

On Cournot–Bertrand competition with differentiated products

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Abstract In this paper, we present a game theoretic framework for Cournot–Bertrand competition based on a nonlinear price function. The competition is between two firms and is assumed to take place in terms of pricing decision and quantity produced. However, the proposed objective function has not been used in literature before, yet the throughput obtained in this paper generalizes some of the existing results in literature. The competitive interaction between firms is described and analyzed using best-reply reaction, proposed adaptive adjustment and bounded rationality approach. The condition of stability of Nash equilibrium (NE) is induced by these approaches. Interestingly, we prove that there exists exactly a unique NE. Furthermore, it is noticed that when firms adopt best-reply and the proposed adaptive adjustment, the firms' strategic variables become asymptotically stable. On the contrary, when the bounded rationality is used both quantity and price behave chaotically due to bifurcation occurred.

Keywords Cournot–Bertrand game · Best-reply · Adaptive adjustment · Bounded rationality · Nash equilibrium · Asymptotic stability · Bifurcation · Chaos

1 Introduction

The most fundamentals papers in the theory of strategic interaction among firms are those published by [Cournot \(1838\)](#), [Bertrand \(1883\)](#). Cournot suggested quantities as strategic variables in which firms may play in market. While Bertrand proposed prices as other strategic variables that some firms may prefer. Since the appearance of these two fundamental papers, a huge number of papers have extended these observations. It has been reported elsewhere

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(Singh and Vives 1984; Vives 1985; Cheng 1985) that quantities induce a lower degree of competition if commodities are substitutes. Therefore, if firms are free to choose their strategic variables, they definitely prefer to compete in quantities rather than prices. There is also another type of game in which one firm competes in quantity and the other firm competes in price. This is called a Cournot–Bertrand game (Kreps and Scheinkman 1983). To the best of our knowledge, Shubik (1955) was the first who has systematically investigated duopoly settings with firms choosing quantities and prices at the same time. However Shubik has formulated the analytical framework for the analysis of this kind of games, he was not able to derive the equilibrium solution of these games. Since then, the literature has reported a lower number of papers in which the equilibrium solution (Nash equilibrium) has analytically obtained and its characteristics such as stability has been discussed. The Cournot–Bertrand competition requires a certain degree of differentiation between products offered by firms so that to avoid one firm from dominating the market by its lower price. In Arya et al. (2008), Hackner (2000), Tremblay et al. (2009), Zanchettin (2006), the authors have argued that in certain cases, Cournot–Bertrand game may be optimal. Moreover, Tremblay et al. (2010) have illustrated that empirical evidence has proven that this kind of competition is abundant. Recently Tremblay and Tremblay (2011) have studied the static prosperities of the Nash equilibrium of a Cournot–Bertrand duopoly according to product differentiation. Naimzada and Tramontana (2012) have studied the dynamic prosperities of a Cournot duopoly game with product differentiation using linear demand and cost functions. The idea of the current paper is motivated by what have been discussed and obtained in Tremblay et al. (2011).

This paper aims to set up a benchmark framework that studies the dynamic characteristics of the Cournot–Bertrand game. We set up a model that is characterized by standard assumptions such as a general nonlinear price function and linear cost function. The price function can be exploited as a linear and non-linear function. Furthermore the assumption that one firm sets price and the other uses quantity as a source of heterogeneity is studied in this paper.

The paper is organized as follows. In Sect. 2 the model is introduced in its static version. The best reply is used to get the Nash equilibrium and to study its stability. A proposed repeated game with instantaneous adjustment is presented in Sect. 3. It shows that the model has only one Nash equilibrium point and is asymptotically stable under certain conditions. In Sect. 4, a repeated game with adaptive adjustment approach is used to investigate the model. Interestingly, this approach demonstrates that the Nash point of the model is asymptotically stable. Furthermore, a repeated game with bounded rationality approach is used to investigate the model in Sect. 5. In addition, it is shown that the model behaves chaotically due to bifurcation occurred. Finally, some conclusions are presented.

