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A decomposition approach to vector equilibrium problems

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Abstract A vector equilibrium problem is generally defined by a bifunction which takes values in a partially ordered vector space. When this space is endowed with a componentwise ordering, the vector equilibrium problem can be decomposed into a family of equilibrium subproblems, each of them being governed by a bifunction obtained from the initial one by selecting some of its scalar components. Similarly to multi-criteria optimization, three types of solutions can be defined for these equilibrium subproblems, namely, weak, strong and proper solutions. The aim of this paper is to show that, under appropriate convexity assumptions, the set of all weak solutions of a vector equilibrium problem can be recovered as the union of the sets of proper solutions of its subproblems.

Keywords Vector equilibrium problem · Multi-criteria optimization problem · Generalized convexity · Scalarization · Decomposition

1 Introduction

Equilibrium problems have been intensively studied in the last two decades due to their wide range of applications. Actually, optimization problems, variational inequalities, saddle point (minimax) problems, Nash equilibria, complementarity problems, and other important problems, can be seen as particular instances of the general equilibrium problem (Blum and Oettli 1994; Iusem and Sosa 2003).

The classical equilibrium problems, as introduced by Muu and Oettli (1992), are governed by real-valued bifunctions, hence they are called scalar equilibrium problems. The vector equilibrium problems are governed by bifunctions which take values in a partially ordered real vector space. They have been investigated by many authors, beginning with Ansari

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(2000)—whose first draft was indeed presented in 1996 at the Second World Congress of Nonlinear Analysts, Athens, Greece, Bianchi et al. (1997) and Oettli (1997).

In this paper we will consider a particular class of vector equilibrium problems which are governed by bifunctions whose outcome space is a finite-dimensional real Euclidean space, endowed with the usual componentwise ordering. This particular setting allows us to decompose any vector equilibrium problem into a family of equilibrium subproblems, each of them being governed by a bifunction obtained from the initial one by selecting some of its scalar components. We investigate the relationship between three types of solutions of these equilibrium subproblems, namely, weak, strong and proper solutions. Our main result shows that, under suitable convexity assumptions, any weak solution of a vector equilibrium problem is a proper (hence strong) solution for at least one of its subproblems.

The paper is organized as follows. In Sect. 2 we introduce some basic notions of multi-criteria optimization and we establish new characterizations of weakly and proper minimal points. Vector equilibrium problems are introduced in Sect. 3. Under appropriate generalized convexity assumptions, we characterize their weakly and proper solutions by means of linear scalarizations. Section 4 is devoted to our main result concerning the decomposition of vector equilibrium problems and its applications in multi-criteria optimization and vector variational inequalities.

2 Multi-criteria optimization

In general, the outcome space of a multi-criteria optimization problem is a real Euclidean space of given dimension. However, by decomposing such a problem into subproblems we should operate with several outcome spaces of different dimensions simultaneously. Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}_+$ be the set of nonnegative real numbers. For every $n \in \mathbb{N}$, we endow the n-dimensional real Euclidean space with three binary relations, defined as follows. For any $u, v \in \mathbb{R}^n$,

$$\begin{aligned} u \leq v &:\Longleftrightarrow v - u \in \mathbb{R}^n_+, \\ u < v &:\Longleftrightarrow v - u \in \operatorname{int} \mathbb{R}^n_+, \\ u \lneq v &:\Longleftrightarrow u \leq v \text{ and } u \neq v, \end{aligned}$$

where $\operatorname{int} S$ stands for the interior of a set $S \subseteq \mathbb{R}^n$. Throughout the paper we will denote by $\operatorname{cl} S$ the closure of S and we adopt the following notational conventions.

For any sets $S \subset \mathbb{R}^n$, $S' \subset \mathbb{R}^n$ and $A \subset \mathbb{R}$, and for any $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$, we denote $S \pm S' = \{u \in \mathbb{R}^n \mid \exists\,(x, x') \in S \times S' \text{ s.t. } u = x \pm x'\}$, $S \pm v = S \pm \{v\}$, $v \pm S' = \{v\} \pm S'$, $A \cdot S = \{u \in \mathbb{R}^n \mid \exists(\alpha, x) \in A \times S \text{ s.t. } u = \alpha x\}$, $A \cdot v = A \cdot \{v\}$ and $\lambda S = \{\lambda\} \cdot S$.

Let S be a nonempty subset of $\mathbb{R}^n$ with $n \geq 2$. As usual in multi-criteria optimization (see, e.g., Göpfert et al. 2003) a point $v = (v_1, \dots, v_n) \in S$ is called:

- *weakly minimal* if there is no $u \in S$ such that $u < v$.
- *strongly minimal* if there is no $u \in S$ such that $u \lneq v$.
- *properly minimal* (in the sense of Geoffrion 1968) if there exists a real number $\mu > 0$ such that for any $u = (u_1, \dots, u_n) \in S$ and $i \in \{1, \dots, n\}$ with $u_i < v_i$ there is $j \in \{1, \dots, n\} \setminus \{i\}$ with $\mu(u_j - v_j) \geq v_i - u_i$.

