuzzy Optimization and Decision Making*, *12*(3), 289–304.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.

Zhang, W. G., Wang, Y. L., Chen, Z. P., & Nie, Z. K. (2007). Possibilistic mean-variance models and efficient frontiers for portfolio selection problem. *Information Sciences*, *177*(13), 2787–2801.

Zhang, W. G., Zhang, X., & Chen, Y. (2011). Portfolio adjusting optimization with added assets and transaction costs based on credibility measures. *Insurance: Mathematics and Economics*, *49*(3), 353–360.

Zhang, W. G., Zhang, X. L., & Xiao, W. L. (2009). Portfolio selection under possibilistic mean-variance utility and a SMO algorithm. *European Journal of Operational Research*, *197*(2), 693–700.