2 The static model

In this section, a market is considered to have two firms that make their strategic choices simultaneously. The strategic variable of firm 1 is q_1 , as in Cournot, while the other firm fixes its price p_2 , as in Bertrand. It is assumed that both firms are in competition and decisions are made simultaneously in an environment with perfect and complete information. According to this information, we propose that both firms fact the following inverse non-linear demand functions:

$$\begin{aligned} p_1 &= a - q_1^\gamma - dq_2^\gamma \\ p_2 &= a - q_2^\gamma - dq_1^\gamma \end{aligned} \quad (1)$$

where $a > 0, \gamma \in \mathbb{R}$. The constant d has significant meaning that is the degree of product differentiation or product substitution. If $d = 0$ this means that the market incorporates two monopolistic firms. If $d = 1$ the two inverse demand functions become identical and thus homogenous commodities are obtained. It is assumed that firm 1 plays in market as in a Cournot duopoly with output q_1 . While the other firm plays as in Bertrand case with price p_2 . Therefore, the system (1) can be rewritten in the two strategic variables, q_1 and p_2 :

$$\begin{aligned} p_1 &= a(1-d) - (1-d^2)q_1^\gamma - dp_2 \\ q_2 &= \sqrt[\gamma]{a - p_2 - dq_1^\gamma} \end{aligned} \tag{2}$$

Both firms use linear production cost function $cq_i, i = 1, 2$, where c is a fixed marginal cost. So, the profits of the two firms are:

$$\pi_i = (p_i - c)q_i, \quad i = 1, 2 \tag{3}$$

Using (2), the first-order conditions that maximize the profits yield the following non-linear best-reply functions as follows:

$$\begin{aligned} r_1(p_2) : q_1 &= \left[\frac{a(1-d) - c}{(1+\gamma)(1-d^2)} + \frac{d}{(1+\gamma)(1-d^2)}p_2 \right]^{1/\gamma} \\ r_2(q_1) : p_2 &= \frac{\gamma a + c}{(1+\gamma)} - \frac{\gamma d}{(1+\gamma)}q_1^\gamma \end{aligned} \tag{4}$$

These firms' best-replies intersect in Nash equilibrium, $NE = (q_1^*, p_2^*)$ as follows:

$$\begin{aligned} q_1^* &= \left[\frac{(1+\gamma-d)(a-c)}{(1+\gamma)^2(1-d^2) + \gamma d^2} \right]^{1/\gamma} \\ p_2^* &= \frac{\gamma a + c}{(1+\gamma)} - \frac{\gamma d}{(1+\gamma)} \left[\frac{(1+\gamma-d)(a-c)}{(1+\gamma)^2(1-d^2) + \gamma d^2} \right] \end{aligned} \tag{5}$$

Proposition 1 *With differentiated products, the Nash equilibrium is stable when*

$$\gamma(1+\gamma)(1-d^2) \left[\frac{(1+\gamma-d)(a-c)}{(1+\gamma)^2(1-d^2) + \gamma d^2} \right]^{\frac{\gamma-1}{\gamma}} > d$$

Proof It is shown in Dixit (1986) that a stable equilibrium in the Cournot–Bertrand model requires that $|\pi_{ii}| > |\pi_{ij}|$, where $\pi_{ii} = \frac{\partial^2 \pi_i}{\partial s_i^2}, \pi_{ij} = \frac{\partial^2 \pi_i}{\partial s_i \partial s_j}, s_i = q_1$ and $s_2 = p_2$. For $\gamma = 1$, those conditions give that $d < \frac{\sqrt{17}-1}{4}$ as in Tremblay and Tremblay (2011). These conditions prove the proposition 1. \square

Now we consider time and assume that both firms take action in discrete time. Due to this some cases are analyzed in the next sections.

3 Best reply with instantaneous adjustment

Usually firms do not have complete information about its opponent choices and instead they use some expectations about that. Indeed, Cournot has proposed that without any other information. A firm may conjecture that its opponent will make the same decision in the

next period as taken the current one (Naimzada and Tramontana 2012). These kinds of expectations are known as static (or naïve) expectations and are used here with both firms. It is assumed that firm 1 expects that its rival will maintain the same price fixed in the next time step, $t + 1$ as in the current t . On the other hand, firm 2 chooses the price considering the quantity produced by firm 1 in $t + 1$ equal to the quantity already produced in t . Based on these expectations, the best-reply with instantaneous adjustment gives the following system:

$$\begin{aligned} q_1' &= r_1(p_2) = \left[\frac{a(1-d)-c}{(1+\gamma)(1-d^2)} + \frac{d}{(1+\gamma)(1-d^2)} p_2 \right]^{1/\gamma} \\ p_2' &= r_2(q_1) = \frac{\gamma a+c}{(1+\gamma)} - \frac{\gamma d}{(1+\gamma)} q_1^\gamma \end{aligned} \tag{6}$$

where, $(\cdot)'$ denotes the unit-time advancement operator. The above system permits only one fixed point that is NE. Therefore, it is important to discuss the stability of this point.