Denote by $\text{w-min}\, S$, $\text{s-min}\, S$ and $\text{p-min}\, S$ the sets of all weakly minimal, strongly minimal, and properly minimal points of S, respectively. It is easily seen that

$$\text{w-min}\, S \supseteq \text{s-min}\, S \supseteq \text{p-min}\, S. \tag{1}$$

These sets can be characterized geometrically. To this aim we recall some notions of convex analysis. A subset C of $\mathbb{R}^n$ is said to be a:

- *convex* set if $(1-t)C + tC = C$ for any $t \in [0, 1]$;
- *cone* if $\mathbb{R}_+ \cdot C = C$ and $0_n \in C$, where 0_n denotes the zero vector of $\mathbb{R}^n$;
- *convex cone* if C is a cone and a convex set, i.e., $0_n \in C = \mathbb{R}_+ \cdot C = C + C$;
- *upward* set if $C + \mathbb{R}^n_+ = C$, i.e., C has the free-disposal property (Debreu 1959).
- *closely* $\mathbb{R}^n_+$*-convex* set if $\mathrm{cl}(C + \mathbb{R}^n_+)$ is convex (Breckner and Kassay 1997).

Obviously $\mathbb{R}^n_+$ is a convex cone and we have

$$\text{w-min}\, S = \left\{v \in S \mid (S - v) \cap (-\mathrm{int}\,\mathbb{R}^n_+) = \emptyset\right\}; \tag{2}$$

$$\text{s-min}\, S = \left\{v \in S \mid (S - v) \cap (-\mathbb{R}^n_+) = \{0_n\}\right\}. \tag{3}$$

According to Theorem 2.1.15 of Podinovskiĭ and Nogin (1982), there exists a family $(\Lambda_\varepsilon)_{\varepsilon \in]0,1/n]}$ of convex cones of $\mathbb{R}^n$, satisfying the following four properties:

$$\bigcap_{\varepsilon \in]0,1/n]} \Lambda_\varepsilon = \mathbb{R}^n_+; \tag{4}$$

$$0_n \notin \mathrm{int}\,\Lambda_\varepsilon, \quad \text{for any } 0 < \varepsilon \leq 1/n; \tag{5}$$

$$\Lambda_\varepsilon \setminus \{0_n\} \subseteq \mathrm{int}\,\Lambda_{\tilde\varepsilon}, \quad \text{for any } 0 < \varepsilon < \tilde\varepsilon \leq 1/n; \tag{6}$$

$$\text{p-min}\, S = \bigcup_{0<\varepsilon\leq 1/n} \{v \in S \mid (S - v) \cap (-\Lambda_\varepsilon) = \{0_n\}\}. \tag{7}$$

Remark 1 Property (5) means actually that $\Lambda_\varepsilon \neq \mathbb{R}^n$ for any $0 < \varepsilon \leq 1/n$, while the properties (4) and (6) show that

$$\mathbb{R}^n_+ \setminus \{0_n\} \subseteq \mathrm{int}\,\Lambda_{\tilde\varepsilon}, \quad \text{for any } 0 < \tilde\varepsilon \leq 1/n. \tag{8}$$

Note that Henig (1982) also introduced a notion of proper efficiency with respect to a general cone C, by means of certain enlarged cones with similar properties. Actually, for the particular cone $C = \mathbb{R}^n_+$, several other concepts of proper efficiency coincide with Geoffrion's one (see, e.g., Guerraggio et al. 1994).

The study of the minimal points of any set S can be reduced to the study of the corresponding minimal points of an upward set, namely $S + \mathbb{R}^n_+$, by means of the following formulae (see, e.g., Lemma 2.2.1 of Podinovskiĭ and Nogin 1982):

$$\text{w-min}\, S = S \cap \text{w-min}(S + \mathbb{R}^n_+);$$
$$\text{s-min}\, S = \text{s-min}(S + \mathbb{R}^n_+);$$
$$\text{p-min}\, S = \text{p-min}(S + \mathbb{R}^n_+).$$

Our next result shows that similar representations hold for weakly minimal and properly minimal points if we replace the upward set $S + \mathbb{R}^n_+$ by its closure.

Theorem 1 *For every subset S of $\mathbb{R}^n$ we have*

$$\text{w-min}\, S = S \cap \text{w-min}\,\mathrm{cl}(S + \mathbb{R}^n_+); \tag{9}$$

$$\text{p-min}\, S = S \cap \text{p-min}\,\mathrm{cl}(S + \mathbb{R}^n_+). \tag{10}$$

Proof According to Proposition 2.2.6 of Luc (1989), for any $B \subseteq A \subseteq \mathbb{R}^n$ we have

$$B \cap \text{w-min}\, A \subseteq \text{w-min}\, B \quad \text{and} \quad B \cap \text{p-min}\, A \subseteq \text{p-min}\, B.$$

Letting $A := \mathrm{cl}(S + \mathbb{R}^n_+)$ and $B := S$ we get the inclusions "$\supseteq$" in (9) and (10).

