Ann Oper Res (2017) 258:479–502
DOI 10.1007/s10479-015-1963-9

The optimal harvesting problem under price uncertainty: the risk averse case

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Published online: 27 August 2015

Abstract We study the exploitation of a one species, multiple stand forest plantation when timber price is governed by a stochastic process. Our model is a stochastic dynamic program with a weighted mean-risk objective function, and our main risk measure is the Conditional Value-at-Risk. We consider two stochastic processes, geometric Brownian motion and Ornstein–Uhlenbeck: in the first case, we completely characterize the optimal policy for all possible choices of the parameters while in the second, we provide sufficient conditions assuring that harvesting everything available is optimal. In both cases we solve the problem theoretically for *every* initial condition. We compare our results with the risk neutral framework and generalize our findings to any coherent risk measure that is affine on the current price.

Keywords Multistage stochastic programming · Optimal harvesting · Forestry · Coherent risk measures

1 Introduction

The presence of uncertainty in decision making can be seen in most fields, and the need to incorporate stochastic elements has been acknowledged by several authors. In forestry models, the most common elements that are considered stochastic are timber prices, interest rates, growth processes, and forest fires. A variety of tools have been proposed to deal with these problems, including Markov decision processes (Lembersky and Johnson 1975), stochastic programming (Boychuk and Martell 1996), simulation (Kim et al. 2009; Carmel et al. 2009) and others. As demonstrated by Lönnstedt and Svensson (2000), forest owners

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are more concerned by direct economic risks than biological hazards so in this work we focus on price variations.

The majority of papers that deal with price uncertainty have risk neutral objective functions such as maximizing expected profit or minimizing expected costs. The literature of risk averse models in forestry is still scarce. In Gong and Löfgren (2003), the authors investigate the effect of risk aversion in a two-period harvest investment decision model and show that the optimal harvesting policy depends on the sign of a marginal variance function. In Alvarez and Koskela (2006), the authors show that higher risk aversion shortens the expected rotation period, and that increased forest value volatility decreases the optimal harvesting threshold, which is not true under risk neutrality.

The work Alvarez and Koskela (2007) is the first to consider risk aversion in stochastic ongoing rotation models. It is shown that under risk aversion the optimal harvesting threshold is lower, which translates into a shorter rotation period. The effect of risk aversion on the length of the rotation period is also studied in Gong and Löfgren (2008) using a model with regeneration costs. In this case, the risk averse rotation period can be either longer or shorter than the risk neutral one depending on these costs. Clarke and Harry (1989) find the rotation period when both stock size and price are stochastic. The single-rotation problem is solved theoretically as an optimal stopping problem and the authors propose a numerical iterative scheme to solve the ongoing-rotation problem. It is worth mentioning that all these papers consider single stand models.

The works Mosquera et al. (2011) and Tahvonen and Kallio (2006) consider multiple stand models in which risk aversion is represented via expected utility. Both papers study how the introduction of risk aversion changes the optimal policy by numerical resolution. In particular, Tahvonen and Kallio (2006) concludes that risk aversion completely changes the optimal harvesting policy, favoring smooth forest configurations and making harvesting dependent of price level and forest owner's time preference and wealth. Also considering a multiple stand model but representing risk aversion via Conditional Value-at-Risk (CVaR), Piazza and Pagnoncelli (2015) prove analytically that risk aversion shortens the optimal rotation period and study how the variation of the model's parameters affect the optimal harvesting policy.

The main goal of this paper is to study theoretically a risk averse formulation of a harvest scheduling problem for a multiple stands forest under stochastic timber prices. While the works of Brazee and Mendelsohn (1988), Thomson (1992) and Yoshimoto and Shoji (1998) assume normality and adopt a geometric Brownian motion (GBM) to represent the evolution of timber prices, others, like Alvarez and Koskela (2005), Gjolberg and Guttormsen (2002), argue that a mean reverting process, or Ornstein–Uhlenbeck (O–U) is a better description of the timber price path due to empirical data that has been collected for several species. It is not our intention to go any further into this discussion, and we refer the reader to Insley and Rollins (2005) for more details. As the issue seems far from being settled, and following Plantinga (1998) and Tahvonen and Kallio (2006), we decided to consider both stochastic processes in our work.

Due to an increase of the available computational power, multistage stochastic problems have gained significant attention recently, in areas such as finance (Blomvall and Shapiro 2006; Valladão et al. 2014), pension funds (Haneveld et al. 2010) and energy (Philpott et al. 2013), among others. We propose a multistage stochastic programming model in which the decision maker optimizes a *weighted mean-risk* objective, maximizing expected return while avoiding decisions that exhibit high variability. We build upon our previous work Piazza and Pagnoncelli (2014) and extend the results for the risk averse case. The dynamics and the assumptions of the model are the same as in our previous work, so we briefly describe the model in Sect. 2 for completeness.

The main contributions of the current paper are as follows. First, we completely characterize the optimal policy in the case where prices follow GBM and, for the O–U case, we find a sufficient condition for the optimality of harvesting every mature tree that can be translated into a reservation price. Second, we establish a connection between the risk-neutral and risk averse worlds, that is, we explain how the inclusion of risk affects the optimal harvest policy. We will work mostly with the CVaR, but closed form results are also derived for the Mean Deviation Risk of order 2. In particular, we extend our results for any risk measure that is affine on the current price.

There are very few benchmark problems in stochastic harvest scheduling, and to the best of our knowledge, our work is the first that finds closed form solutions when multiple stand forest models and stochastic prices are considered. Furthermore, coherent risk measures have been widely used in applications, and we believe this is one of the first papers in the literature to incorporate them into forestry.

The rest of the paper is as follows: In Sect. 2 we give a complete description of our model. In Sect. 3 we write dynamic programming equations for the model and discuss the introduction of risk measures in a dynamic framework. With the assumption that prices follow a GBM, we completely characterize the optimal policy in Sect. 4 and compare the results obtained with the risk neutral case. In Sect. 5 we assume prices follow an O–U process and obtain sufficient conditions involving the current price. In Sect. 6 we generalize some of our results for risk measures other than the CVaR. Section 7 concludes the paper and discuss possible extensions. All proofs can be found in the "Appendix".

2 Model formulation

The forest growth model we use was introduced by Rapaport et al. (2003) and considers a one species forest plantation of total area S with maturity age of n years. In contrast with the case of wild forests, the state of a forest plantation may be described by specifying the areas occupied by trees of different ages, making the assumption that trees are planted within a pre-specified and constant distance of each other.

For each period $t \in \mathbb{N}$ we denote $x_{a,t} \geq 0$ the area of trees of age $a = 1, \ldots, n$ in year t, and $\bar{x}_t \geq 0$ the area occupied by trees beyond maturity (older than n). Using a single state variable to represent the over-mature trees conveys the underlying assumption that the growth of trees is negligible beyond maturity. Each period we must decide how much land $c_t \geq 0$ to harvest. Assuming that only mature trees can be harvested we must have

$$0 \leq c_t \leq \bar{x}_t + x_{n,t}, \tag{1}$$

and then the area not harvested in that period will comprise the over-mature trees at the next step, namely, $\bar{x}_{t+1} = \bar{x}_t + x_{n,t} - c_t$. We neglect natural mortality at every age, again an assumption valid in managed forest plantations but not in wild forests. Hence, the transition between age classes is given by $x_{a+1,t+1} = x_{a,t}\ \forall a = 1, \ldots, n-1$. The harvested area is immediately allocated to new seedlings that will comprise the one year old trees in the following year: $x_{1,t+1} = c_t$.

We represent the *state* of the tree population by the vector state

$$\mathbb{X} = (\bar{x}, x_n, \ldots, x_1)^T,$$

and the dynamics described in the previous paragraph can be represented by

$$\mathbb{X}_{t+1} = A\mathbb{X}_t + Bc_t, \tag{2}$$

where

$$A = \begin{pmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The expression of constraint (1) in terms of the defined matrices is

$$0 \leq c_t \leq CA\mathbb{X}_t, \quad \text{with} \quad C = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \end{pmatrix}. \tag{3}$$

The order of events is the following: at every period t the decision maker observes the state of the forest and the current price and decides how much she will harvest, c_t, obtaining a benefit $p_t c_t$. Her decision changes the current state of the system from $\mathbb{X}_t$ to $\mathbb{X}_{t+1}$ according to (2). Then timber price p_{t+1} is observed, leaving her in position of deciding the following harvest c_{t+1}.

3 Weighted mean-risk formulation

Decision makers usually want to maximize expected returns while avoiding risk. To achieve this goal we propose a *weighted mean-risk* objective function $\mathcal{R}$ given by

$$\mathcal{R}[X] := \lambda\mathbb{E}[X] + (1 - \lambda)\rho[X], \tag{4}$$

where ρ is some risk measure, X represents losses and $\lambda \in [0, 1]$ balances the two objectives. In Piazza and Pagnoncelli (2014) the authors find the optimal policy for the case $\lambda = 1$.