Proposition 2 *The Nash equilibrium NE of the above system is globally asymptotically stable if $d < \frac{1+\gamma}{\sqrt{\gamma+(1+\gamma)^2}}$ and $\gamma > 0$.*

Proof To prove this proposition, the Jacobian of system (6) is given as follows:

$$J = \begin{bmatrix} 0 & \frac{d}{\gamma(1+\gamma)(1-d^2) \left(\frac{(a-c)(\gamma-d+1)}{\gamma d^2+(1-d^2)(1+\gamma)^2} \right)^{\frac{\gamma-1}{\gamma}}} \\ -\frac{\gamma^2 d}{1+\gamma} \left(\frac{(a-c)(\gamma-d+1)}{\gamma d^2+(1-d^2)(1+\gamma)^2} \right)^{\frac{\gamma-1}{\gamma}} & 0 \end{bmatrix} \tag{7}$$

with a trace equal to 0 and a determinant equal to:

$$\text{Det} = \frac{\gamma d^2}{(1-d^2)(1+\gamma)^2} \tag{8}$$

The eigenvalues hence become,

$$\begin{aligned} \mu_1 &= \frac{d\sqrt{\gamma}}{(1+\gamma)\sqrt{d^2-1}} \\ \mu_2 &= \frac{-d\sqrt{\gamma}}{(1+\gamma)\sqrt{d^2-1}} \end{aligned} \tag{9}$$

with these eigenvalues the following cases are obtained:

Case I if $|d| < 1$, then the eigenvalues are purely imaginary and conjugate with modulus:

$$|m| = \frac{d\sqrt{\gamma}}{(1+\gamma)\sqrt{1-d^2}} \tag{10}$$

This means that $|m| < 1$ gives $d < \frac{1+\gamma}{\sqrt{\gamma+(1+\gamma)^2}}$ and consequently NE is asymptotically stable.

Case II if $d \in \mathbb{R} - (-1, 1)$, then the eigenvalues are real numbers with opposite sign. In this case, the NE is asymptotically stable provided that $d > \sqrt{1+\gamma}$ and $\gamma > -1$. \square

Now let us make some numerical simulations to show the qualitative behavior of the NE of the above system. From an economic perspective, proposition 1 (case I) shows that when a firm attempts to increase profits by reducing the degree of competition through an increase in product differentiation d above the value $\frac{1+\gamma}{\sqrt{\gamma+(1+\gamma)^2}}$, it also tends to destabilize the Nash

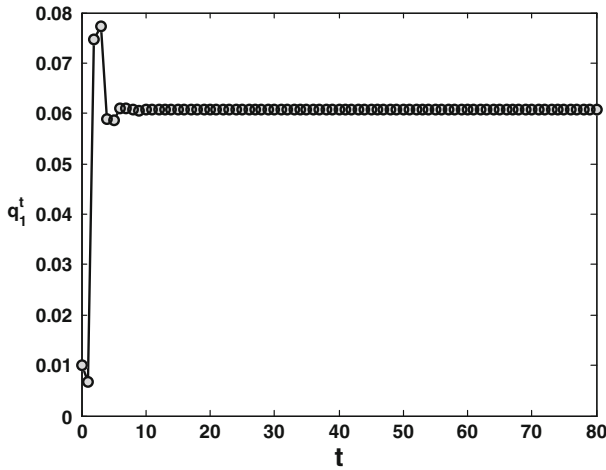


Fig. 1 The stability of $q_{1,t}$: $a = 1, \gamma = 0.22, d = 0.7, c = 0.1$ (case I)

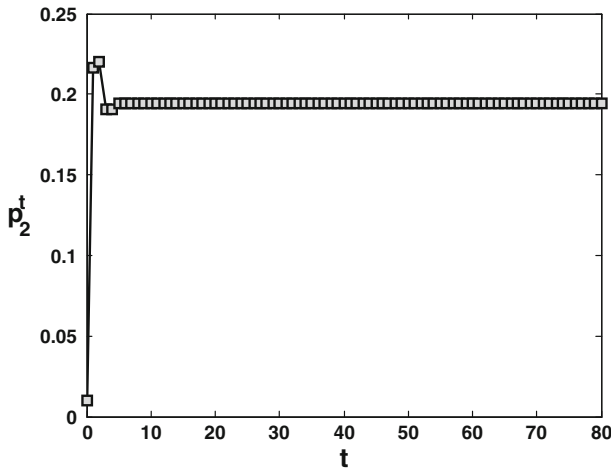


Fig. 2 The stability of $p_{2,t}$: $a = 1, \gamma = 0.22, d = 0.7, c = 0.1$ (case I)

equilibrium. The same observation is for the case II. Figures 1 and 2 depict for some value parameters the throughput of case I while Figs. 3 and 4 depict the throughput of case II. It is worth mentioning here that if $\gamma = 1$, the results found by Naimzada and Tramontana (2012) become a special case of this paper.