Now suppose to the contrary that inclusion "$\subseteq$" in (9) is not true. Then, since w-min $S \subseteq S$, it would exists $v \in (\text{w-min } S) \setminus \text{w-min}\, \mathrm{cl}(S + \mathbb{R}^n_+)$. Taking into account that $v \in \text{w-min } S \subseteq S \subseteq \mathrm{cl}(S + \mathbb{R}^n_+)$, we infer the existence of a point $w \in \mathrm{cl}(S + \mathbb{R}^n_+)$ such that $w < v$, i.e., $w \in v - \mathrm{int}\, \mathbb{R}^n_+$. Since $v - \mathrm{int}\, \mathbb{R}^n_+$ is open, it is a neighborhood of w, hence its intersection with $S + \mathbb{R}^n_+$ is nonempty. Choose $s \in S$ and $u \in \mathbb{R}^n_+$ such that $s + u \in v - \mathrm{int}\, \mathbb{R}^n_+$. Then $s \in v - u - \mathrm{int}\, \mathbb{R}^n_+ \subseteq v - \mathbb{R}^n_+ - \mathrm{int}\, \mathbb{R}^n_+ = v - \mathrm{int}\, \mathbb{R}^n_+$, which yields $s < v$, contradicting the assumption that $v \in$ w-min S. Thus (9) holds.

In order to prove the inclusion "$\subseteq$" in (10), consider an arbitrary $v^0 \in$ p-min S. Taking into account that $v^0 \in S$, we just have to show that $v^0 \in \text{p-min}\, \mathrm{cl}(S + \mathbb{R}^n_+)$. Since $v^0 \in$ p-min S, we infer by (7) the existence of some $\tilde{\varepsilon} \in]0, 1/n]$ such that $(S - v^0) \cap (-\Lambda_{\tilde{\varepsilon}}) = \{0_n\}$. By (5) it follows that $(S - v^0) \cap (-\mathrm{int}\, \Lambda_{\tilde{\varepsilon}}) = \emptyset$, hence

$$S \cap (v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}}) = \emptyset. \tag{11}$$

Choose a number $\varepsilon \in]0, \tilde{\varepsilon}[$. Supposing to the contrary that $v^0 \notin \text{p-min}\, \mathrm{cl}(S + \mathbb{R}^n_+)$, we infer by (7), applied to $\mathrm{cl}(S + \mathbb{R}^n_+)$ in the role of S, that

$$v^0 \notin \left\{ v \in \mathrm{cl}(S + \mathbb{R}^n_+) \mid \left(\mathrm{cl}(S + \mathbb{R}^n_+) - v\right) \cap (-\Lambda_\varepsilon) = \{0_n\} \right\}.$$

Since $v^0 \in S \subseteq \mathrm{cl}(S + \mathbb{R}^n_+)$, it follows that $\left(\mathrm{cl}(S + \mathbb{R}^n_+) - v^0\right) \cap (-\Lambda_\varepsilon) \neq \{0_n\}$, hence $\left(\mathrm{cl}(S + \mathbb{R}^n_+) - v^0\right) \cap (-\Lambda_\varepsilon \setminus \{0_n\}) \neq \emptyset$. By the choice of ε, we can deduce by (6) that $\left(\mathrm{cl}(S + \mathbb{R}^n_+) - v^0\right) \cap (-\mathrm{int}\, \Lambda_{\tilde{\varepsilon}}) \neq \emptyset$, i.e., $\mathrm{cl}(S + \mathbb{R}^n_+) \cap \left(v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}}\right) \neq \emptyset$. Thus we can choose a point $w^0 \in \mathrm{cl}(S + \mathbb{R}^n_+) \cap (v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}})$. Since $v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}}$ is a neighborhood of w^0, we can find a point

$$\tilde{w} := \tilde{v} + \tilde{u} \in (S + \mathbb{R}^n_+) \cap (v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}})$$

with $\tilde{v} \in S$ and $\tilde{u} \in \mathbb{R}^n_+$. By (11) we infer that $\tilde{u} \neq 0_n$. In view of (8), it follows that $v^0 - \tilde{v} \in \tilde{u} + \mathrm{int}\, \Lambda_{\tilde{\varepsilon}} \subseteq \mathbb{R}^n_+ \setminus \{0_n\} + \mathrm{int}\, \Lambda_{\tilde{\varepsilon}} \subseteq \mathrm{int}\, \Lambda_{\tilde{\varepsilon}} + \mathrm{int}\, \Lambda_{\tilde{\varepsilon}} = \mathrm{int}\, \Lambda_{\tilde{\varepsilon}}$, the last equality being true since $\Lambda_{\tilde{\varepsilon}}$ is a convex cone. It follows that $\tilde{v} \in v^0 - \mathrm{int}\, \Lambda_{\tilde{\varepsilon}}$, which contradicts (11) because $\tilde{v} \in S$. Thus (10) holds. □