The variance is a classical risk measure, and it dates back to the work Markowitz (1952). It has been used extensively in many fields and it is a popular risk measure in forestry (Reeves and Haight 2000; Gong 1998; Gong and Löfgren 2003; Ollikainen 1993). Another popular measure is the Value-at-Risk (VaR), defined by

$$\text{VaR}_\alpha[X] = \inf\{x : \mathbb{P}(X \leq x) \geq 1 - \alpha\}, \quad \alpha \in (0, 1).$$

The VaR is the left side $(1 - \alpha)$ quantile of the distribution of X and it is also used in forestry (Zhong-wei and Yan 2009; Roessiger et al. 2011).

3.1 The Conditional Value-at-Risk

A risk measure that gained increasing popularity since the publication of Rockafellar and Uryasev (2000) is the Conditional Value-at-Risk (CVaR). For a continuous random variable X, the CVaR is defined as the average of losses above the VaR level:

$$\text{CVaR}_\alpha[X] = \mathbb{E}[X|X > \text{VaR}_\alpha[X]] = \inf\left\{t \in \mathbb{R} : t + \alpha^{-1}\mathbb{E}[X - t]_+\right\}. \tag{5}$$

In the seminal work of Artzner et al. (1999), the authors defined a set of axioms that a risk measure must satisfy in order to be called *coherent*. While the CVaR satisfies all those properties, nor the variance or the VaR are coherent. We are going to use the CVaR extensively throughout our paper since it will be our prototype example of risk measure.

The majority of papers in forestry use expected utility to model risk aversion. Although risk measures and expected utility criteria are related through dual representations of the risk functional (see example 6.14 of Shapiro et al. 2009), both have its own advantages and drawbacks. Our goal here is to apply risk measures to forestry and a comparison of both approaches is out of the scope of this paper.

3.2 Dynamic programming formulation

In this paper we deal with risk averse stochastic dynamic problems and, in order to write dynamic programming equations, we need to extend the concept of conditional expectation to accommodate arbitrary risk measures. If X and Y are random variables, we can consider the value of $\mathcal{R}$ at the conditional distribution of X given $Y = y$, which we denote as $\mathcal{R}_{|Y}[X]$ and refer to as a *conditional risk mapping* (Ruszczynski and Shapiro 2005). Following Shapiro (2009), we consider a nested risk averse formulation based on conditional risk mappings:

$$\begin{cases} V_0(\mathbb{X}_0, p_0) = \text{Min}_{c_0} \left\{ -p_0 c_0 + \delta \mathcal{R}_{|p_0} \Big[\text{Min}_{c_1} -p_1 c_1 + \delta \mathcal{R}_{|p_1} \Big[\cdots \Big] \Big] \right\} \\ \text{s.t.} \quad (2) \text{ and } (3), \end{cases} \tag{6}$$

where $\mathcal{R}_{|p_t}$ is a conditional risk mapping for $t \in \mathcal{T}$ and $\delta \in (0, 1)$ is the discount factor. In this paper we will work with $\mathcal{R}_{|p_t}$ of the form (4). Throughout the paper we will consider two different types of sets $\mathcal{T}$: (i) $\mathcal{T} = \{1, \ldots, T\}$ for the finite horizon case, where T is the time horizon of the problem, and, (ii) $\mathcal{T} = \mathbb{N}$ for the infinite horizon case, with particular attention to the convergence properties of the objective function of (6). Details can be found in Ruszczynski and Shapiro (2005), where the authors show that the corresponding dynamic programming equations are,

$$V_t(\mathbb{X}_t, p_t) = \underset{c_t \in [0, CA\mathbb{X}_t]}{\text{Min}} \left\{ -p_t c_t + \delta \mathcal{R}_{|p_t} \left[V_{t+1}(A\mathbb{X}_t + Bc_t, p_{t+1}) \right] \right\}, \tag{7}$$

for all $t \in \mathcal{T}$ and in the finite horizon case $V_T(\mathbb{X}_T, p_T) = -p(T)c^*(T)$, with $c^*(T) = CA\mathbb{X}(T)$ which corresponds to harvest the maximum possible amount.

If the conditional risk mappings are positive homogeneous (see Shapiro 2009), it is easy to check that if

$$-p_{T-1} - \delta \mathcal{R}_{|p_{T-1}}[-p_T] \leq 0, \tag{8}$$

the optimal solution for $t = T - 1$ is $c^*_{T-1} = CA\mathbb{X}_{T-1}$.[1]

Remark 1 In the general case, the determination of the optimal policy for an arbitrary t is more difficult, as we do not know future policies. However, we will see that in other time stages, conditions analogous to (8), i.e.,

$$-p_t - \delta \mathcal{R}_{|p_t}[-p_{t+1}] \leq 0 \quad \text{with} \quad t \in \mathcal{T}, \tag{9}$$

play a fundamental role in the characterization of the optimal policy in this article. Observe that these conditions do not consider the entire future, but only the price of timber *one period ahead* of time.

4 Geometric Brownian motion

We first study problem (6) when $\rho = \text{CVaR}_\alpha$ and the dynamics of prices follow a geometric Brownian motion (GBM). This dynamics has been extensively used to model asset prices

[1] When $-p_{T-1} - \delta \mathcal{R}_{|p_{T-1}}[-p_T] = 0$ the optimum of (7) is reached for *any* $c \in [0, CA\mathbb{X}_{T-1}]$, hence, for this particular value we may adopt the convention $c^*_{T-1} = CA\mathbb{X}_{T-1}$.

in financial markets and therefore represents a natural choice for timber prices. It is a well known process, hence, we define it without further detail,

$$dp_t = \mu p_t dt + \sigma p_t dW_t \quad \text{(GBM)}, \tag{10}$$

where $\mu \in \mathbb{R}$ is the *drift* of the GBM, $\sigma > 0$ is the constant variance and W_t denotes the Wiener process.

For the GBM, the price p_{t+1} conditional on p_t follows a lognormal distribution, and it is possible to compute $\text{CVaR}_{\alpha|p_t}[-p_{t+1}]$ explicitly:

$$\text{CVaR}_{\alpha|p_t}[-p_{t+1}] = -p_t \frac{e^{\mu}}{\alpha} \Phi(z_{1-\alpha} - \sigma),$$

where Φ is the cumulative distribution function of the normal random variable with mean 0 and variance 1 and $z_\alpha = \Phi^{-1}(1-\alpha)$ (see Theorem 6.2 of Shapiro et al. 2009).

It is easy to see that (9) is equivalent to

$$p_t[-1 + \lambda\delta e^{\mu} + (1-\lambda)\delta e^{\mu}\kappa] \leq 0, \quad \text{with } 0 < \kappa = \frac{1}{\alpha}\Phi(z_{1-\alpha} - \sigma) < 1. \tag{11}$$

Therefore, condition (9) is equivalent to

$$\delta e^{\mu}[\lambda + (1-\lambda)\kappa] \leq 1,$$

that depends only on the parameters of the problem $(\delta, \lambda, \alpha)$ and those of the price process (μ, σ). This implies that when prices follow a GBM, condition (9) is satisfied for all p_t or for none.

However, throughout this section we work with the strict inequality version of the condition above

$$\delta e^{\mu}[\lambda + (1-\lambda)\kappa] < 1, \tag{12}$$

to assure convergence of the objective function in the infinite horizon case (see the proof of Lemma 1).

We are now in condition to state the main result of this section, the characterization of the *greedy policy*, i.e., $c_t^* = CA\mathbb{X}_t$ for all t, when prices follow a GBM. The theorem encompasses both finite and infinite horizon cases, and we present a brief technical lemma showing that the value function is well defined in the latter case.

Lemma 1 *Consider problem* (6) *with* $\mathcal{T} = \mathbb{N}$ *and* $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)CVaR_\alpha$ *assume prices evolve according to* (10)*. If Condition* (12) *holds, the value function* V_0 *is well defined.*

Theorem 1 *Consider problem* (6) *with* $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)CVaR_\alpha$ *and assume prices evolve according to* (10)*. If condition* (12) *holds, the greedy policy is optimal.*

4.1 Discussion and insights

In this section we compare the optimality conditions for the risk neutral and risk averse cases. In Piazza and Pagnoncelli (2014), it is shown that in the risk neutral setting under GBM, the greedy policy is optimal if and only if

$$\delta e^{\mu} < 1. \tag{13}$$

Besides the obvious observation that when $\lambda = 1$ the risk neutral case is retrieved, other interesting interpretations of condition (12) can be found. Since λ and κ belong to [0, 1], we have the intuitive result that whenever the greedy policy is optimal for the risk neutral case it will be automatically optimal for the risk averse case for the same choice of parameters.

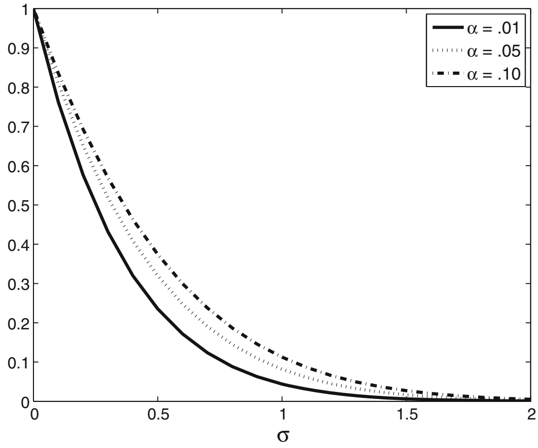

Fig. 1 κ as a function of σ for $\alpha = 0.10$, 0.05 and 0.01

In Sect. 4.3 we show how the solution changes with λ; we now concentrate in the effect of the other parameters. In Fig. 1 we can see that for values of σ close to zero, that is, in the case where GBM is less volatile, the constant κ is close to one and conditions (12) and (13) become essentially equivalent. For larger values of σ, that is, for higher volatility, we observe that κ is closer to zero and, unless the drift μ is sufficiently high, the optimal policy for a risk averse decision maker will be greedy.