4 The repeated game with adaptive adjustments

Recently, several works have considered a more realistic mechanism through which firms are willing to use adaptive adjustment. In Naimzada and Tramontana (2012), it is assumed that firms are reluctant to change their current production or price. The reason for that is firms are affected by some kind of anchoring which prevents them from changing too much their strategic variables from one period to the next Naimzada and Tramontana (2012). In

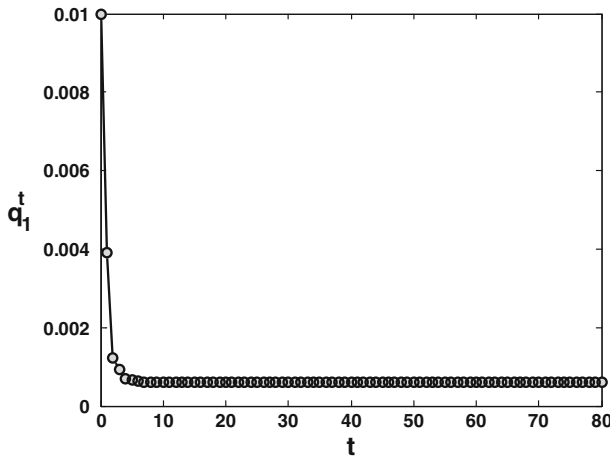


Fig. 3 The stability of $q_{1,t}$: $a = 1, \gamma = 0.22, d = 2, c = 0.1$ (case II)

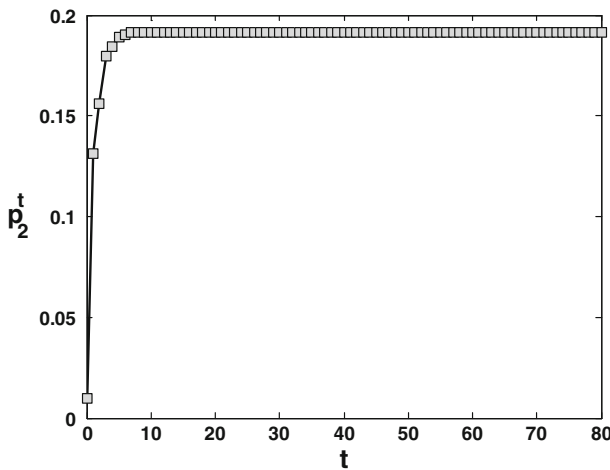


Fig. 4 The stability of $p_{2,t}$: $a = 1, \gamma = 0.22, d = 2, c = 0.1$ (case II)

this section, we suppose that the static game described by (4) is repeated a finite number of times in such a way that both firms move with a weighted average of the true output in the previous period ($q_{1,t}, p_{2,t}$) and the optimal output in the previous period that is given by the best replies (4). Using such weight, ω where $0 \leq \omega \leq 1$ with incomplete information makes both firms adjust slowly. Based on that an adaptive adjustment system of equations is given as follows:

$$\begin{aligned}
 q_{1,t+1} &= \omega q_{1,t} + (1 - \omega) \left[\frac{a(1-d)-c}{(1+\gamma)(1-d^2)} + \frac{d}{(1+\gamma)(1-d^2)} p_{2,t} \right]^{1/\gamma} \\
 p_{2,t+1} &= \omega p_{2,t} + (1 - \omega) \left[\frac{\gamma a+c}{(1+\gamma)} - \frac{\gamma d}{(1+\gamma)} q_{1,t}^\gamma \right]
 \end{aligned}
 \tag{11}$$

With this weight the change in strategic variables from one period of time to the next will be to some extent small. It is noticed that if $\omega = 0$ both firms adjust the value of their strategic

variables to the best reply and therefore the strength of anchoring is going to be weakened. On the opposite, when $\omega = 1$ the anchoring will be strong enough such that both firms will never alter their adopted strategies. Therefore, the system described by (11) admits the following NE.