Remark 2 Proposition 2.2.6 of Luc (1989), which has been used by us in the proof of Theorem 1, shows also that for any sets $B \subseteq A \subseteq \mathbb{R}^n$ we have

$$B \cap \text{s-min } A \subseteq \text{s-min } B.$$

In particular, by letting $A := \mathrm{cl}(S + \mathbb{R}^n_+)$ and $B := S$ for any set $S \subseteq \mathbb{R}^n$, we get

$$\text{s-min } S \supseteq S \cap \text{s-min}\, \mathrm{cl}(S + \mathbb{R}^n_+). \tag{12}$$

However, the inverse inclusion does not hold in general. For instance, consider

$$S := \{(1, 0)\} \cup]0, 1[\times]0, 1[\subseteq \mathbb{R}^2.$$

It is easily seen that s-min $S = \{(1, 0)\}$ while $\text{s-min}\, \mathrm{cl}(S + \mathbb{R}^2_+) = \{(0, 0)\}$, hence the equality in (12) does not hold.

We end this section by presenting some characterizations of weak minimality and proper minimality by means of linear scalarization. The following lemma is a counterpart of some classical results by Hurwicz (1958) and Geoffrion (1968).

In the sequel $\langle \cdot, \cdot \rangle$ will represent the usual inner product in $\mathbb{R}^n$.

Lemma 1 *For any nonempty set $S \subseteq \mathbb{R}^n$ the following assertions hold:*

1° $\operatorname{argmin}_{v\in S}\langle c, v\rangle \subseteq$ w-min S *for every* $c \in \mathbb{R}^n$ *with* $c \gneq 0_n$.
2° $\operatorname{argmin}_{v\in S}\langle c, v\rangle \subseteq$ p-min S *for every* $c \in \mathbb{R}^n$ *with* $c > 0_n$.
3° *If S is convex, then*

$$\text{w-min}\, S = \bigcup_{c\gneq 0_n} \operatorname*{argmin}_{v\in S}\langle c, v\rangle; \tag{13}$$

$$\text{p-min}\, S = \bigcup_{c> 0_n} \operatorname*{argmin}_{v\in S}\langle c, v\rangle. \tag{14}$$

Corollary 1 *If $S \subseteq \mathbb{R}^n$ is nonempty and closely $\mathbb{R}^n_+$-convex, then* (13) *and* (14) *hold.*

Proof Let us show first that for every $c \in \mathbb{R}^n$ with $c \geq 0_n$ we have

$$\operatorname*{argmin}_{v\in S}\langle c, v\rangle = S \cap \operatorname*{argmin}_{v'\in \mathrm{cl}(S+\mathbb{R}^n_+)}\langle c, v'\rangle. \tag{15}$$

Indeed, the inclusion "$\supseteq$" in (15) is true since $S \subseteq \mathrm{cl}(S+\mathbb{R}^n_+)$. In order to prove the inclusion "$\subseteq$", consider any $\bar{v} \in \operatorname{argmin}_{v\in S}\langle c, v\rangle$. Then $\bar{v} \in S \subseteq \mathrm{cl}(S+\mathbb{R}^n_+)$ and

$$\langle c, \bar{v}\rangle \leq \langle c, v\rangle, \ \forall\, v \in S. \tag{16}$$

We just have to prove that $\langle c, \bar{v}\rangle \leq \langle c, v'\rangle$ for an arbitrary point $v' \in \mathrm{cl}(S+\mathbb{R}^n_+)$. There exist two sequences of points, $(v^k)_{k\in\mathbb{N}}$ in S and $(u^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n_+$, such that $v' = \lim_{k\to\infty}(v^k + u^k)$. For each $k \in \mathbb{N}$, by applying (16) for $v := v^k$ and recalling that $c \in \mathbb{R}^n_+$, we infer that $\langle c, \bar{v}\rangle \leq \langle c, v^k\rangle \leq \langle c, v^k + u^k\rangle$. Passing to the limit, we deduce the desired relation: $\langle c, \bar{v}\rangle \leq \langle c, v'\rangle$. Thus (15) holds.

Since S is closely $\mathbb{R}^n_+$-convex, the set $\mathrm{cl}(S+\mathbb{R}^n_+)$ is convex. By Lemma 1 (3°), applied for $\mathrm{cl}(S+\mathbb{R}^n_+)$ in the role of S, we deduce that

$$\text{w-min}\,\mathrm{cl}(S+\mathbb{R}^n_+) = \bigcup_{c\gneq 0_n} \operatorname*{argmin}_{v'\in \mathrm{cl}(S+\mathbb{R}^n_+)}\langle c, v'\rangle;$$

$$\text{p-min}\,\mathrm{cl}(S+\mathbb{R}^n_+) = \bigcup_{c> 0_n} \operatorname*{argmin}_{v'\in \mathrm{cl}(S+\mathbb{R}^n_+)}\langle c, v'\rangle.$$