Let us now fix the value of σ and turn our attention to the role of the risk aversion parameter α. In Fig. 1, as we increase α, that is, as the decision maker becomes less risk averse, the multiplicative factor κ increases. In the limiting case, where $\alpha = 1$, it can be seen from (11) that $\kappa = 1$ and the risk neutral policy is retrieved.

It may seem that the consideration of both parameters α and λ is redundant as they both are related to the level of risk aversion. This idea is reinforced by the fact that extreme values of α and λ produce the same result, i.e., both $\alpha = 0$ and $\lambda = 1$ yield the risk neutral solution. But, they are conceptually different. Indeed, the value of α establishes which fraction of the worst possible outcomes are taken into consideration, while λ assigns the weight these outcomes will have in the objective function. Besides, for other risk measures the only parameter controlling risk aversion is λ.

4.2 Another optimal policy

We show that Condition (9) is tight for the optimality of the greedy policy in the sense that if it does not hold then the optimal policy is not greedy. If (9) does not hold, not only Lemma 1 does not apply, but we can prove that the value function is not defined whenever $\mathcal{T} = \mathbb{N}$.[2] Hence, we only consider $\mathcal{T} = \{0, \ldots, T\}$. It is natural to think that the decision maker should postpone the harvest as much as possible. Hence, before the final time T, harvesting should be stopped altogether in order to have the maximum surface available at time T. However, observe that every land plot harvested and planted n or more time steps before T will contain mature trees available for harvesting at time T. Hence, it is convenient to harvest every mature tree at time $T - n$, since there is enough time for seedlings to mature before reaching T. Repeating this reasoning we can conjecture that the only time steps when harvesting is

[2] Details available from the authors upon request.

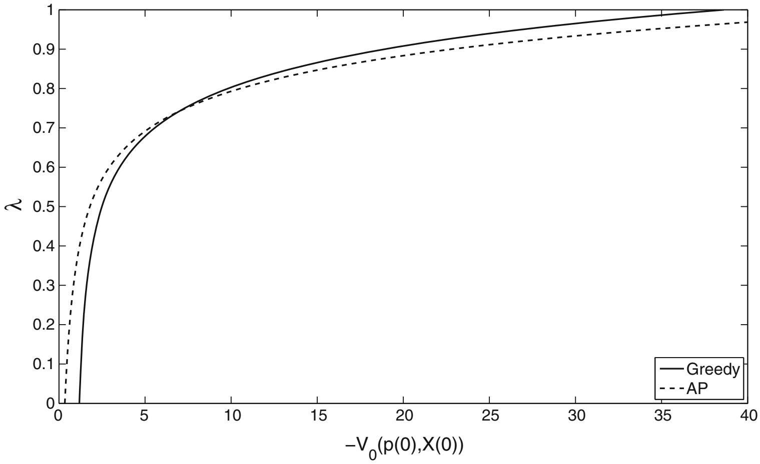

Fig. 2 Efficient frontier for $\lambda \in [0, 1]$

allowed are $T - kn$ for $k = 0, 1, \dots, \lfloor T/n \rfloor$ and that everything available then should be harvested. This is to say,

$$c_t = \begin{cases} CA\mathbb{X}_t & \text{if } t = T - kn \\ 0 & \text{else.} \end{cases}$$

We call this harvesting policy the *accumulating policy* (AP), and we have the following result.

Theorem 2 *Consider problem* (6) *with* $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)CVaR_\alpha$ *and assume prices evolve according to* (10)*. If condition* (12) *does not hold, then the AP is optimal in the finite horizon case.*

4.3 Efficient frontier

Following Markowitz (1952) and Zheng (2009), we construct an efficient frontier by varying the value of λ when $\rho =$ CVaR. For $\lambda \in [0, 1]$, we calculate the total value at time $t = 0$ under the greedy and accumulating policies. From Theorem 1 we know that the greedy policy will be optimal for $\delta e^\mu[\lambda + (1-\lambda)\kappa]$ less than one, and, beyond that value, Theorem 2 indicates that the accumulating policy is optimal.

For a value of λ in the y axis, Fig. 2 shows the negative of the total value at time $t = 0$ under the greedy policy (continuous line) and the accumulated policy (dashed line) in the x-axis. The figure is generated with parameters $\mu = 0.2$, $\sigma = 0.4$, $p(0) = 1$, $\delta = 0.95$, $\alpha = 0.1$ and with $T = 20$ periods for the initial state $\mathbb{X}(0) = (0, 0, 0, 1)$. With these values, condition (12) gives a value of $\lambda \leq 0.742$. Observe that the two lines intersect and the change of optimal policies occurs for λ between 0.74 and 0.75, as we predicted.

5 Ornstein–Uhlenbeck process

We now study problem (6) when prices follow a mean-reverting Ornstein–Uhlenbeck process (O–U) and focus on $\rho =$ CVaR. It is well-know that GBM does not capture some behaviors of price movement such as mean-reversion, which is best emulated by the O–U process.

Since O–U is a well-known process, we define it without further detail.

$$dp_t = \eta(\bar{p} - p_t)dt + \sigma dW_t \qquad \text{(O–U)}, \tag{14}$$

where $\eta > 0$ is the rate of mean-reversion to an equilibrium $\bar{p}$, $\sigma > 0$ is the constant variance and W_t denotes the Wiener process. A closed expression for the value of the price p_{t+1} given the price p_t can be written as follows:

$$p_{t+1} = e^{-\eta} p_t + (1 - e^{-\eta})\bar{p} + \int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW_s.$$

In an O–U process, it can be shown (see for example Maller et al. 2009) that at every time period t the random variable p_t follows a normal distribution[3] with conditional mean $\mathbb{E}_{|p_t}[p_{t+1}] = \bar{p} + (p_t - \bar{p})e^{-\eta}$, and conditional variance given by $\text{Variance}_{|p_t}[p_{t+1}] = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta})$.

If X follows a normal distribution with mean μ and variance σ, the $\text{CVaR}_\alpha[X]$ can be explicitly computed:

$$\text{CVaR}_\alpha[X] = \mu + \frac{\sigma}{\alpha\sqrt{2\pi}} e^{-z_\alpha^2/2}, \tag{15}$$

where $z_\alpha = \Phi^{-1}(1-\alpha)$ and Φ is the cumulative distribution function of the normal random variable with mean 0 and variance 1 (see Theorem 6.2, Shapiro et al. 2009).

Using (15) we can calculate the CVaR at each time period as follows:

$$\text{CVaR}_{\alpha|p_t}[-p_{t+1}] = -p_t e^{-\eta} - \bar{p}(1 - e^{-\eta}) + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{1 - e^{-2\eta}}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha}.$$

Using that $\delta e^{-\eta} \in (0, 1)$, condition (9) can be expressed as

$$p(t) \geq \frac{\delta}{1 - \delta e^{-\eta}} \left[\bar{p}(1 - e^{-\eta}) - \frac{(1-\lambda)\sigma}{\sqrt{2\pi}} \sqrt{\frac{1 - e^{-2\eta}}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha} \right], \tag{16}$$

where the right hand side can be interpreted as a *reservation price*, i.e., a price threshold above which it is optimal to harvest everything available.

In the previous section, when prices follow a GBM, we have condition (12) that does not depend on p_t and assures the satisfaction of (9) for all t. We do not have anything similar when prices follow an O–U process, as we see that condition (16) depends on p_t. Hence, when solving Bellman equation (7) for t, we have no information of what may happen at $t + 1$ or after and we have to consider every possible situation.

However, we will show in Theorem 3 that condition (16) is sufficient to assure that the greedy policy is optimal at t, i.e., $c_t = CA\mathbb{X}_t$. We must stress here that (16) is *not* necessary and that we do not have any information about what the optimal policy is when it does not hold.

To simplify notation we name

$$a = e^{-\eta} \text{ and } b = \bar{p}(1 - e^{-\eta}) - \frac{\sigma(1-\lambda)}{\sqrt{2\pi}} \sqrt{\frac{1 - e^{-2\eta}}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha}, \tag{17}$$

[3] Even though the arithmetic O–U can lead to negative values, the process is frequently used to model the evolution of prices (see, for example, Alvarez and Koskela 2005; Gjolberg and Guttormsen 2002). The discussion of which process best represents timber prices is far from being settled. We refer the reader to Dixit and Pindyck (1994), Insley and Rollins (2005) and the references therein for more information.

and hence $\mathcal{R}_{|p_t}[-p_{t+1}] = -ap_t - b$. The idea of the proof consists in showing that the coefficient of c in (7) is composed of terms of the form

$$\Delta^{m_i}_{j_i}(p_t) = \delta^{m_i} \left\{ -p_t a^{m_i}(1-\delta^{j_i} a^{j_i}) - \frac{b}{1-a}\left[1 - \delta^{j_i} - a^{m_i}(1-\delta^{j_i} a^{j_i})\right]\right\} \tag{18}$$

for some values of m_i and j_i such that $j_i \in \{0, \ldots, n-1\}$.