$$\begin{aligned} \bar{q}_1 &= \left[\frac{a(1-d)-c}{(1+\gamma)(1-d^2)} + \frac{d(1+\gamma)(\gamma a+c)(1-d^2)-\gamma d^2[a(1-d)-c]}{(1+\gamma)(1-d^2)[(1+\gamma)^2(1-d^2)+\gamma d^2]} \right]^{1/\gamma} \\ \bar{q}_2 &= \frac{(1+\gamma)(\gamma a+c)(1-d^2)-\gamma d[a(1-d)-c]}{(1+\gamma)^2(1-d^2)+\gamma d^2} \end{aligned} \tag{12}$$

Proposition 3 *The Nash equilibrium of the system described by (11) is globally asymptotically stable if $\omega^2 + \frac{\gamma d^2(1-\omega)^2}{\omega^2(1-d^2)(1+\gamma)^2} < 0$ such that $\gamma \in [-0.194, \infty)$.*

Proof To prove this proposition, the Jacobian of system (11) is given as follows:

$$J = \begin{bmatrix} \omega & \frac{d(1-\omega)}{\gamma \omega(1+\gamma)(1-d^2) \left(\frac{(a-c)(\gamma-d+1)}{\gamma d^2+(1-d^2)(1+\gamma)^2} \right)^{\left(\frac{\gamma-1}{\gamma}\right)}} \\ \frac{-\gamma^2 d(1-\omega)}{\omega(1+\gamma)} \left(\frac{(a-c)(\gamma-d+1)}{\gamma d^2+(1-d^2)(1+\gamma)^2} \right)^{\left(\frac{\gamma-1}{\gamma}\right)} & \omega \end{bmatrix} \tag{13}$$

The eigenvalues hence become,

$$\begin{aligned} \sigma_1 &= \omega + \frac{d(1-\omega)}{\omega(1+\gamma)} \sqrt{\frac{\gamma}{d^2-1}} \\ \sigma_2 &= \omega - \frac{d(1-\omega)}{\omega(1+\gamma)} \sqrt{\frac{\gamma}{d^2-1}} \end{aligned} \tag{14}$$

with these eigenvalues the following cases are obtained:

Case I if $|d| < 1$, then the eigenvalues are imaginary and conjugate with modulus:

$$|M| = \sqrt{\omega^2 + \frac{\gamma d^2(1-\omega)^2}{\omega^2(1-d^2)(1+\gamma)^2}} \tag{15}$$

This means that $|m| < 1$ gives $\omega^2 + \frac{\gamma d^2(1-\omega)^2}{\omega^2(1-d^2)(1+\gamma)^2} < 0$ such that $\gamma \in [-0.194, \infty)$ and consequently NE is asymptotically stable.

Case II if $d \in \mathbb{R} - (-1, 1)$, then the eigenvalues are real numbers. In this case, the NE is asymptotically stable if $-1 < \frac{d}{\omega(1+\gamma)} \sqrt{\frac{\gamma}{d^2-1}} < 1 + \omega$. This means that we should utilize values for γ and d such that this condition is satisfied. \square

Now as simulated before, we make some numerical simulation to illustrate the throughput of the above two cases. Figures 5, 6 and 7 depict for some value parameters the throughput of case I while Figs. 8, 9 and 10 depict the throughput of case II.

5 The repeated game with bounded rationality

On the other hand, if a player does not have complete knowledge of the profit function, it can use some local estimation of the marginal profit in order to follow the steepest slope

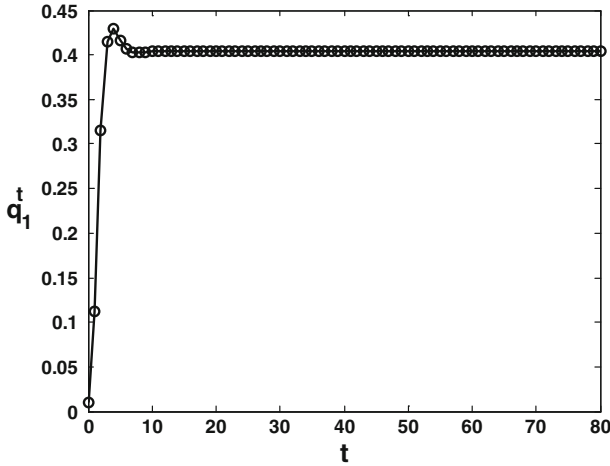


Fig. 5 The stability of $q_{1,t}$: $a = 1, \gamma = 0.8, d = 0.7, c = 0.1, \omega = 0.3$ (case I)