By combining these two relations with Theorem 1 and relation (15), we infer

$$\text{w-min}\, S = S \cap \text{w-min}\,\mathrm{cl}(S+\mathbb{R}^n_+) = \bigcup_{c\gneq 0_n}\Big(S \cap \operatorname*{argmin}_{v'\in \mathrm{cl}(S+\mathbb{R}^n_+)}\langle c, v'\rangle\Big) = \bigcup_{c\gneq 0_n} \operatorname*{argmin}_{v\in S}\langle c, v\rangle;$$

$$\text{p-min}\, S = S \cap \text{p-min}\,\mathrm{cl}(S+\mathbb{R}^n_+) = \bigcup_{c> 0_n}\Big(S \cap \operatorname*{argmin}_{v'\in \mathrm{cl}(S+\mathbb{R}^n_+)}\langle c, v'\rangle\Big) = \bigcup_{c> 0_n} \operatorname*{argmin}_{v\in S}\langle c, v\rangle.$$

Thus both relations (13) and (14) hold true. □

Remark 3 Under the hypothesis of Corollary 1, relation (13) can be also recovered as a particular instance of Theorem 3.2 in the paper by Breckner and Kassay (1997).

3 Vector equilibrium problems and their scalarization

Throughout this section D will be a nonempty subset of a real linear space X.

Given a real-valued bifunction $\varphi : D \times D \to \mathbb{R}$, satisfying the property that

$$\varphi(x, x) = 0 \ \text{ for all } \ x \in D, \tag{17}$$

the *scalar equilibrium problem* governed by φ (as defined by Muu and Oettli 1992), consists in finding the elements $x \in D$ satisfying

$$\varphi(x, y) \geq 0 \ \text{ for all } \ y \in D.$$

Due to (17) the set of all solutions of the scalar equilibrium problem is given by

$$\begin{aligned} \text{eq}(D \mid \varphi) &:= \{x \in D \mid \varphi(x, y) \geq 0, \ \forall\, y \in D\} \\ &= \{x \in D \mid \nexists\, y \in D \text{ s.t. } \varphi(x, y) < 0\} \\ &= \{x \in D \mid \varphi(x, D) \cap (-\text{int}\,\mathbb{R}_+) = \emptyset\} \\ &= \{x \in D \mid 0 = \min \varphi(x, D)\}\,, \end{aligned} \tag{18}$$

where $\varphi(x, D) := \{\varphi(x, y) \mid y \in D\}$ and $\min \varphi(x, D) := \min\limits_{y \in D} \varphi(x, y)$ for any $x \in D$.

Consider now a vector-valued bifunction, $f = (f_1, \ldots, f_n) : D \times D \to \mathbb{R}^n$ $(n \geq 2)$, which satisfies the property that

$$f(x, x) = 0_n \ \text{ for all } \ x \in D. \tag{19}$$

By adapting the representation (18) to the minimality concepts introduced in Sect. 2 (i.e., letting $v := 0_n$ and $S := f(x, D)$ in (2), (3) and (7) for any $x \in D$), we can introduce the *vector equilibrium problem* governed by f, which consists in finding the elements of the following three sets:

$$\begin{aligned} \text{w-eq}(D \mid f) &:= \{x \in D \mid 0_n \in \text{w-min} f(x, D)\} \\ &= \{x \in D \mid \nexists\, y \in D \text{ s.t. } f(x, y) < 0_n\} \\ &= \left\{x \in D \mid f(x, D) \cap (-\text{int}\,\mathbb{R}^n_+) = \emptyset\right\}; \end{aligned} \tag{20}$$

$$\begin{aligned} \text{s-eq}(D \mid f) &:= \{x \in D \mid 0_n \in \text{s-min} f(x, D)\} \\ &= \{x \in D \mid \nexists\, y \in D \text{ s.t. } f(x, y) \lneqq 0_n\} \\ &= \left\{x \in D \mid f(x, D) \cap (-\mathbb{R}^n_+) = \{0_n\}\right\}; \end{aligned} \tag{21}$$

$$\begin{aligned} \text{p-eq}(D \mid f) &:= \{x \in D \mid 0_n \in \text{p-min} f(x, D)\} \\ &= \{x \in D \mid f(x, D) \cap (-\Lambda_\varepsilon) = \{0_n\} \text{ for some } \varepsilon \in]0, 1/n]\}\,. \end{aligned} \tag{22}$$

The elements of these three sets will be called *weak solutions*, *strong solutions*, and *proper solutions* of the vector equilibrium problem governed by f. Notice that, in more general ordered spaces, Ansari (2000) and Bianchi et al. (1997) initiated the study of vector equilibrium problems by considering weak solutions. Later on, strong and proper solutions of vector equilibrium problems were investigated in many papers (see, e.g., Capătă 2011 or Bigi et al. 2012 and the references therein).