Instead of checking the sign of $\Delta^m_j(p_t)$ for every value of m and j, we use the equivalence

$$\Delta^m_j(p_t) \leq 0 \iff p_t \geq \frac{b}{1-a}\left[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] \tag{19}$$

and prove in the next lemma that $\Delta^m_j(p_t) \leq 0$ when condition (16) holds.

Lemma 2 *If $a \in (0, 1)$ and $b \geq 0$, condition (16) implies that*

$$\Delta^m_j(p_t) \leq 0, \quad \textit{for all } m \textit{ and } j \in \mathbb{N}.$$

When price follows an O–U process, we have that $a < 1$, but for some values of the parameters we could have $b < 0$. Hence, the second hypothesis of Lemma 2 has to be explicitly required in the statement of the main result of this section.

Lemma 3 *Consider problem (6) with $\mathcal{T} = \mathbb{N}$ with $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)CVaR_\alpha$ and assume prices evolve according to (14). The value function V_0 is well defined.*

Theorem 3 *Consider problem (6) with $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)CVaR_\alpha$ and assume prices follow (14). If $b \geq 0$ and condition (16) holds, then $c_t = CA\mathbb{X}_t$ is optimal at time t.*

The right-hand side of (16) can be used as a reservation price, as we know that if p_t is above that value it is optimal to harvest every mature tree. The condition is sufficient but not necessary: if (16) does not hold, we are not able to discard the greedy policy.

5.1 Discussion and insights

In Piazza and Pagnoncelli (2014) the risk neutral version of this problem was studied, and the following reservation price was found

$$p_r = \frac{\delta}{1-\delta e^{-\eta}}\,\bar{p}\,(1-e^{-\eta}),$$

which is exactly the positive term of the right hand side of (16). Therefore, for every value of the parameters λ, η, α and σ, the greedy policy is more likely to be optimal in the risk averse framework than in the risk neutral one.

If we choose $\alpha = 1$ in the calculation of the CVaR we obtain the usual expected value operator. Indeed, observe that the term $e^{-z_\alpha^2/2}/\alpha$ in (16) goes to zero when α approaches 1, implying that resulting reservation price would be p_r, retrieving the solution for the risk neutral harvesting problem.

It is straightforward to see that if the volatility σ goes to zero we have that the reservation price also coincides with p_r. This result is rather intuitive and it mimics what we obtained for the GBM case: if the volatility is small the sufficient conditions for the optimality of the greedy policy are essentially the same under risk neutrality and risk averseness. However, when the volatility σ increases the two cases are significantly different.

When η goes to infinity, b goes to $\bar{p}$ and a goes to zero. The reservation price simplifies to $\delta\bar{p}$. In this case one can expect that the optimal policy will be greedy for any t since the

Table 1 Parameters regions

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	(i)	(ii)	(v)
$b < 0$	(iii)	(iv)	(v)

speed of the mean reversion is so high that the price is essentially equal to $\bar{p}$ for all time periods and therefore greater than the reservation price $\delta\bar{p}$. This is consistent with the result for constant deterministic prices presented in Rapaport et al. (2003) that establishes that the greedy policy is optimal for all time periods.

In the right hand side of (16), the term $\sqrt{\frac{1-e^{-2\eta}}{2\eta}}$ is strictly decreasing with η and it lies between zero and one. When the speed of the mean reversion η vanishes, the whole expression converges to 1. Since the first term in the expression of b goes to zero as η approaches zero, we have in this case that $b = \frac{-\sigma(1-\lambda)}{\sqrt{2\pi}\alpha} e^{-z_\alpha^2/2} < 0$. It is interesting to note that in this case Theorem 3 does not apply. We deal with this particular situation in the next section.

6 Extension to affine weighted mean-risk measures

It is worth investigating whether the results of the previous section can be generalized for other values of a and b. This would eliminate the hypothesis $b \geq 0$ of Theorem 3, and, more importantly, it would allow the extension of the results to other price processes and to risk measures other than CVaR_α.

The proof of Theorem 3 relies heavily in the fact that

$$\mathcal{R}_{|p_t}[-p_{t+1}] = -ap_t - b$$

and is valid for $(a, b) \in (0, 1) \times \mathbb{R}_+$. In the following we study the extension of this theorem to $(a, b) \in (0, 1/\delta) \times \mathbb{R}$. We start with the generalized version of Lemma 3.

Lemma 4 *Consider problem (6) with $\mathcal{T} = \mathbb{N}$ and assume that $\mathcal{R}_{|p_t}[-p_{t+1}] = -ap_t - b$. The value function $V_0(\cdot, \cdot)$ is well defined if $a \in (0, 1/\delta)$.*

We divide the semi-plane of parameters in five regions as shown in Table 1 and study the variation of the right-hand side of (19) in each of these regions. The following theorem summarizes the sufficient conditions for the optimality of the greedy policy, that can be found with this method.

Theorem 4 *Consider problem (6) and assume $\mathcal{R}_{|p_t}[-p_{t+1}] = -ap_t - b$.*

1. *In region* (i), $p_t \geq b\delta/(1 - \delta a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}_t$.
2. *In region* (ii), $p_t \geq b\delta/(1 - \delta a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}_t$.
3. *In region* (iii), *no sufficient condition assuring* $c^*(t) = CA\mathbb{X}_t$ *is found if* $\mathcal{T} = \mathbb{N}$. *In the finite horizon case* $p_t \geq \frac{b}{1-a}\left[1 - \frac{1-\delta}{a^{T-t}(1-\delta a)}\right]$ *is sufficient.*
4. *In region* (iv), $p_t \geq b/(1 - a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}_t$.

Remark 2 Although regions (i) and (ii) deliver the same sufficient condition, we decided to keep them separated as case (i) is exactly Theorem 3. The theorem does not provide information about the region (iii) when $\mathcal{T} = \mathbb{N}$, because the method of proof used delivers the condition $p_t = \infty$. However, this method can be applied in the finite horizon case to retrieve a meaningful condition. Concerning region (v), not only Lemma 4 does not apply, but it is possible to prove that the objective function is not well defined.[4]

Theorem 1 is a corollary of Theorem 4. Indeed, when prices follow a GBM and we use $\mathcal{R} = \lambda\mathbb{E} + (1-\lambda)\text{CVaR}$, we have $a = \lambda e^{\mu} + (1-\lambda)e^{\mu}\kappa$ and $b = 0$. When $b = 0$, by simple inspection we see that conditions (i) and (ii) reduce to $p_t \geq 0$, that is always satisfied. Furthermore, $a < 1/\delta$ is equivalent to Condition (12).

Corollary 1 *If $b = 0$ and $p_t \geq 0$ for all t (as is the case for a GBM price process) and $0 < a < 1/\delta$, the greedy policy is always optimal.*

In the next subsection we compute explicit values of a and b for another risk measure, showing the relevance of Theorem 4.

6.1 Mean deviation risk

Another important risk measure is the Mean Deviation Risk (MDR) of order p:

$$\text{MDR}[X] := \mathbb{E}[X] + c\mathbb{E}\left[|X - \mathbb{E}[X]|^p\right])^{1/p},$$

where $p \in [1, +\infty)$ and $c > 0$. The MDR has similarities with the expression of the variance, and under mild hypothesis, e.g, nonatomic probability measures, is a coherent risk measure for $p > 1, c \geq 0$, and $p = 1, c \in [0, 1/2]$ (Shapiro et al. 2009). For GBM and O–U we have respectively:

$$\text{MDR}_{|p_t}(-p_{t+1}) = -p_t e^{\mu} - c p_t e^{\mu}\left(\mathbb{E}\left[\left|e^{-\frac{\sigma^2}{2}+\sigma W} - 1\right|^p\right]\right)^{1/p},$$

$$\text{MDR}_{|p_t}(-p_{t+1}) = -p_t e^{-\eta} - \bar{p}(1-e^{-\eta}) + c\left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^p\right]\right)^{1/p}.$$

The MDR is much less tractable than the CVaR. However, for the particular case of $p = 2$ and a O–U process, we are able to obtain explicit expressions using Itô's isometry[5]:

$$\text{MDR}_{|p_t}(-p(t+1)) = -p_t e^{-\eta} - \bar{p}(1-e^{-\eta}) + c\sigma\left(\frac{1}{2\eta} - \frac{e^{-2\eta}}{2\eta}\right)^{1/2}.$$

Therefore, when the MDR of order 2 is used as a weighted risk objective as in (4), it is affine in p_t for O–U and the coefficients are

$$a = e^{-\eta}, \quad b = \bar{p}(1-e^{-\eta}) - c\sigma\left(\frac{1-e^{-2\eta}}{2\eta}\right)^{1/2}.$$

7 Conclusions

We study a harvest scheduling problem under price uncertainty. We depart from the usual risk neutral framework and incorporate weighted mean-risk measures in the objective function. Considering both the finite and infinite time horizon frameworks, we obtain conditions on the parameters of the model that characterize the optimal policy.