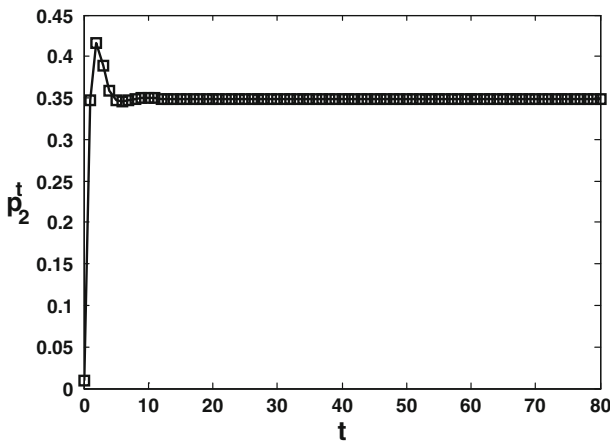


Fig. 6 The stability of $p_{2,t}$: $a = 1, \gamma = 0.8, d = 0.7, c = 0.1, \omega = 0.3$ (case I)

of the profit function. Such limited information makes players unable to completely solve the optimization problem $\max \pi_i (q_{1,t}, p_{2,t}), i = 1, 2$ by considering expectations about the strategic choice (quantity or price) that the competitor will choose for the next period, but they are able to get a correct estimate of their own local slope, i.e. the partial derivatives of the profit computed at the current state of production and price. This will help each player to increase or decrease the quantity produced and price at time $t + 1$ depending on whether its own marginal profit at time t is positive or negative. This adjustment mechanism is described as follows:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k (q_{1,t}) \frac{\partial \pi_1}{\partial q_{1,t}}, \\ p_{2,t+1} &= p_{2,t} + k (p_{2,t}) \frac{\partial \pi_2}{\partial p_{2,t}} \end{aligned} \tag{16}$$

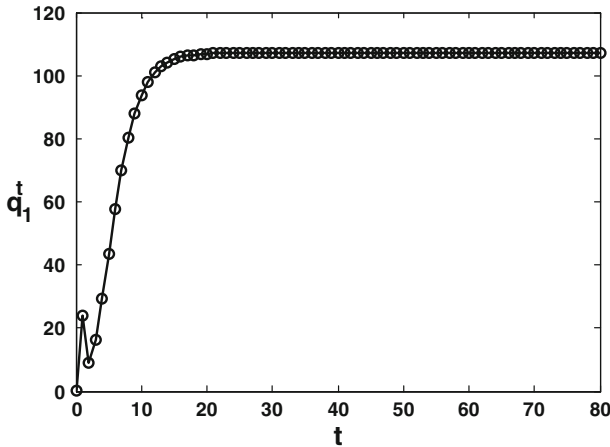


Fig. 7 The stability of $q_{1,t}$: $a = 1, \gamma = -0.19401, d = 0.7, c = 0.1, \omega = 0.3$ (case I)

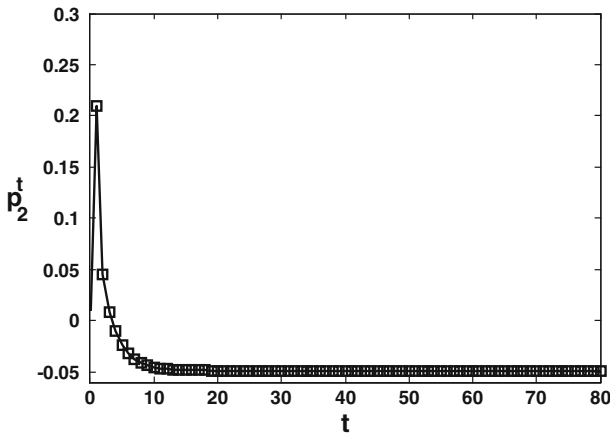


Fig. 8 The instability of $p_{2,t}$: $a = 1, \gamma = -0.19401, d = 0.7, c = 0.1, \omega = 0.3$ (case II)

where, k is known as the speed of adjustment. Using (2) and (3), one gets the following dynamical system:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k \left[a(1-d) - c - dp_{2,t} - (1-d^2)q_{1,t}^\gamma \right], \\ p_{2,t+1} &= p_{2,t} + k \left[a - p_{2,t} - dq_{1,t}^\gamma \right]^{1/\alpha} \end{aligned} \tag{17}$$

The system admits the following Nash equilibrium point:

$$\begin{aligned} \bar{q}_1 &= \left(\frac{a(1-2d)-c}{1-2d^2} \right)^{1/\gamma} \\ \bar{q}_2 &= \frac{dc+a(1-d)}{1-2d^2} \end{aligned} \tag{18}$$

Analytically, it is tedious for any mathematical packages such as Mathematica, Maple and Matlab to calculate the Jacobian matrix of the system (17) at Nash equilibrium. Instead we investigate the stability or instability of Nash equilibrium through numerical simulations.