Lemma 2 *The following inclusions hold:*

$$\text{p-eq}(D \mid f) \subseteq \text{s-eq}(D \mid f) \subseteq \text{w-eq}(D \mid f).$$

Proof For every $x \in D$ we have $\text{p-min} f(x, D) \subseteq \text{s-min} f(x, D) \subseteq \text{w-min} f(x, D)$ in view of (1) applied for $S := f(x, D)$. Thus, whenever $x \in \text{p-eq}(D \mid f)$ it follows by (22) that $0_n \in \text{p-min} f(x, D) \subseteq \text{s-min} f(x, D)$, hence $x \in \text{s-eq}(D \mid f)$. Similarly, if $x \in \text{s-eq}(D \mid f)$, then according to (21) we have $0_n \in \text{s-min} f(x, D) \subseteq \text{w-min} f(x, D)$, hence $x \in \text{w-eq}(D \mid f)$. □

Theorem 2 *For any $c \in \mathbb{R}^n$ let $\langle c, f\rangle : D \times D \to \mathbb{R}$ be the real-valued bifunction defined for all $(x, y) \in D \times D$ by*

$$\langle c, f\rangle(x, y) := \langle c, f(x, y)\rangle.$$

The following assertions hold:

1° $\mathrm{eq}(D \mid \langle c, f\rangle) \subseteq \text{w-eq}(D \mid f)$ *whenever* $c \gneq 0_n$.
2° $\mathrm{eq}(D \mid \langle c, f\rangle) \subseteq \text{p-eq}(D \mid f)$ *whenever* $c > 0_n$.
3° *If the set $f(x, D)$ is closely $\mathbb{R}^n_+$-convex for every $x \in D$, then*

$$\text{w-eq}(D \mid f) = \bigcup_{c \gneq 0_n} \mathrm{eq}(D \mid \langle c, f\rangle); \tag{23}$$

$$\text{p-eq}(D \mid f) = \bigcup_{c > 0_n} \mathrm{eq}(D \mid \langle c, f\rangle). \tag{24}$$

Proof Consider an arbitrary $c \in \mathbb{R}^n$. By applying (18) for $\varphi := \langle c, f\rangle$ and taking into account that $0_n \in f(x, D)$ for any $x \in D$ due to (19), we deduce that

$$\begin{aligned} \mathrm{eq}(D \mid \langle c, f\rangle) &= \left\{x \in D \;\middle|\; 0 = \min_{y\in D}\langle c, f(x, y)\rangle\right\} \\ &= \left\{x \in D \;\middle|\; \langle c, 0_n\rangle = \min_{y\in D}\langle c, f(x, y)\rangle\right\} \\ &= \left\{x \in D \;\middle|\; 0_n \in \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle\right\}. \end{aligned} \tag{25}$$

In order to prove 1° and 2°, consider a point $x \in \mathrm{eq}(D \mid \langle c, f\rangle)$. By (25) and Lemma 1 (1° and 2°) applied for $S := f(x, D)$ it follows that, whenever $c \gneq 0_n$, we have $0_n \in \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle \subseteq \text{w-min}\, f(x, D)$, hence $x \in \text{w-eq}(D \mid f)$ by (20). Similarly, if $c > 0_n$, then we have $0_n \in \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle \subseteq \text{p-min}\, f(x, D)$, hence $x \in \text{p-eq}(D \mid f)$ in view of (22).

Now let us prove 3°. Assume that $f(x, D)$ is closely $\mathbb{R}^n_+$-convex for every $x \in D$. By (20), (22), (25) and Corollary 1, applied for $S := f(x, D)$ with $x \in D$, we infer

$$\begin{aligned} \text{w-eq}(D \mid f) &= \{x \in D \mid 0_n \in \text{w-min}\, f(x, D)\} \\ &= \left\{x \in D \;\middle|\; 0_n \in \textstyle\bigcup_{c\gneq 0_n} \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle\right\} \\ &= \textstyle\bigcup_{c\gneq 0_n} \left\{x \in D \;\middle|\; 0_n \in \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle\right\} \\ &= \textstyle\bigcup_{c\gneq 0_n} \mathrm{eq}(D \mid \langle c, f\rangle); \\ \text{p-eq}(D \mid f) &= \{x \in D \mid 0_n \in \text{p-min}\, f(x, D)\} \\ &= \left\{x \in D \;\middle|\; 0_n \in \textstyle\bigcup_{c>0_n} \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle\right\} \\ &= \textstyle\bigcup_{c>0_n} \left\{x \in D \;\middle|\; 0_n \in \operatorname{argmin}_{v\in f(x,D)}\langle c, v\rangle\right\} \\ &= \textstyle\bigcup_{c>0_n} \mathrm{eq}(D \mid \langle c, f\rangle). \end{aligned}$$

Thus both relations (23) and (24) hold. □

Recall (see, e.g., Jeyakumar 1985 and Breckner and Kassay 1997) that a vector-valued function $g : D \to \mathbb{R}^n$ is said to be :

– $\mathbb{R}^n_+$-*subconvexlike* if there is $e \in \operatorname{int} \mathbb{R}^n_+$ such that

$$(1 - t)g(D) + tg(D) +]0, \infty[\cdot e \subseteq g(D) + \mathbb{R}^n_+, \;\; \forall\, t \in]0, 1[;$$

– $\mathbb{R}^n_+$-*convex* if its epigraph, defined by means of the ordering $\leq$ of $\mathbb{R}^n$ as

$$\text{epi}(g) := \{(x, v) \in D \times \mathbb{R}^n \mid g(x) \leq v\},$$

is a convex subset of the linear space $X \times \mathbb{R}^n$.