[4] Details available from the authors upon request.

[5] The calculations are done in the "Appendix" of this manuscript.

Our theoretical results characterize completely the optimal policy for the GBM case and provide a closed expression of a reservation price for the O–U case. We focus first on the Conditional Value-at-Risk, and later extend our results for any affine risk measure. We also prove that the Mean-Deviation Risk is affine so our results apply directly to this case.

Future work includes considering a more complex forest growth model. For example, taking into account natural mortality needs only a very simple modification in the matrix that defines the growth dynamics. However, proofs may need a significant adaptation.

Considering multi-species forests would allow us to compare our results with an important part of the present literature. The traditional paradigm indicates that a risk averse decision maker would favor a more homogeneous land allocation, establishing more ecologically friendly forests. Indeed, through numerical experiments, Hildebrandt et al. (2010), Knoke et al. (2005) and Roessiger et al. (2011) show that higher risk aversion implies higher mixture of species.

Acknowledgements This research was partially supported by Programa Basal PFB 03, CMM. B.K. Pagnoncelli acknowledges the financial support of FONDECYT under projects 11130056 and 1120244. A. Piazza acknowledges the financial support of FONDECYT under Project 11090254 and of CONICYT Anillo ACT1106 and CONICYT REDES 140183.

Appendix

Preliminaries

Before getting into the proof of the lemmas and theorems stated in the main body of the paper, we present some definitions that will be necessary throughout the proofs.

In (7), the optimal control c_t in each period depends only on the current state of the forest and price through the *decision function* $c_t = \pi_t(\mathbb{X}_t, p_t)$. A sequence $\Pi = \{\pi_t\}_{t\in\mathcal{T}}$ is called a *policy*. Of course, a policy is *feasible* if $\mathbb{X}_t$ and $c_t = \pi_t(\mathbb{X}_t, p_t)$ satisfy (2) and (3) for every possible value of $\mathbb{X}_t$ and p_t at every instant $t \in \mathcal{T}$. Observe that the non-negativity of the state variables $\bar{x}_t$ and $x_{a,t}$ for $a = 1, \ldots, n$ is assured by (2) and (3).

The expected benefit of a given policy Π for an initial state $\mathbb{X}_0$ and an initial price p_0 is

$$Q_0^{\Pi}(\mathbb{X}_0, p_0) = -p_0\pi_0(\mathbb{X}_0, p_0) + \delta\mathcal{R}_{|p_0}\left[-p_1\pi_1(\mathbb{X}_1, p_1) + \delta\mathcal{R}_{|p_1}[\cdots]\right], \tag{20}$$

if the time horizon is infinite. Correspondingly, we denote $Q_{0,T}^{\Pi}(\mathbb{X}_0, p_0)$ the expected benefit of policy Π whenever $\mathcal{T} = [1, \ldots, T]$.

Problem (6) can be stated as the problem of finding a feasible policy that minimizes (20),

$$V_0(\mathbb{X}_0, p_0) = \begin{cases} \min_{\Pi} & Q_0^{\Pi}(\mathbb{X}_0, p_0) \\ \text{s.t.} & \Pi \text{ is a feasible policy,} \end{cases} \tag{21}$$

or analogously, $V_{0,T}(\mathbb{X}_0, p_0) = \min_{\Pi} Q_{0,T}^{\Pi}(\mathbb{X}_0, p_0)$ s. t. Π is a feasible policy.

In the sequel, we will also use the expected discounted benefit from an intermediate step

$$Q_t^{\Pi}(\mathbb{X}_t, p_t) = -p_t c_t + \delta\mathcal{R}_{|p_t}\left[-p_{t+1}c_{t+1} + \delta\mathcal{R}_{|p_{t+1}}[\cdots]\right]$$

and the corresponding value function

$$V_t(\mathbb{X}_t, p_t) = \begin{cases} \min_{\Pi} & Q_t^{\Pi}(\mathbb{X}_t, p_t) \\ \text{s.t.} & \Pi \text{ is a feasible policy,} \end{cases}$$

as well as the analogous definitions in the finite case.

Appendix 1: Proof of Lemma 1

For given initial state and price $\mathbb{X}_0$, p_0 we consider the cost resulting of the application of policy Π (not necessarily optimal) up to T: $Q^{\Pi}_{0,T}(\mathbb{X}_0, p_0)$. To lighten the notation we will use $Q^{\Pi}_{0,T}$ instead of $Q^{\Pi}_{0,T}(\mathbb{X}_0, p_0)$ as $\mathbb{X}_0$ and p_0 remain constant throughout the proof. The expression of $Q^{\Pi}_{0,T}$ is a slight modification of (20)

$$Q^{\Pi}_{0,T} = -p_0 c_0 + \delta \mathcal{R}_{|p_0}[-p_1 c_1 + \cdots + \delta \mathcal{R}_{|p_{T-1}}[-p_T\, c_T]],$$

Due to the fact that $p_t c_t \geq 0$ for all t (when prices follow a GBM) and the monotonicity of any coherent risk measure we know that $Q^{\Pi}_{0,T} \geq Q^{\Pi}_{0,T+1}$, hence the sequence $Q^{\Pi}_{0,T}$ either converges to the limit or diverges to $-\infty$ when $T \to \infty$. We now prove that $Q^{\Pi}_{0,T}$ is bounded below for all T,

$$Q^{\Pi}_{0,T} = -p_0 c_0 - \delta e^{\mu} \chi p_0 c_1 - \cdots - (\delta e^{\mu} \chi)^T p_0 c_T \geq -p_0 S \frac{1 - (\delta e^{\mu} \chi)^{T+1}}{1 - \delta e^{\mu} \chi},$$

where S represents the total surface of the forest and $\chi = \lambda + \kappa(1 - \lambda)$.

If $\delta e^{\mu} \chi < 1$ we get $Q^{\Pi}_{0,T} > -p_0 S \frac{1}{1-\delta e^{\mu}\chi} > -\infty$ for all T. This implies that the sequence $Q^{\Pi}_{0,T}$ converges when T goes to infinity. This limit is the value associated to the policy Π denoted as Q^{Π}_0,

$$Q^{\Pi}_0 = -p_0 \sum_{t=0}^{\infty} (\delta e^{\mu} \chi)^t c_t > -\infty \tag{22}$$

As the bound on Q^{Π}_0 does not depend on the policy Π, we conclude

$$V_0(\mathbb{X}_0, p_0) = \underset{\Pi}{\text{Min}}\, Q^{\Pi}_0 > -\infty.$$

Appendix 2: Proof of Theorem 1

To prove that the greedy policy is optimal, we check that the benefit associated with it, Q^{GP}, satisfies the Bellman equation (7) from any initial condition. The formula of Q^{GP} is obtained using (22), where $\chi = \lambda + \kappa(1 - \lambda)$.

We consider first the infinite time horizon case. If the initial state is $\mathbb{X}_t = (\bar{x}, x_n, \ldots, x_2, x_1)$, it is easy to see that the harvests associated to the GP are $c_{t+in} = \bar{x} + x_n$ for all $i \in \mathbb{N}$ and $c_{t+in+j} = x_{n-j}$ for $j = 1, \ldots, n-1$ and $i \in \mathbb{N}$, and hence,

$$Q^{GP}_t(\mathbb{X}_t, p_t) = -p_t \sum_{i=0}^{\infty} \left((\delta e^{\mu} \chi)^{in} \bar{x} + \sum_{j=0}^{n-1} (\delta e^{\mu} \chi)^{in+j} x_{n-j} \right). \tag{23}$$

Given $c_t = c \in [0, CA\mathbb{X}_t]$, the state at $t+1$ is $\mathbb{X}_{t+1} = (\bar{x} + x_n - c, x_{n-1}, \ldots, x_1, c)$ and the value associated to the GP is

$$Q^{GP}_{t+1}(\mathbb{X}_{t+1}, p_{t+1}) = -p_{t+1} \sum_{i=0}^{\infty} \Big((\delta e^{\mu} \chi)^{in} (\bar{x} + x_n - c) + \sum_{j=0}^{n-2} (\delta e^{\mu} \chi)^{in+j} x_{n-j-1} + (\delta e^{\mu} \chi)^{in+n-1} c \Big).$$

Inserting $V_{t+1} = Q^{GP}_{t+1}$ into the rhs of the Bellman equation (7), the argument of the Min operator is

$$\begin{aligned}\Phi(c) &= -p_t c + \delta \mathcal{R}_{|p_t}\Bigg[-p_{t+1}\sum_{i=0}^{\infty}\Big((\delta e^{\mu}\chi)^{in}(\bar{x}+x_n-c) \\ &\quad + \sum_{j=0}^{n-2}(\delta e^{\mu}\chi)^{in+j}x_{n-j-1} + (\delta e^{\mu}\chi)^{in+n-1}c\Big)\Bigg] \\ &= -p_t c - p_t\sum_{i=0}^{\infty}\Big((\delta e^{\mu}\chi)^{in+1}(\bar{x}+x_n-c) \\ &\quad + \sum_{j=0}^{n-2}(\delta e^{\mu}\chi)^{in+j+1}x_{n-j-1} + (\delta e^{\mu}\chi)^{in+n}c\Big) \end{aligned} \tag{24}$$

The coefficient affecting c in $\Phi(c)$ is

$$\text{coeff}(c) = -p_t\left(1 - \sum_{i=0}^{\infty}(\delta e^{\mu}\chi)^{in+1} + \sum_{i=0}^{\infty}(\delta e^{\mu}\chi)^{in+n}\right) = -p_t(1-\delta e^{\mu}\chi)\sum_{i=0}^{\infty}(\delta e^{\mu}\chi)^{in} > 0$$

As $\text{coeff}(c) < 0$, the minimum in (7) is attained when $c = \bar{x} + x_n$. Inserting this value of c in (24) yields

$$\begin{aligned}\Phi(\bar{x}+x_n) &= -p_t(\bar{x}+x_n) - p_t\sum_{i=0}^{\infty}\left(\sum_{j=0}^{n-2}(\delta e^{\mu}\chi)^{in+j+1}x_{n-j-1} + (\delta e^{\mu}\chi)^{in+n}(\bar{x}+x_n)\right) \\ &= -p_t\sum_{i=0}^{\infty}\left((\delta e^{\mu}\chi)^{in}(\bar{x}+x_n) + \sum_{j=1}^{n-1}(\delta e^{\mu}\chi)^{in+j}x_{n-j}\right) = Q^{GP}(\mathbb{X}_t, p_t),\end{aligned}$$

showing that (7) holds and that the GP is optimal.