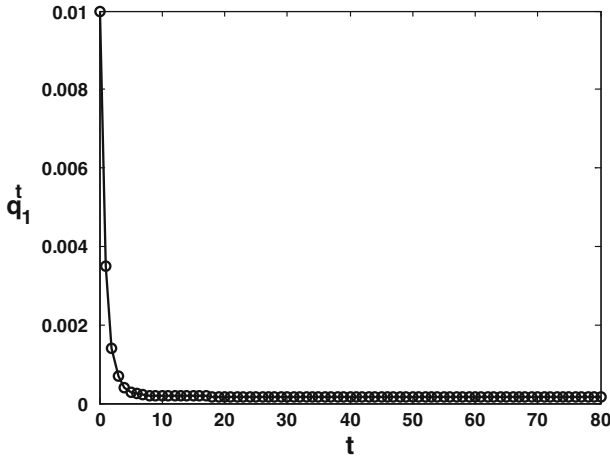


Fig. 9 The stability of $q_{1,t}$: $a = 1, \gamma = 0.2, d = 2.7, c = 0.1, \omega = 0.3$ (case II)

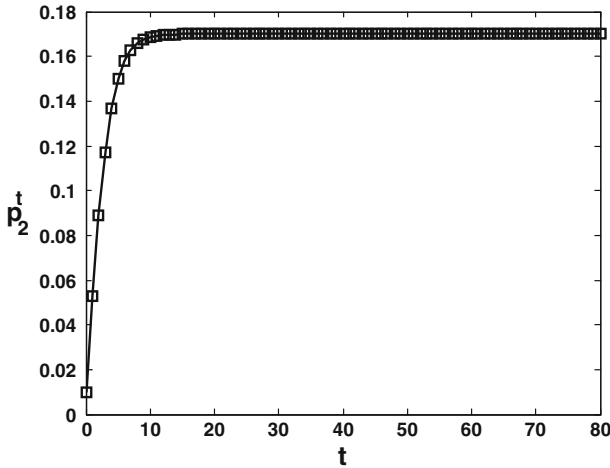


Fig. 10 The stability of $p_{2,t}$: $a = 1, \gamma = 0.2, d = 2.7, c = 0.1, \omega = 0.3$ (case II)

When the values of the system’s parameters are $a = 1, \gamma = 1, d = 0.1, c = 0.1, k \in [2.5, 3.5]$, Figs. 11 and 12 show that the quantity q_1^t and the price p_2^t go from stable state to chaos through bifurcation. Moreover, it shows that when both firms use linear prices functions ($\gamma = 1$) and the degree of differentiation d between firms is small, the strategic variables of the firms are stable until arrive to chaos due to bifurcation.

On the contrary, when firms adopt nonlinear prices with low level of differentiation the Nash equilibrium becomes unstable. Figures 13 and 14 simulate this output. Based on these findings, one should be aware that bounded rationality approach is less powerful in tackling this kind of Cournot–Bertrand competition when comparison with the proposed adaptive adjustment. The theoretical importance of this result is that our analysis is the first to analyze and discover the weakness of bounded rationality when applied on price/quantity game. Furthermore, the literature has shown that almost all the price functions used even in Cournot

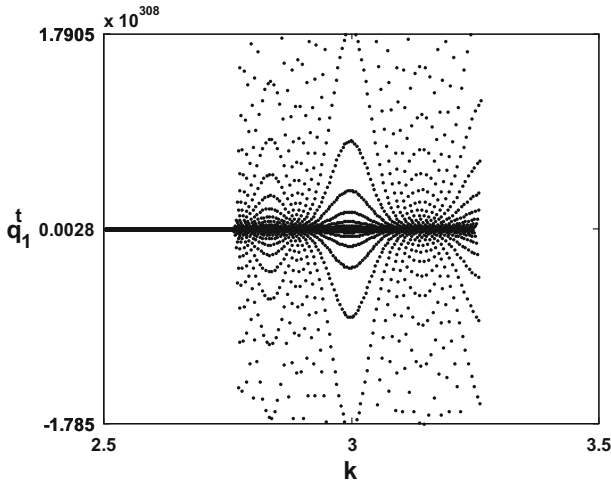


Fig. 11 The first part of system (17): k vs. $q_{1,t}$ at the parameters, $a = 1, \gamma = 1, d = 0.1, c = 0.1, k \in [2.5, 3.5]$