The next lemma gathers known results (see, e.g., Breckner and Kassay 1997).

Lemma 3 *For any function* $g = (g_1, \ldots, g_n) : D \to \mathbb{R}^n$ *the following assertions hold:*

1° *g is $\mathbb{R}^n_+$-convex if and only if D is convex and the scalar components of g are convex in classical sense, i.e., for any $i \in \{1, \ldots, n\}$ the function $g_i : D \to \mathbb{R}$ satisfies*

$$g_i((1-t)x' + tx'') \leq (1-t)g_i(x) + tg_i(x), \ \forall x', x'' \in D, \ \forall t \in [0, 1].$$

2° *g is $\mathbb{R}^n_+$-subconvexlike if and only if its range, $g(D)$, is a closely $\mathbb{R}^n_+$-convex set, i.e., the set* $\text{cl}(g(D) + \mathbb{R}^n_+)$ *is convex.*

3° *If g is $\mathbb{R}^n_+$-convex, then $g(D) + \mathbb{R}^n_+$ is a convex set, hence g is $\mathbb{R}^n_+$-subconvexlike.*

Remark 4 In view of Lemma 3, the hypothesis of Theorem 2 (3°), namely that $f(x, D)$ is closely $\mathbb{R}^n_+$-convex for every $x \in D$, means actually that the bifunction f is $\mathbb{R}^n_+$-subconvexlike in its second argument, i.e., for every $x \in D$ the function $g = f(x, \cdot) : D \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-subconvexlike.

Notice that under the hypothesis of Theorem 2 (3°) the set D is not necessarily convex. For instance, $D := \{0, 1\} \subseteq X := \mathbb{R}$ is not convex, but the constant bifunction $f : D \times D \to \mathbb{R}^n$, defined by $f(x, y) = 0_n$ for all $(x, y) \in D \times D$, is $\mathbb{R}^n_+$-subconvexlike in its second argument.

Corollary 2 *Assume that D is convex. If f is $\mathbb{R}^n_+$-convex in its second argument, i.e., for every $x \in D$ the function $f(x, \cdot) : D \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-convex, then* (23) *and* (24) *hold.*

Proof Directly follows from Theorem 2, in view of Lemma 3 and Remark 4. □

4 Decomposition of vector equilibrium problems

As in the previous section, we assume that $f = (f_1, \ldots, f_n) : D \times D \to \mathbb{R}^n$ is a vector-valued bifunction satisfying the property (19), where D is a nonempty subset of a real linear space X.

For convenience we introduce the index set $I_n := \{1, \ldots, n\}$. Given any selection of indices, $I := \{i_1 < \cdots < i_k\} \subseteq I_n$, the notation f_I will represent the function

$$f_I = (f_{i_1}, \ldots, f_{i_k}) : D \times D \to \mathbb{R}^k. \tag{26}$$

By analogy to multi-criteria optimization problems, which can be decomposed into a family of optimization subproblems associated to certain selections of criteria (see, e.g., Popovici 2005), we will associate to each nonempty set of indices $I \subseteq I_n$ the *equilibrium subproblem* governed by f_I, defined as follows.

If $I = \{i\}$ is a singleton, then we consider the scalar equilibrium problem governed by f_i, whose solution set is

$$\text{eq}(D \mid f_I) := \text{eq}(D \mid f_i).$$

If I has cardinality $|I| = k \geq 2$, then we consider the vector equilibrium problem governed by f_I, whose weak solutions, strong solutions and proper solutions are the elements of the corresponding sets

$$\text{w-eq}(D \mid f_I) := \{x \in D \mid 0_k \in \text{w-min}\, f_I(x, D)\},$$
$$\text{s-eq}(D \mid f_I) := \{x \in D \mid 0_k \in \text{s-min}\, f_I(x, D)\},$$
$$\text{p-eq}(D \mid f_I) := \{x \in D \mid 0_k \in \text{p-min}\, f_I(x, D)\},$$

according to (20), (21) and (22), respectively.