The proof for the finite horizon case is similar but more involved. Indeed, let t be expressed as $T-(kn+j)$, where $k = \lfloor\frac{T-t}{n}\rfloor$ and $j \in \{0,\ldots,n-1\}$ is the remainder of the integer division of $(T-t)$ by n. This way of expressing t puts in evidence that after completing k cycles there will be still j time steps to go until reaching the end of the horizon. Thus, the expected benefit associated to the GP is

$$\begin{aligned}Q^{GP}_t(\mathbb{X}_t, p_t) &= -p_t\Bigg[\sum_{i=0}^{k-1}(\delta e^{\mu}\chi)^{in}\left(\bar{x} + \sum_{l=0}^{n-1}(\delta e^{\mu}\chi)^{l}x_{n-l}\right) \\ &\quad + (\delta e^{\mu}\chi)^{kn}\left(\bar{x} + \sum_{l=0}^{j}(\delta e^{\mu}\chi)^{l}x_{n-l}\right)\Bigg].\end{aligned}$$

The first term of the rhs represents the expected benefit of the k completed cycles, while the second term corresponds to the last j steps. Again, we need to check that Q^{GP} satisfies (7) and that the minimum is attained for $c = CA\mathbb{X}_t$. We leave the details to the reader.

Appendix 3: Proof of Theorem 2

To prove that the accumulating policy is optimal, we check that the benefit associated with it (Q^{AP}) satisfies the dynamic programming equation (7). Let t be expressed as $T-(kn+j)$, where $k=\lfloor \frac{T-t}{n} \rfloor$ and $j \in \{0,\ldots,n-1\}$ is the remainder of the integer division of $(T-t)$ by n. After some computations we can prove that

$$Q_t^{AP}(\mathbb{X}_t, p_t) = -p_t\Big[(\delta e^{\mu}\chi)^k(\bar{x} + \sum_{l=0}^{j} x_{n-l}) + \sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in+j} S\Big] \tag{25}$$

where S represents the total surface of the forest and $\chi = \lambda + \kappa(1-\lambda)$. We point out that for $k=0$, we follow the convention $\sum_1^0(\cdot) = 0$.

For the rest of the proof we divide the study into two cases depending on the value of j: (i) $j>0$ and (ii) $j=0$.

(i) Here we have $t+1 = T-(k'n+j')$ where $k'=k$ and $j'=j-1 \in \{0,\ldots,n-2\}$ and $Q_{t+1}^{AP}(A\mathbb{X}_t + Bc, p_{t+1})$ can be expressed as

$$-p_{t+1}\Big[(\delta e^{\mu}\chi)^{j-1}(\bar{x}+x_n-c+\sum_{l=0}^{j-1} x_{n-l-1}) + \sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in+j-1} S\Big],$$

where $\chi = \lambda + (1-\lambda)\kappa$. Inserting $V = Q^{AP}$ into the right-hand side of the dynamic programming equation (7), the argument of the min operator, $\Phi(c)$, is

$$\begin{aligned}
&- p_t c + \delta\mathcal{R}_{|p_t}\big[- p_{t+1}\big[(\delta e^{\mu}\chi)^{j-1}(\bar{x} + x_n - c + \sum_{l=0}^{j-1} x_{n-l-1}) + \sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in+j-1} S\big]\big] \\
&= - p_t c - p_t\big[(\delta e^{\mu}\chi)^{j}(\bar{x} + x_n - c + \sum_{l=0}^{j-1} x_{n-l-1}) + \sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in+j} S\big]\big) \\
&= - p_t c(1-(\delta e^{\mu}\chi)^j) - p_t\big[(\delta e^{\mu}\chi)^{j}(\bar{x} + x_n + \sum_{l=1}^{j} x_{n-l}) + \sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in+j} S\big] \\
&= - p_t c(1-(\delta e^{\mu}\chi)^j) + Q_t^{AP}(\mathbb{X}_t, p_t).
\end{aligned}$$

As the coefficient of c is non-negative, the minimum is attained when $c=0$ and $\Phi(0)$ is exactly $Q_t^{AP}(\mathbb{X}_t, p_t)$, showing that equation (7) holds.

(ii) Case $t = T-kn$. In this case, we have $t+1 = T-[(k-1)n+n-1)]$ and $Q_{t+1}^{AP}(A\mathbb{X}_t + Bc, p_{t+1})$ can be expressed as

$$\begin{aligned}
&- p_{t+1}\Big[(\delta e^{\mu}\chi)^{n-1}(\bar{x} + x_n - c + \sum_{l=0}^{n-2} x_{n-l-1} + c) + \sum_{i=1}^{k-1}(\delta e^{\mu}\chi)^{in+n-1} S\Big] \\
&= - p_{t+1}\left[(\delta e^{\mu}\chi)^{n-1} S + \sum_{i=2}^{k}(\delta e^{\mu}\chi)^{in-1} S\right] = -p_{t+1}\sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in-1} S.
\end{aligned}$$

Inserting again $V = Q^{AP}$ into the right-hand side of the Bellman's equation (7), the argument of the min operator is

$$\Phi(c) = -p_t c + \delta\mathcal{R}_{|p_t}\left[-p_{t+1}\sum_{i=1}^{k}(\delta e^{\mu}\chi)^{in-1} S\right].$$

The coefficient of c is negative, and thus, the minimum is attained when $c = \bar{x} + x_n$. So we have,

$$\Phi(\bar{x} + x_n) = -p_t(\bar{x} + x_n) - p_t \sum_{i=1}^{k} (\delta e^{\mu} \chi)^{in} S.$$

The right-hand side is exactly (25) when $j = 0$, hence we have $\Phi(\bar{x} + x_n) = Q_t^{AP}(\cdot, \cdot)$ and equation (7) is satisfied.

In both cases, we have shown that $Q_t^{AP}(\cdot, \cdot)$ satisfies equation (7), hence it is the value function and the proposed policy is optimal.

Appendix 4: Proof of Lemma 2

Due to (19), we only need to show that

$$\frac{\delta b}{1 - \delta a} \geq \frac{b}{1 - a}\left[1 - \frac{1 - \delta^j}{a^m(1 - \delta^j a^j)}\right]. \tag{26}$$

Using that

$$\frac{\delta b}{1 - \delta a} = \frac{b}{1 - a}\left[1 - \frac{1 - \delta}{1 - \delta a}\right],$$

we have that (26) is equivalent to

$$\begin{aligned}
\frac{1}{1-a}\left[\frac{1-\delta}{1-\delta a}\right] \leq \frac{1}{1-a}&\left[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] \\
&\iff \frac{1}{1-a}\left[\frac{a^m(1-\delta^j a^j)}{1-\delta a}\right] \leq \frac{1}{1-a}\left[\frac{1-\delta^j}{1-\delta}\right] \\
&\iff \frac{1}{1-a}\left[a^m \sum_{l=0}^{j-1} (\delta a)^l\right] \leq \frac{1}{1-a}\left[\sum_{l=0}^{j-1} \delta^l\right].
\end{aligned}$$

Given that $a \in (0, 1)$, the last inequality is always valid.