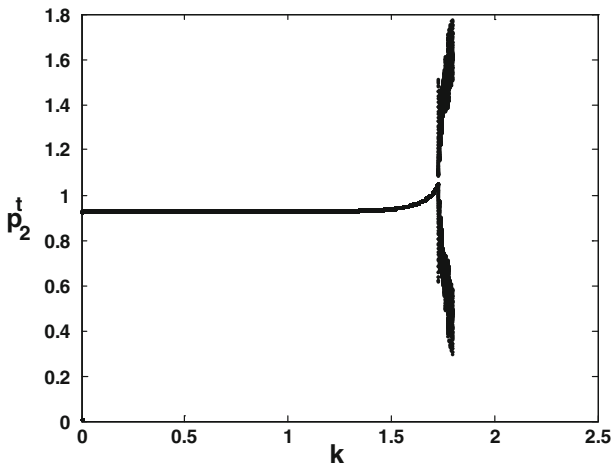


Fig. 12 The second part of system (17): k vs. p_2^t at the parameters, $a = 1, \gamma = 1.1, d = 0.1, c = 0.1, k \in [0, 3.5]$

or Bertrand games are either linear or quadratic. This paper introduces a generalized non-linear price function.

6 Conclusions

In this paper, the dynamical characteristics of a Cournot–Bertrand competition based on an unknown nonlinear price function have been analyzed. However, the proposed price function has not been used previously, and the results found in this paper generalize some of the existing results in literature [see for example, [Naimzada and Tramontana \(2012\)](#)]. The competition interaction between firms has been analyzed using best-reply reactions, instan-

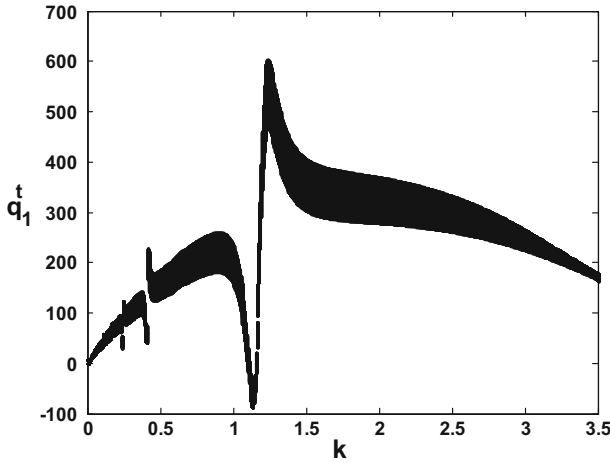


Fig. 13 The first part of system (17): k vs. q_1^t at the parameters, $a = 1, \gamma = -1.1, d = 0.1, c = 0.1, k \in [0, 3.5]$

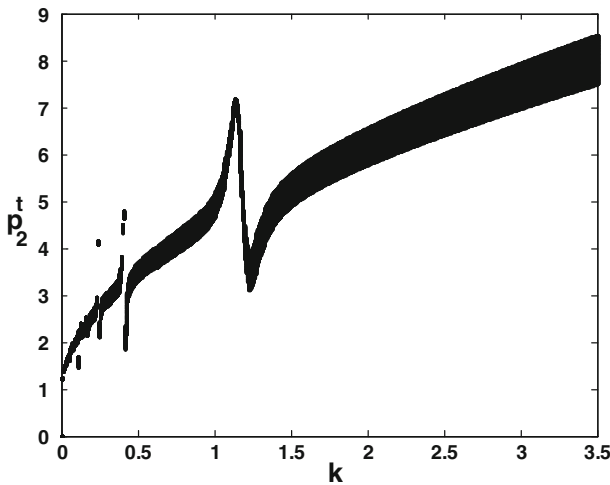


Fig. 14 The second part of system (17): k vs. p_2^t at the parameters, $a = 1, \gamma = -1.1, d = 0.1, c = 0.1, k \in [0, 3.5]$

taneous and adaptive adjustment, and bounded rationality. Our results have shown that when firms have adopted either the best-reply or the proposed adaptive adjustment, Nash equilibrium becomes asymptotically stable. On the contrary, with bounded rationality, the Nash equilibrium becomes chaotic due to bifurcation.

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