Lemma 4 *For every nonempty subset I of I_n we have*

$$\text{p-eq}(D \mid f_I) \subseteq \text{s-eq}(D \mid f_I) \subseteq \text{w-eq}(D \mid f_I) \subseteq \text{w-eq}(D \mid f). \tag{27}$$

Proof Let $I \subseteq I_n$ be a nonempty set. The first two inclusions in (27) hold by Lemma 2 applied to f_I in the role of f. In order to prove the third inclusion, let $x \in \text{w-eq}(D \mid f_I)$. Suppose to the contrary that $x \notin \text{w-eq}(D \mid f)$. Then we have $0_n \notin \text{w-min}\, f(x, D)$ by (20). Since $0_n = f(x, x) \in f(x, D)$, we infer the existence of some $y \in D$ such that $f(x, y) < 0_n$, i.e., $f_i(x, y) < 0$ for all $i \in I_n$. It follows that $f_I(x, y) < 0_k$, contradicting the hypothesis that $x \in \text{w-eq}(D \mid f_I)$. □

Theorem 3 *If the bifunction f is $\mathbb{R}^n_+$-subconvexlike in its second argument, then*

$$\text{w-eq}(D \mid f) = \bigcup_{\emptyset \neq I \subseteq I_n} \text{p-eq}(D \mid f_I). \tag{28}$$

Proof Lemma 4 shows that the inclusion "$\supseteq$" in (28) holds, even in absence of any convexity assumption.

In order to prove the inclusion "$\subseteq$" assume that f is $\mathbb{R}^n_+$-subconvexlike in its second argument. In view of Remark 4, this means that $\text{cl}(f(x, D) + \mathbb{R}^n_+)$ is convex for every $x \in D$. Consider an arbitrary element $\tilde{x} \in \text{w-eq}(D \mid f)$. Since the hypothesis of Theorem 2 (3°) is fulfilled, we can deduce by (23) the existence of $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that $c \gneq 0_n$ and $\tilde{x} \in \text{eq}(D \mid \langle c, f \rangle)$. Consider the set

$$\tilde{I} = \{i \in I_n \mid c_i > 0\}.$$

Since $c \gneq 0_n$, the set $\tilde{I}$ is nonempty. Denoting by m its cardinality, it can be written as $\tilde{I} = \{i_1, \ldots, i_m\}$ with $1 \le i_1 < \cdots < i_m \le n$. Let us introduce a function $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m) : D \times D \to \mathbb{R}^m$ and a vector $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_m) \in \mathbb{R}^m$, defined for every $j \in I_m = \{1, \ldots, m\}$ as

$$\tilde{f}_j = f_{i_j} \quad \text{and} \quad \tilde{c}_j = c_{i_j}.$$

Observe that $\langle c, f \rangle = \langle \tilde{c}, \tilde{f} \rangle$, where the inner products are defined in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Recalling that $\tilde{x} \in \text{eq}(D \mid \langle c, f \rangle)$, we actually have $\tilde{x} \in \text{eq}(D \mid \langle \tilde{c}, \tilde{f} \rangle)$. Since $\tilde{c} > 0_m$, we infer by 2° in Theorem 2 (applied for $n := m$, $f := \tilde{f}$ and $c := \tilde{c}$) that $\tilde{x} \in \text{p-eq}(D \mid \tilde{f})$. Taking into account that, by the notational convention (26), we actually have $\tilde{f} = f_{\tilde{I}}$, it follows that $\tilde{x} \in \text{p-eq}(D \mid f_{\tilde{I}}) \subseteq \bigcup_{\emptyset \neq I \subseteq I_n} \text{p-eq}(D \mid f_I)$. Thus inclusion "$\subseteq$" in (28) is true. □

Corollary 3 *Assume that D is convex. If the bifunction f is $\mathbb{R}^n_+$-convex in its second argument, then* (28) *holds.*

Proof Directly follows from Theorem 3, in view of Lemma 3 and Remark 4. □

Corollary 4 *Under the hypotheses of Theorem 3 (in particular, Corollary 3), we have*

$$\text{w-eq}(D \mid f) = \bigcup_{\emptyset \neq I \subseteq I_n} \text{s-eq}(D \mid f_I). \tag{29}$$

Proof By Theorem 3 (in particular, Corollary 3) and Lemma 4 we deduce that w-eq$(D \mid f) = \bigcup_{\emptyset \neq I \subseteq I_n}$ p-eq$(D \mid f_I) \subseteq \bigcup_{\emptyset \neq I \subseteq I_n}$ s-eq$(D \mid f_I) \subseteq \bigcup_{\emptyset \neq I \subseteq I_n}$ w-eq$(D \mid f_I) \subseteq$ w-eq$(D \mid f)$, hence the equality (29) holds. □

Remark 5 Given a vector-valued function, $F : D \to \mathbb{R}^n$, we can define a bifunction $f : D \times D \to \mathbb{R}^n$ for all $(x, y) \in D \times D$ by

$$f(x, y) := F(y) - F(x).$$

In this case the vector equilibrium problem governed by f becomes a multi-criteria minimization problem. Thus, as a direct consequence of Corollary 3, we recover a well-known result in multi-criteria optimization with applications in location theory, namely Corollary 1 of Lowe et al. (1984).

Similarly, by an appropriate choice of the bifunction f, certain types of vector variational inequalities can be also formulated as vector equilibrium problems. In particular, Theorem 13 of Popovici and Rocca (2012) as well as Theorem 4.1 of Popovici and Rocca (2013) can be seen as counterparts of Corollary 3.

Further extensions of our results could be established by considering arcwise convexity, as in the paper by La Torre and Popovici (2010).

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