Appendix 5: Proof of Lemma 3

Given initial state and price $\mathbb{X}_0$, p_0 we denote by $Q_{0,T}^{\Pi}(\mathbb{X}_0, p_0)$ the cost resulting of the application of a (not necessarily optimal) policy Π up to T. As in Lemma 1 we denote $Q_{0,T}^{\Pi} = Q_{0,T}^{\Pi}(\mathbb{X}_0, p_0)$. Knowing that $\mathcal{R}_{p_t}[-p_{t+1}] = -ap_t - b$, where a and b are defined in (17), we can write Q_T^{Π} as:

$$\begin{aligned}
Q_{0,T}^{\Pi} &= -p_0 c_0 + \delta \mathcal{R}_{|p_0}[-p_1 c_1 + \delta \mathcal{R}_{|p_1}[-p_2 c_2 + \cdots \\
&\quad + \delta \mathcal{R}_{|p_{T-3}}[-p_{T-2} c_{T-2} + \delta \mathcal{R}_{|p_{T-2}}[-p_{T-1} c_{T-1} + \delta \mathcal{R}_{|p_{T-1}}[-p_T\, c_T]]]]] \\
&= -p_0 c_0 + \delta \mathcal{R}_{|p_0}[-p_1 c_1 + \delta \mathcal{R}_{|p_1}[-p_2 c_2 + \cdots \\
&\quad + \delta \mathcal{R}_{|p_{T-3}}[-p_{T-2} c_{T-2} + \delta \mathcal{R}_{|p_{T-2}}[-p_{T-1}(c_{T-1} + \delta a c_T) - \delta b c_T]]]] \\
&= -p_0 c_0 + \delta \mathcal{R}_{|p_0}[-p_1 c_1 + \delta \mathcal{R}_{|p_1}[-p_2 c_2 \cdots \\
&\quad + \delta \mathcal{R}_{|p_{T-3}}[-p_{T-2}(c_{T-2} + \delta a c_{T-1} + (\delta a)^2 c_T) - b\delta c_{T-1} - b\delta^2(a+1) c_T]]]
\end{aligned}$$

$$\vdots$$
$$= -p_0 \sum_{t=0}^{T} (\delta a)^t c_t - b \sum_{t=1}^{T} \delta^t c_t \frac{1-a^t}{1-a}$$

We will show that $\lim_{T\to\infty} Q^{\Pi}_{0,T}$ exists and define the value associated with policy Π for the infinite time horizon as the value of that limit. To this end we compute

$$\begin{aligned} |Q^{\Pi}_{0,T+\tau} - Q^{\Pi}_{0,T}| &\le p_0 \sum_{t=T+1}^{T+\tau} c_t (\delta a)^t + |b| \sum_{t=T+1}^{T+\tau} \delta^t c_t \frac{1-a^t}{1-a} \\ &\le p_0 \sum_{t=T+1}^{\infty} c_t (\delta a)^t + |b| \sum_{t=T+1}^{\infty} \delta^t c_t \frac{1}{1-a} \\ &\le (\delta a)^{T+1} p_0 S \frac{1}{1-\delta a} + \delta^{T+1} S |b| \frac{1}{1-\delta} \frac{1}{1-a}. \end{aligned} \tag{27}$$

As the last expression converges to 0 when $T \to \infty$ for all $\tau \in \mathbb{N}$, we conclude that $\lim_{T\to\infty} Q^{\Pi}_{0,T}$ exists.

Appendix 6: Proof of Theorem 3

We state $\mathbb{X}_{t+1}$ and equation (7) in terms of $\mathbb{X}_t$ and c as follows.

$$\mathbb{X}_t = \begin{pmatrix} \bar{x}_t \\ x_{n,t} \\ x_{n-1,t} \\ \vdots \\ \vdots \\ x_{1,t} \end{pmatrix} \longrightarrow \mathbb{X}_{t+1} = A\mathbb{X}_t + Bc = \begin{pmatrix} \bar{x}_t + x_{n,t} - c \\ x_{n-1,t} \\ x_{n-2,t} \\ \vdots \\ \vdots \\ c \end{pmatrix},$$

$$V_t(\mathbb{X}_t, p_t) = \min_c \big\{ -p_t c + \delta \mathcal{R}_{|p_t}(V_{t+1}(\mathbb{X}_{t+1}, p_{t+1})) \big\}.$$

The main idea of the proof is to consider the role played by c in all the possible expressions of $V_{t+1}(\cdot,\cdot)$. This is not an easy task, but despite all the possible harvesting policies, the coefficient of c has a particular structure: it is the sum of terms of the form $\Delta^{m_i}_{j_i}(p_t)$ (as defined in (18)), for some values of $m_i \in \mathbb{N}$ and $j_i \in \{0,\dots,n-1\}$ plus possibly one negative term $\Gamma^m(p_t) = -\delta^m[p_t a^m + b\sum_{l=0}^{m-1} a^l]$.[6]

Indeed, from $t+1$ on, two different situations can arise: (i) nothing is harvested in the next n steps or (ii) the first harvest occurs at $t = j_0$ with $1 \le j_0 < n$.

In case (i), the state at $t+n$ will be

$$\mathbb{X}_{t+n} = (S - c, c, 0, \dots, 0)^T.$$

It is easy to see that the influence of c extinguishes as the constraint on the harvest is $c_{t+n} \le S$. We do not know the complete expression of $V_{t+1}(\cdot,\cdot)$ but we do know that the coefficient of c is simply $\Gamma^0(p_t) = -p_t$ with no $\Delta^m_j(p_t)$ terms.

[6] We assume that at every step from $t+1$ onwards, we either harvest nothing at all or everything available. Due to the linearity of the forestry model, this assumption is equivalent to requiring that the coefficient of c in (7) is never zero, but having a zero coefficient is an event with zero probability.

In case (ii), the first harvest after t takes place at $t + j_0$ and (7) can be written as:

$$V_t(\mathbb{X}_t, p_t) = \min_c \Big\{ - p_t c + \delta \mathcal{R}_{|p_{t+1}}[\delta \mathcal{R}_{|p_{t+2}}[\ldots \delta \mathcal{R}_{p_{t+j_0-1}}[-p_{t+j_0}(\bar{x}_t + \cdots + x_{n-j_0,t} - c) + \delta \mathcal{R}_{|p_{t+j_0}}[V_{t+j_0+1}(\mathbb{X}_{t+j_0+1}, p_{t+j_0+1})]]]]\Big\}.$$

Hence, the first term of the coefficient of c is of the form

$$\Delta^0_{j_0}(p_t) = -p_t(1 - \delta^{j_0} a^{j_0}) - \frac{b}{1-a}(-\delta^{j_0} + \delta^{j_0} a^{j_0}).$$

There might be more terms including c in the expression of $V_{t+j_0+1}(\cdot,\cdot)$. For a complete characterization of the coefficient of c we refer the reader to Piazza and Pagnoncelli (2014) where the analogous result in the risk neutral case is presented. The construction of the coefficient of c follows the same lines, the reader only needs to substitute the operator $\mathbb{E}_{|p_t}$ for $\mathcal{R}_{|p_t}$.

The number of terms comprising the coefficient of c, may or not be finite. In the infinite case, Lemma 3 implies that the sum converges.

The proof is completed by showing that the coefficient of c is negative. But, Lemma 2 shows that $\Delta^m_j \le 0$ when condition (16) holds, which finishes the proof.

Appendix 7: Proof of Lemma 4

For values of $a \in (0, 1)$ the proof presented for Lemma 3 is valid. For values of $a \in [1, 1/\delta)$ we need to modify the proof from (27) onwards.

We have that

$$|Q^{\Pi}_{T+\tau} - Q^{\Pi}_T| \le p_0 \sum_{t=T+1}^{T+\tau} c_t(\delta a)^t + |b| \sum_{t=T+1}^{T+\tau} \delta^t c_t \sum_{j=0}^{t-1} a^j.$$

Using that $1 \le a$ and $c_t \le S$ for all t we get

$$|Q^{\Pi}_{T+\tau} - Q^{\Pi}_T| \le (\delta a)^{T+1} p_0 S \frac{1}{1-\delta a} + S|b| \sum_{t=T+1}^{\infty} t(a\delta)^t.$$

As the sum $\sum_{t=1}^{\infty} t(a\delta)^t$ converges whenever $a\delta < 1$, its T-tail must go to zero when T goes to infinity.

Finally, we have that the right hand side of the inequality above converges to 0 when $T \to \infty$ and we conclude that $\lim_{T\to\infty} Q^{\Pi}_T$ exists.

Appendix 8: Proof of Theorem 4

This theorem is a generalization of Theorem 3 from $(a, b) \in (0, 1) \times \mathbb{R}_+$ to $(a, b) \in (0, 1/\delta) \times \mathbb{R}$. The proof of Theorem 3 consist in characterizing the coefficient of c in the Bellman equation (7). It is shown that this coefficient is the sum of infinite terms of the form $\Delta^m_j(p_t)$. This construction does not depend on the value of a and b but relies exclusively in the fact that the conditional risk measure is affine on p_t. Hence, this part of the proof extends directly to the more general setting of this theorem.

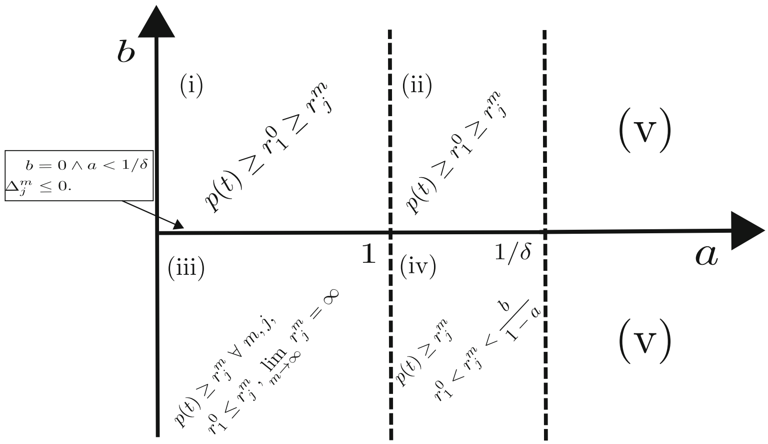

Fig. 3 Semi-plane of parameters a and b

Table 2 Parameters regions

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	(i)	(ii)	(v)
$b < 0$	(iii)	(iv)	(v)

The characterization of the coefficient' sign relies on Lemma 2 presented in Sect. 5 that gives a sufficient condition assuring that

$$\Delta_j^m(p_t) \leq 0 \quad \text{for all } \; m \leq T - t \quad \text{and for all } \; j \in \{1, \ldots, n\}. \tag{28}$$

Lemma 2 is valid for $(a, b) \in (0, 1) \times \mathbb{R}_+$. In the following we study the extension of this lemma to $(a, b) \in (0, 1/\delta) \times \mathbb{R}$. We start by noticing that (18) is not valid for $a = 1$. We will use the following representation of $\Delta_j^m(p_t)$,

$$\begin{cases} \delta^{m_i}\left\{-p_t a^{m_i}(1-\delta^{j_i}a^{j_i}) - \frac{b}{1-a}\left[1-\delta^{j_i} - a^{m_i}(1-\delta^{j_i}a^{j_i})\right]\right\} & \text{if } \; a \neq 1 \\ \delta^m\{-p_t(1-\delta^j) - b[m - \delta^j(m+j)]\} & \text{if } \; a = 1, \end{cases}$$

The parameters' semi-plane is divided in five regions as shown in Fig. 3 and Table 2.[7]

In the following, we look for conditions implying Condition (28), as this is sufficient to prove that the coefficient of c_t^* is negative and hence $c_t^* = CA\mathbb{X}_t$. We will see that the following conditions imply Condition (28):

1. In region (i), $p_t \geq b\delta/(1 - a\delta)$ (this is Lemma 2).
2. In region (ii), $p_t \geq b\delta/(1 - a\delta)$.[8]
3. In region (iii), no sufficient condition assuring Condition (28) is found in the infinite horizon case.
4. In region (iv), $p_t \geq b/(1 - a)$

[7] In the particular case that $1 - \delta a = 0$ we observe that $\Delta_j^m(p_t)$ does not depend of p_t and that Condition (28) can be verified a priori. The study of this particular case is straightforward and we omit it.

[8] Although, regions (i) and (ii) yield the same sufficient condition, we keep them as independent regions because the proof we present needs to separate the cases $a \leq 1$ and $a > 1$, and because this theorem applied in region (i) corresponds to Theorem 3 (see Remark 2).

Let us denote by $r_j^m(a,b)$ to the rhs of (19) when $a \neq 1$ and the corresponding expression for $a = 1$, i.e.,

$$r_j^m(a,b) = \begin{cases} \frac{b}{1-a}\left[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] & \text{if } a \neq 1 \\ \frac{-b}{1-\delta^j}[m - \delta^j(m+j)] & \text{if } a = 1 \end{cases}$$

In the following, we prove the properties summarized in Fig. 3. We observe in the first place that

$$\Delta_j^m(p_t) \leq 0 \iff p_t \geq r_j^m(a,b) \qquad \text{if } 1 - \delta a > 0 \tag{29}$$

We start by determining whether $r_1^0(a,b) = \frac{b\delta}{1-\delta a}$ bounds $r_j^m(a,b)$ (below or above) for all m and $j = 1, \ldots, n$. We study the case $a \neq 1$, leaving the easier particular case $a = 1$ to the reader. We also make the observation that if $b = 0$ then $r_j^m = 0$ for all m and j.

$$\begin{aligned} & r_1^0(a,b) \lesseqqgtr r_j^m(a,b) \\ & \iff \frac{b\delta}{1-\delta a} = \frac{b}{1-a}\left[1 - \frac{1-\delta}{(1-\delta a)}\right] \lesseqqgtr \frac{b}{1-a}\left[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] \\ & \iff \frac{-b}{1-a}\left[\frac{1-\delta}{(1-\delta a)}\right] \lesseqqgtr \frac{-b}{1-a}\left[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] \\ & \iff \frac{b}{a-1}\left[\frac{1-\delta}{1-\delta a}\right] \lesseqqgtr \frac{b}{a-1}\left[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] \end{aligned} \tag{30}$$

If $\operatorname{sign}(b)\operatorname{sign}(a-1) > 0$, i.e., in regions (ii) or (iii), (30) is equivalent to

$$\begin{aligned} & a^m \frac{1-\delta^j a^j}{1-\delta a} \lesseqqgtr \frac{1-\delta^j}{1-\delta} \\ & \iff a^m \sum_{l=0}^{j-1} (\delta a)^j \lesseqqgtr \sum_{l=0}^{j-1} \delta^j \end{aligned}$$

- If $a > 1$ (region (ii)), the inequality above holds with "$\geq$". Hence, $r_1^0(a,b) \geq r_j^m(a,b)$.
- If $a < 1$ (region (iii)), it holds with "$\leq$". Hence, $r_1^0(a,b) \leq r_j^m(a,b)$.

If $\operatorname{sign}(b)\operatorname{sign}(a-1) < 0$, i.e., in regions (i) or (iv), (30) is equivalent to

$$\begin{aligned} & \frac{1-\delta^j}{1-\delta} \lesseqqgtr a^m \frac{1-\delta^j a^j}{1-\delta a} \\ & \sum_{l=0}^{j-1} \delta^j \lesseqqgtr \sum_{l=0}^{j-1} \delta^j a^m \sum_{l=0}^{j-1} (\delta a)^j \end{aligned}$$

- If $a > 1$ (region (iv)), the inequality above holds with "$\leq$". Hence, $r_1^0(a,b) \leq r_j^m(a,b)$.
- If $a < 1$ (region (i)), it holds with "$\geq$". Hence, $r_1^0(a,b) \geq r_j^m(a,b)$.

Table 3 summarizes these results.

Putting this information together with that of (29), we conclude that to assure Condition (28) it is sufficient to have the conditions indicated in Table 4.

In regions (i) and (ii) we are ready to give a sufficient condition assuring Condition (28):

$$p_t \geq r_1^0(a,b) = \frac{\delta b}{1-\delta a}.$$

Table 3 Bounds for $r_j^m(a, b)$

	$0 < a \leq 1$	$1 < a < 1/\delta$
$b \geq 0$	$r_1^0 \geq r_j^m$	$r_1^0 \geq r_j^m$
$b < 0$	$r_1^0 \leq r_j^m$	$r_1^0 \leq r_j^m$

Table 4 Bounds for p_t

	$0 < a \leq 1$	$1 < a < 1/\delta$
$b \geq 0$	$p_t \geq r_1^0$	$p_t \geq r_1^0$
$b < 0$	$p_t \geq r_j^m \forall m, j$	$p_t \geq r_j^m \forall m, j$

To reach some conclusion in regions (iii) and (iv) we need some extra information of $r_j^m(a, b)$.

In region (iv) it is very easy to check that $r_j^m(a, b) \leq b/(1-a)$. Hence $p_t \geq b/(1-a)$ is sufficient to assure Condition (28).

In region (iii) is a bit different. We have that $\lim_{m\to\infty} r_j^m(a, b) = +\infty$, hence p_t cannot be greater than r_j^m for all m. Hence, no conclusion can be drawn in the infinite horizon case. However, in the finite horizon case Condition (28) only requires having $p_t \geq r_j^m$ for $m \leq T-t$. Furthermore, some calculation shows that $r_j^{m+1} > r_j^m$ and $r_j^m > r_{j+1}^m$. Hence, we can propose a condition depending on the value of $T-t$: $p_t \geq r_1^{T-t} = \frac{b}{1-a}\left[1 - \frac{1-\delta}{a^{T-t}(1-\delta a)}\right]$.

Appendix 9: Mean deviation risk calculation for O–U

For the particular case of $p = 2$ and a O–U process, the conditional MDR is given by:

$$\begin{aligned} \mathrm{MDR}_{|p_t}(-p(t+1)) &= -p_t e^{-\eta} - \bar{p}(1 - e^{-\eta}) \\ &\quad + c\left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^2\right]\right)^{1/2}. \end{aligned}$$

In order to calculate the stochastic integral we apply Itô's isometry:

$$\mathbb{E}\left(\left|\int_0^T G(t, W_t) dW_t\right|^2\right) = \mathbb{E}\left(\int_0^T |G(t, W_t)|^2 \, dt\right), \qquad (31)$$

for a stochastic process $G(t, W_t) \in \mathbb{L}^2(0, T)$. Using (31) and noting that in our case the process G is deterministic, we have

$$\begin{aligned} \left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^2\right]\right)^{1/2} &= \left(\mathbb{E}\left[\int_t^{t+1} (\sigma e^{\eta(s-(t+1))})^2 ds\right]\right)^{1/2} \\ &= \sigma\left(\mathbb{E}\left[\int_t^{t+1} e^{2\eta(s-(t+1))} ds\right]\right)^{1/2} = \sigma\left(\frac{1}{2\eta} - \frac{e^{-2\eta}}{2\eta}\right)^{1/2}. \end{aligned}$$

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